Identities involving binomial-coefficients,
Bernoulli- and Stirlingnumbers

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Intro & Notation

Abstract: an overview about the concept of this collection of articles is given as well as notational conventions.

Contents:

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1. Introduction

In this collection of articles I present some known identities about binomial-coefficients, Bernoulli-, Stirlingnumbers (and some other), as well as some heuristics about compositions of that numbers in context with geometric series (powerseries), harmonic- and more general: zeta-series.

I present the results in terms of a "*toolbox*" of matrices and vectors of infinite dimension.

Most of the matrices are of lower triangular shape, so that the common matrix-operations like addition, right-multiplication and inversion, even finding of Eigensystems etc are based on finite operations. Convergence/divergence is then an issue when leftmultiplication is applied and/or infinite square-matrices like the Vandermondematrix ZV are used - such situations must be considered specifically.

Thus a chapter about divergent summation (for instance Cesaro and Eulersummation) is appended where the matrix-representation can express some of the common techniques in a very concise and instructive manner. Some selected results of divergent sums are already included, which exhibit some much interesting relations, for instance the sums of bernoulli-numbers and some variants of that.

Expressing relations in terms of matrix-products is a rich source of identities, which are usually expressed as identities of sums of products of coefficients (like the "sum of products of binomial-coefficients and bernoulli-numbers"), since a matrix-product of infinite dimensioned matrices gives infinitely many such sums for each column (and each row) of the result-vector/matrix in one shot.

Over this collection of articles I'll often save the effort to express such relations in the conventional summation-notation; they may be simply reproduced; in some cases however, where the expressions are very common or special interesting, I'll write them out for convenience of the reader.

1.1. notational conventions

The *toolbox* contains the following vectors and matrices, with the following conventions:

1. all matrices and vectors are understood as of *infinite dimension*
2. vectors are primarily assumed as *column-vector*
3. the transpose-symbol *"~"* is used (as in the openly available number-theoretic program Pari/GP) for convenient translation of the formulae into the programming language, and to prevent confusion with the apostroph for the derivative, which shall also be used in some chapters.
4. the indices r,c for rows and columns are always assumed as *beginning at zero*
5. the superscript prefix ^{d} is added, if a vector is assumed as the coefficients of a diagonal matrix. With very common diagonal matrices (***I***,***J***,***F***) and where the context is clear/ should be clear by the requirements of matrixdimension in multiplications, I often leave this symbol for brevity of the formula.
6. the elementwise-product of two matrices ("Hadamard"- multiplication) is denoted by "☼":

   $A ☼ B = C$

   If mixed operations like elementwise multiplication and division in a formula I'll append the appropriate standardoperator:

   $A ☼* B \; ☼/ \; C= D$
7. matrices are generally assumed as lower triangular matrices (few exceptions)

1.1.1. Vectors

Basic vectors are

Vandermondevectors $V(x) = [1, x, x^2, x^3,...] \sim$
used for operations on formal powerseries

harmonic/Zeta-like vectors $Z(s) = [1,1/2^s,1/3^s,1/4^s,...] \sim$
used for composition with coefficients of dirichlet-series

Summing vector $V(1) = Z(0) = [1,1,1,1,...]\sim$

Factorials $Fac(s) = [1, 1, 2!^s, 3!^s, 4!^s,...] \sim$
in some chapters I use only F and F^{-1} for the diagonal matrix Fac(1) and Fac(-1)

Bernoulli-numbers $B = [\beta_0, \beta_1, \beta_2,...]$ *where β_k are the k'th bernoulli-numbers*
Bernoulli-numbers $B_+ = [\beta_0, \beta_1, \beta_2,...]$ *where* $\beta_1 = +1/2$

I also use for notational convenience ***J*** and ***I*** for the vectors resp diagonalmatrices

Identity-matrix $I = diag(1,1,1,...)$
alternating identity $J = {}^dV(-1) = diag([1,-1,1,...])$

1.1.2. Matrices

Basic lower-triangular matrices of number-theoretic coefficients are the following (more detailed description in the resp. chapter):

Pascalmatrix $P := P_{r,c} = binomial(r,c)$ *if* $r>=c$
column-signed $P_j := P * J := P_{j\ r,c} = (-1)^c * binomial(r,c)$ *if* $r>=c$
row-signed ${}_jP := J * P := {}_jP_{r,c} = (-1)^r * binomial(r,c)$ *if* $r>=c$

matrices representing the Bernoulli-polynomials
$BY := BY_{r,c} = \beta_{r-c} * binomial(r,c)$ *if* $r>=c$
$BY_m := BY$, *using the standard setting* $\beta_1 = -1/2$
$BY_p :=$ *similar to BY, only using* $\beta_1 = +1/2$

G-matrices $G := G_{r,c} = \beta_{r-c} * binomial(r,c)/(c+1)$ *if* $r>=c$
$G_m := G$, *using the standard setting* $\beta_1 = -1/2$
$G_p :=$ *similar to G, only using* $\beta_1 = +1/2$

Stirling-matrices
1'st kind $St1 := st1_{r,c} = stirling\_kind1_{r,c}$ *if* $r>=c$
2'nd kind $St2 := st2_{r,c} = stirling\_kind2_{r,c}$ *if* $r>=c$
row/colum-shifted versions (first column/row is zero, only top-left-element=1)
1'st kind $S1 := s1_{r+1,c+1} = stirling\_kind1_{r,c}$ *if* $r>=c>0$
2'nd kind $S2 := s2_{r+1,c+1} = stirling\_kind2_{r,c}$ *if* $r>=c>0$

the Vandermondematrix ZV as column-concatenation of Z-vectors Z(0),Z(-1),Z(-2),...
$ZV := zv_{r,c} = (r+1)^c$
$VZ := vz_{r,c} = (c)^r$ *// beginning at c,r=zero*

a Toeplitzmatrix Toeplitz(x) is by common definition a square-matrix
$Toeplitz(x) = Toeplitz(x)[r,c] = x^{r-c}$
Here it occurs most often in a hadamard-product with a triangular matrix, for instance with P
$Toeplitz(x) = Toeplitz(x)[r,c] = x^{r-c}$ *if* $r>=c$

1.1.3. Other shorthands

For the here most common binomial-coefficient *binomial(r,c)* I use for brevity

bi(r,c) := binomial(r,c)
ch(r,c) := binomial(r,c) *// I'll delete this abbreviation while rewriting the articles*

1.2. Examples for operating with vectors and matrices

The sum of a powerseries in x:

$$\sum_{k=0}^{inf} x^k = V(1)\sim * V(x) = 1/(1 - x)$$

$$V(1)\sim * V(x) = \Sigma_{r=0..inf} (1 * x^r) = 1/(1-x)$$

$$\begin{bmatrix} 1 \\ x \\ x\hat{}2 \\ x\hat{}3 \\ x\hat{}4 \\ x\hat{}5 \\ ... \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & ... \end{bmatrix} \quad \begin{bmatrix} 1/(1-x) \end{bmatrix}$$

Note: in all matrix-display I'll omit the "…" and limit-notation because of programming-overhead in Pari/GP

Simple sign-inversion:

$$J * V(x) = V(-x)$$

$$\begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \\ 32 \end{bmatrix}$$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ . & -1 & . & . & . & . \\ . & . & 1 & . & . & . \\ . & . & . & -1 & . & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & -1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -2 \\ 4 \\ -8 \\ 16 \\ -32 \end{bmatrix}$$

Using ***P*** and ***V****(x)* means to apply the binomial-theorem:

$$P * V(x) = V(1+x)$$

$$\Sigma_{c=0..r} \text{binomial}(r,c) * x^c = (1+x)^r$$

$$\begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \\ 32 \end{bmatrix}$$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 3 \\ 9 \\ 27 \\ 81 \\ 243 \end{bmatrix}$$

Use of column-signed Pascalmatrix $\boldsymbol{P_j}$

$$P_j * V(x) = V(1-x)$$

$$\Sigma_{c=0..r} (-1)^c \text{binomial}(r,c) * x^c = (1-x)^r$$

$$\begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \\ 32 \end{bmatrix}$$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & -1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ 1 & -3 & 3 & -1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Example eigenvector-relations:

*P_j * V(1/2) = V(1/2)*
saying V(1/2) is an eigenvector of P_j

$\Sigma_{c=0..r}$ $(-1)^c$ binomial(r,c)* $1/2^c$ = $1/2^r$

$$\begin{bmatrix} 1 \\ 1/2 \\ 1/4 \\ 1/8 \\ 1/16 \\ 1/32 \end{bmatrix}$$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & -1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ 1 & -3 & 3 & -1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1/2 \\ 1/4 \\ 1/8 \\ 1/16 \\ 1/32 \end{bmatrix}$$

Powers of matrices:

Powers of P

P^n * V(x) = V(n + x)

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 3 & 1 & . & . & . & . \\ 9 & 6 & 1 & . & . & . \\ 27 & 27 & 9 & 1 & . & . \\ 81 & 108 & 54 & 12 & 1 & . \\ 243 & 405 & 270 & 90 & 15 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 16 \\ 64 \\ 256 \\ 1024 \end{bmatrix}$$

Powers of P

P^x * V(1) = V(1+x)

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ x & 1 & . & . & . & . \\ x^2 & 2*x & 1 & . & . & . \\ x^3 & 3*x^2 & 3*x & 1 & . & . \\ x^4 & 4*x^3 & 6*x^2 & 4*x & 1 & . \\ x^5 & 5*x^4 & 10*x^3 & 10*x^2 & 5*x & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1+x \\ (1+x)^2 \\ (1+x)^3 \\ (1+x)^4 \\ (1+x)^5 \end{bmatrix}$$

Summation of powerseries (includes to handle also their derivatives)

*lim $_{r\text{->}oo}$ (1/2*V(1/2)~ * P)= V(1) ~*

$$\begin{bmatrix} & 1 & & & & \\ & 1 & 1 & & & \\ & 1 & 2 & 1 & & \\ & 1 & 3 & 3 & 1 & \\ & 1 & 4 & 6 & 4 & 1 \\ & 1 & 5 & 10 & 10 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 1/4 & 1/8 & 1/16 & 1/32 & 1/64 \end{bmatrix} \begin{bmatrix} \text{lim r->oo=} & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

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a slightly more general expression of power-series-summation:

lim $_{r\to oo}$ *(1/x*V(1/x) ~ * P* $^{x-1}$ *)= V(1) ~*

P^2

1				
2	1			
4	4	1		
8	12	6	1	
16	32	24	8	1
32	80	80	40	10

[1/3 1/9 1/27 1/81 1/243 1/729] [lim r->oo= 1 1 1 1 1]

To get a notation, where "input" and "output" of an operation (multiplication with a matrix-operator) has the same form *V(x)* it is sometimes useful to introduce definitions of a row/column-shift of common matrixes, for instance $P_1 = DR^{-1} * P$

lim $_{r\to oo}$ *V(1/2)~ ** P_1 *= V(1) ~*

P1

1	.	.	.	.	.
0	1	.	.	.	.
0	1	1	.	.	.
0	1	2	1	.	.
0	1	3	3	1	.
0	1	4	6	4	1

[1 1/2 1/4 1/8 1/16 1/32] [lim r->oo= 1 1 1 1 1 1]

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a slightly more general expression of power-series-summation:

lim $_{r\to oo}$ *(V(1/x) ~ ** P_1 *)= V(1/(x-1)) ~*
example:x=3

P1

1	.	.	.	.	.
0	1	.	.	.	.
0	1	1	.	.	.
0	1	2	1	.	.
0	1	3	3	1	.
0	1	4	6	4	1

[1 1/3 1/9 1/27 1/81 1/243] [lim r->oo= 1 1/2 1/4 1/8 1/16 1/32]

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2. References

[Project-Index] http://go.helms-net.de/math/binomial/index

[Intro/Notation] http://go.helms-net.de/math/binomial/00_0_intro.pdf

[ListOfMatrices] http://go.helms-net.de/math/binomial/00_1_ListOfMatrices.pdf

[binomialmatrix] http://go.helms-net.de/math/binomial/01_1_binomialmatrix.pdf

[signed binomial] http://go.helms-net.de/math/binomial/01_2_signedbinomialmatrix.pdf

[Stirlingmatrix] http://go.helms-net.de/math/binomial/01_3_stirlingmatrix.pdf

[Gaussmatrix] http://go.helms-net.de/math/binomial/01_5_gaussmatrix.pdf

[GenBernRec] (Generalized Bernoulli-recursion)
http://go.helms-net.de/math/binomial/02_1_GeneralizedBernoulliRecursion.pdf

[SumLikePow] (Sums of like powers)
http://go.helms-net.de/math/binomial/04_3_SummingOfLikePowers.pdf

[Erdoes] http://go.helms-net.de/math/binomial/10_1_erdoes.pdf

[Hasse] http://go.helms-net.de/math/binomial/10_2_recihasse.pdf

[InvVandermonde] http://go.helms-net.de/math/binomial/10_3_InverseVandermonde.pdf

Projekt ***Bernoulli-numbers***, first versions of the above, contain a ***first rough exploratory*** course but are already cover most topics and contain also the basic material about G_p and G_m which is still missing in the above list:

[Bernoulli] http://go.helms-net.de/math/binomial/bernoulli_en.pdf

[Summation] http://go.helms-net.de/math/binomial/pmatrix.pdf

[Matexp] Matrixexponential Alan Edelman & Gilbert Strang, MIT
http://web.mit.edu/18.06/www/pascal-work.pdf

[Laguerre] Laguerrematrix
http://mathworld.wolfram.com/LaguerrePolynomial.html

[Roots] "Polynomials From Pascal's Triangle" Mathforum at Drexel
http://mathpages.com/home/kmath304.htm

[Toeplitzmatrix] Toeplitz-matrices Wikipedia
http://en.wikipedia.org/wiki/Toeplitz_matrix

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