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1-1 Binomial/Pascalmatrix P

Abstract: The most basic identities for all chapters are shown here. As far as proofs are widely obtainable I omit them here; some are own derivations with the possible, but unlikely chance of being in error. The discussion of the row~ or column-signed versions of P is separated in another article, due to the importance of some aspects (like Eigenvector-decompositions and its consequences), which do not apply to the unsigned version.

Contents:

Note: I'm using "*bi(r,c)*" as a shorthand for "*binomial(r,c)*" here, where always *bi(r,c)=0* *if c>r* reflecting the triangular form of ***P***.

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1. Basic definitions and identities

1.1. The "Pascal"~/ binomialmatrix

The Pascalmatrix contains just the binomial-coefficients in a lower triangle:

(1.1.1.) $P := P_{r,c} = binomial(r,c)$

$(\text{if } c>r\ P_{r,c} = 0)$

$$P = \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix}$$

1.2. Matrix-logarithm

The Pascalmatrix can be seen as matrix-exponential of the subdiagonal-matrix of the natural numbers. Using the matrix-function *Sd(d,vec)* providing the entries of *vec* in the *d*'th subdiagonal, it is

$L = Sd(1,Z(-1))$

$$L = \begin{bmatrix} . & . & . & . & . & . \\ 1 & . & . & . & . & . \\ . & 2 & . & . & . & . \\ . & . & 3 & . & . & . \\ . & . & . & 4 & . & . \\ . & . & . & . & 5 & . \end{bmatrix}$$

(1.2.1.) $P = \exp(L)$

$$\exp\left(\begin{bmatrix} . & . & . & . & . & . \\ 1 & . & . & . & . & . \\ . & 2 & . & . & . & . \\ . & . & 3 & . & . & . \\ . & . & . & 4 & . & . \\ . & . & . & . & 5 & . \end{bmatrix}\right) = \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix}$$

Proof: see chapter "Details/Proofs", Proof_MatExp

1.3. Powers

1.3.1. Matrix-multiples / integer powers

The integer powers can be expressed

* either by iterative computation of the appropriate powers
* or by pre- and postmultiplication with a powerseries vector (similarity scaling)
* or by elementwise multiplication ("Hadamard-multiplication") with a Toeplitz-matrix
* or using the multiple of its matrix-logarithm.

The last three ways are essentially the same. Because of the triangular structure all ways can also be used for infinite dimension.

(1.3.1.1.) $P^n = P * P^{n-1}$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix} * \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & . & . & . & . & . \\ 2 & 1 & . & . & . & . \\ 4 & 4 & 1 & . & . & . \\ 8 & 12 & 6 & 1 & . & . \\ 16 & 32 & 24 & 8 & 1 & . \\ 32 & 80 & 80 & 40 & 10 & 1 \end{bmatrix}$$

The result can also be seen as the Hadamard-product of $\boldsymbol{P}$ with the triangular *Toeplitzmatrix* of the powerseries vector $\boldsymbol{V}(n)$ or equivalently as pre- and postmultiplied

(1.3.1.2.) P^n $= P ☼ Toeplitz(n) \quad = P ☼ (V(n) * V(1/n)\sim)$
$= {}^dV(n) * P * {}^dV(1/n)$
$= \exp(n * L)$

$$\underset{P}{\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix}} ☼ \underset{V2}{\begin{bmatrix} 1 & . & . & . & . & . \\ 2 & 1 & . & . & . & . \\ 4 & 2 & 1 & . & . & . \\ 8 & 4 & 2 & 1 & . & . \\ 16 & 8 & 4 & 2 & 1 & . \\ 32 & 16 & 8 & 4 & 2 & 1 \end{bmatrix}} = \underset{P\hat{}2}{\begin{bmatrix} 1 & . & . & . & . & . \\ 2 & 1 & . & . & . & . \\ 4 & 4 & 1 & . & . & . \\ 8 & 12 & 6 & 1 & . & . \\ 16 & 32 & 24 & 8 & 1 & . \\ 32 & 80 & 80 & 40 & 10 & 1 \end{bmatrix}}$$

Proof: see *powers*

1.3.2. Reciprocal ("inverse" for finite dimension)

The reciprocal can be determined in different ways: it can be computed by iteratively solving the matrix-equation $\boldsymbol{P} * \boldsymbol{P}^{-1} = \boldsymbol{I}$ or using the rules of exponentiating and $\boldsymbol{P}$'s matrix-logarithm

$$L = \log(P), \qquad P^{-1} = \exp(-L)$$

since the inverse/reciprocal is the power to the exponent -1. The result is, that the reciprocal is the following product of $\boldsymbol{J}$ and $\boldsymbol{P}$:

(1.3.2.1.) $P^{-1} = J*P*J$

$\Sigma_{k=1..r} [(-1)^{c-k} * bi(r,k) * bi(k,c)] = \delta_{r,c}$
where δ is the Kronecker-delta
$= 0 \quad$ if $r<>c$
$= 1 \quad$ if $r=c$ ...

$$* \underset{P\hat{}\text{-}1}{\begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ -1 & 3 & -3 & 1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ -1 & 5 & -10 & 10 & -5 & 1 \end{bmatrix}}$$

$$\underset{P}{\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & 1 & . & . & . \\ . & . & . & 1 & . & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & 1 \end{bmatrix}}$$

(1.3.2.2.) $P^{-1} = P ☼ Toeplitz(-1)$
$= {}^dV(-1) * P * {}^dV(-1)$
$= J * P * J$

$$\underset{P}{\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix}} ☼ \underset{V1}{\begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 1 & -1 & 1 & . & . & . \\ -1 & 1 & -1 & 1 & . & . \\ 1 & -1 & 1 & -1 & 1 & . \\ -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}} = \underset{P\hat{}\text{-}1}{\begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ -1 & 3 & -3 & 1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ -1 & 5 & -10 & 10 & -5 & 1 \end{bmatrix}}$$

(for more information about the *triangular unit matrix* , which occurs in the latter example, see there)

1.3.3. General powers

General powers of ***P*** can be computed, since the general power is the exponential of scalar multiples of the matrix-logarithm.

*Example: exp(2****L****) =* $\mathbf{P}^2$

*2 L =Sd(1,2*Z(-1))*

P^2 *=exp(2 L)*

$$\exp\left(\begin{bmatrix} . & . & . & . & . & . \\ 2 & . & . & . & . & . \\ . & 4 & . & . & . & . \\ . & . & 6 & . & . & . \\ . & . & . & 8 & . & . \\ . & . & . & . & 10 & . \end{bmatrix}\right) = \begin{bmatrix} 1 & . & . & . & . & . \\ 2 & 1 & . & . & . & . \\ 4 & 4 & 1 & . & . & . \\ 8 & 12 & 6 & 1 & . & . \\ 16 & 32 & 24 & 8 & 1 & . \\ 32 & 80 & 80 & 40 & 10 & 1 \end{bmatrix}$$

Arbitrary complex powers using the parameter *a* with

(1.3.3.1.) P^a *= P ☼ Toeplitz(a)*
*= P ☼ (V(a) * V(1/a)~)*

are also equivalent to

(1.3.3.2.) P^a *= exp(L*a)*

The logarithm-representation makes thing very clear, so the general case is explicitely shown here:

(1.3.3.3.) $P^a = {}^dV(a) * P * {}^dV(a)^{-1}$

*L =Sd(1,a*Z(-1))*

$$\begin{bmatrix} . & . & . & . & . & . \\ a & . & . & . & . & . \\ . & 2*a & . & . & . & . \\ . & . & 3*a & . & . & . \\ . & . & . & 4*a & . & . \\ . & . & . & . & 5*a & . \end{bmatrix}$$

P^a *=exp(L*a)*
= P ☼ Toeplitz(a)

$$\exp\left(\begin{bmatrix} . & . & . & . & . & . \\ a & . & . & . & . & . \\ . & 2*a & . & . & . & . \\ . & . & 3*a & . & . & . \\ . & . & . & 4*a & . & . \\ . & . & . & . & 5*a & . \end{bmatrix}\right) = \begin{bmatrix} 1 & . & . & . & . & . \\ a & 1 & . & . & . & . \\ a^2 & 2*a & 1 & . & . & . \\ a^3 & 3*a^2 & 3*a & 1 & . & . \\ a^4 & 4*a^3 & 6*a^2 & 4*a & 1 & . \\ a^5 & 5*a^4 & 10*a^3 & 10*a^2 & 5*a & 1 \end{bmatrix}$$

1.4. P as operator on powerseries

Using powerseries the pascalmatrix behaves like an operator: input and output form instances of the same type: powerseries. The imagination of P as an application of an operator helps much in understanding some complicated relations, for instance summing up to zeta()-values but even summing zeta()-values to rational numbers can easily be described, not to mention problems concerning sums of like powers and the like. More in these aspects are in the various articles.

1.5. Links to more specific variants

1.5.1. The signed version

The column- and row-signed versions have very interesting properties, which are discussed in a separate article due to their relevance. For instance, they have an interesting Eigensystem, which uncovers a neat relation to the matrix of coefficients, with which Jacob Bernoulli solved the problem of summing like powers.

(1.5.2.) $P_J := P_{J\ r,c} = (-1)^c * binomial(r,c) \quad // \text{ if } r>=c$
(1.5.3.) $\quad = P * J$

$$Pj = \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & -1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ 1 & -3 & 3 & -1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix}$$

(1.5.4.) ${}_JP := P_{J\ r,c} = (-1)^r * binomial(r,c) \quad // \text{ if } r>=c$
(1.5.5.) $\quad = J * P$

$$jP = \begin{bmatrix} 1 & . & . & . & . & . \\ -1 & -1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ -1 & -3 & -3 & -1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ -1 & -5 & -10 & -10 & -5 & -1 \end{bmatrix}$$

See :

1.5.6. Power- and exponential-series

The powerseries of ***P*** itself pops up as a very interesting entitiy after one had stepped to see the Pascal-matrix as an operator on powerseries. What, if I apply it two times? What if many times? What if I sum all powers of it? The result of the alternating powerseries of ***P*** is a matrix, called ***ETA***, which performs alternating summing of like powers, and also its supplement, the matrix ***ZETA*** is introduced, which performs summing of like powers - the both matrices are somehow implementations of a discrete integration-operator.

Also ***PE***, the sum of the exponential-series of ***P*** itself *(exp(exp(**L**))* seems to be an interesting object in some contexts. In the Online Encyclopedia of Integer Sequences (OEIS) the columns of ***PE*** occur as sequences, which describe combinatorical problems, seemingly without observed relation to each other. ***PE*** and its inverse $\boldsymbol{PE}^{-1}$ connect then such sequences to a common framework.

see:

1.5.7. Hierarchy of orders of binomial-matrices P_k

Modifying the matrixlogarithm of ***P*** in meaningful ways leads to two different hierarchies of matrices, where ***P*** is then the instance of order *1*.

One is the hierarchy of *powers of the entries* of ***L***, where the second order matrix is then closely related to the "Laguerre"-matrix, and the zero-order matrix implements a very basic exponential "operator".

The other is the hierarchy of *powers of L itself*; the zero'th order is of little interest since it is simply the unit-matrix scaled by the exponential of *1*; but the second order matrix ***GS*** occurs in connection with the Gauss-normal function and allows to formally describe coefficients for the Erf-function and higher integrals (where, however, the occuring series are divergent)

See:

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2. Operations using P with vectors and matrices (rowsums/columnsums etc)

2.1. Rightmultiplication

2.1.1. Rowsums and right-multiplication with powerseries (binomial theorem)

The rowsums are known as powers of 2:

(2.1.2.) $\Sigma_{c=0..r}\ bi(r,c) = 2^r$

Summing expressed as matrix-multiplication

(2.1.3.) $P * V(1) = V(2)$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \\ 32 \end{bmatrix}$$

More general than the simple rowsums, the right-multiplication by a powerseries means to apply the binomial-theorem.

The sums of powerseries weighted by the binomial-coefficients produce just a ***shift*** of the base by *1*.

Example: (the right-multiplicator is here given as a list of columns to have a group of examples in one table)

(2.1.4.) $\Sigma_{c=0..r}\ n^c\ binomial(r,c) = (n+1)^r$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 & 3 \\ 4 & 1 & 0 & 1 & 4 & 9 \\ -8 & -1 & 0 & 1 & 8 & 27 \\ 16 & 1 & 0 & 1 & 16 & 81 \\ -32 & -1 & 0 & 1 & 32 & 243 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 4 & 9 & 16 \\ -1 & 0 & 1 & 8 & 27 & 64 \\ 1 & 0 & 1 & 16 & 81 & 256 \\ -1 & 0 & 1 & 32 & 243 & 1024 \end{bmatrix}$$

This binomial-relation is valid for any complex exponent *s*:

(2.1.5.) $\Sigma_{c=0..r}\ s^c\ bi(r,c) = (s+1)^r$

(2.1.6.) $P * V(s) = V(1+s)$ *// for all complex s*

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix} * \begin{bmatrix} 1 \\ s \\ s\verb|^|2 \\ s\verb|^|3 \\ s\verb|^|4 \\ s\verb|^|5 \end{bmatrix} = \begin{bmatrix} (s+1)\verb|^|0 \\ (s+1)\verb|^|1 \\ (s+1)\verb|^|2 \\ (s+1)\verb|^|3 \\ (s+1)\verb|^|4 \\ (s+1)\verb|^|5 \end{bmatrix}$$

Also, referring to the chapter about complex powers we may state:

(2.1.7.) $P^{-s} * V(s) = V(0)$

(2.1.8.) $P^t * V(s) = V(t+s)$ *// for all complex s and t*

2.1.9. right-multiplication with unsigned and alternating harmonic/zeta-series

The right-multiplication with a zeta-like-series does not give a smooth result like the above. However, with nonnegative integer argument s for the exponent of the ***alternating*** zeta-series an interesting result can be found. The resulting matrix $\boldsymbol{X}$ is then the factorial-scaled transposed matrix of the Stirling-numbers of 2'nd kind. *(See more about this in the article* Stirlingmatrix*)*.

for an integer exponent n

(2.1.10.) $\Sigma_{c=0..r} (-1)^c c^n \, bi(r,c) = (-1)^r *r!*St2_{r,n}$

for all columns of ZV

(2.1.11.) $P * {}^dZV = J * {}^dF(1) * St2\sim$

$$aZV = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -2 & -4 & -8 & -16 & -32 \\ 1 & 3 & 9 & 27 & 81 & 243 \\ -1 & -4 & -16 & -64 & -256 & -1024 \\ 1 & 5 & 25 & 125 & 625 & 3125 \\ -1 & -6 & -36 & -216 & -1296 & -7776 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & -1 & -3 & -7 & -15 & -31 \\ . & . & 2 & 12 & 50 & 180 \\ . & . & . & -6 & -60 & -390 \\ . & . & . & . & 24 & 360 \\ . & . & . & . & . & -120 \end{bmatrix}$$

The decomposition of $\boldsymbol{X}$ shows the transpose of $\boldsymbol{ST_2}$, the triangular matrix of Stirlingnumbers of 2'nd kind:

$$J = \begin{bmatrix} 1 & . & . & . & . & . \\ . & -1 & . & . & . & . \\ . & . & 1 & . & . & . \\ . & . & . & -1 & . & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & -1 \end{bmatrix} * \; ! = \begin{bmatrix} 1 & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & 2 & . & . & . \\ . & . & . & 6 & . & . \\ . & . & . & . & 24 & . \\ . & . & . & . & . & 120 \end{bmatrix} * \; St2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 3 & 7 & 15 & 31 \\ . & . & 1 & 6 & 25 & 90 \\ . & . & . & 1 & 10 & 65 \\ . & . & . & . & 1 & 15 \\ . & . & . & . & . & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & -1 & -3 & -7 & -15 & -31 \\ . & . & 2 & 12 & 50 & 180 \\ . & . & . & -6 & -60 & -390 \\ . & . & . & . & 24 & 360 \\ . & . & . & . & . & -120 \end{bmatrix} = X$$

2.1.12. right-multiplication with binomial series

The right-multiplication with its transpose yields the binomial-matrix itself, but in square-array-form:

(2.1.13.) $P * P\sim = X$,
where
$X:=X_{r,c} = bi(r+c,c)$

$$P\sim = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 2 & 3 & 4 & 5 \\ . & . & 1 & 3 & 6 & 10 \\ . & . & . & 1 & 4 & 10 \\ . & . & . & . & 1 & 5 \\ . & . & . & . & . & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix} \qquad PP' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 6 & 10 & 15 & 21 \\ 1 & 4 & 10 & 20 & 35 & 56 \\ 1 & 5 & 15 & 35 & 70 & 126 \\ 1 & 6 & 21 & 56 & 126 & 252 \end{bmatrix}$$

This operation represents known sums-of-products of binomialcoefficients, for instance, expressing the diagonal of the result:

(2.1.14.) $\Sigma_{c=0..r} bi(r,c)^2 = bi(2r,r)$

2.2. Leftmultiplication

2.2.1. left-multiplication with powerseries

Applying powerseries as a left-multiplicator introduces the problem of convergence/divergence. In later chapters this will be dealt in more detail.

For *s>1* I give some examples for powerseries-summation using *1/s*V(1/s)* (one row of the result is one example) (proof see "details/leftmultiplication")

(2.2.2.) $\lim \Sigma_{r=0..oo}\ bi(r,c) / s^{r+1} = 1/(s-1)^{c+1}$

(2.2.3.) $\lim_{rows->oo} (1/s*V(1/s) \sim * P)=1/(s-1)\ V(1/(s-1))\sim$

P

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 1/4 & 1/8 & 1/16 & 1/32 & 1/64 \\ 1/3 & 1/9 & 1/27 & 1/81 & 1/243 & 1/729 \\ 1/4 & 1/16 & 1/64 & 1/256 & 1/1024 & 1/4096 \\ 1/5 & 1/25 & 1/125 & 1/625 & 1/3125 & 1/15625 \\ 1/6 & 1/36 & 1/216 & 1/1296 & 1/7776 & 1/46656 \\ 1/7 & 1/49 & 1/343 & 1/2401 & 1/16807 & 1/117649 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1/2 & 1/4 & 1/8 & 1/16 & 1/32 & 1/64 \\ 1/3 & 1/9 & 1/27 & 1/81 & 1/243 & 1/729 \\ 1/4 & 1/16 & 1/64 & 1/256 & 1/1024 & 1/4096 \\ 1/5 & 1/25 & 1/125 & 1/625 & 1/3125 & 1/15625 \\ 1/6 & 1/36 & 1/216 & 1/1296 & 1/7776 & 1/46656 \end{bmatrix}$$

Since powers of ***P*** can be written as Hadamard-products with geometric-series-coefficients, the effect of powers of ***P*** can again nicely be derived for iterated multiplication (and even fractional or complex powers) by applying the appropriate powerseries-vector ***V****()* as new coefficients.

If two parameters *s* and *t* are chosen to give convergent result then

(2.2.4.) $\lim_{rows->oo} (1/s*V(1/s) \sim * P^{1/t}) \quad = t/(s*t-1)*V(t/(s*t-1))\sim$

(2.2.5.) $\lim_{rows->oo} (1/s*V(1/s) \sim * P^{t}) \quad = 1/(s-t)*V(1/(s-t))\sim$

Example:

P^t

$$\begin{bmatrix} 1 & . & . & . \\ t & 1 & . & . \\ t\verb|^|2 & 2*t & 1 & . \\ t\verb|^|3 & 3*t\verb|^|2 & 3*t & 1 \\ t\verb|^|4 & 4*t\verb|^|3 & 6*t\verb|^|2 & 4*t \\ t\verb|^|5 & 5*t\verb|^|4 & 10*t\verb|^|3 & 10*t\verb|^|2 \end{bmatrix}$$

$$\begin{bmatrix} 1/s & 1/s\verb|^|2 & 1/s\verb|^|3 & 1/s\verb|^|4 & 1/s\verb|^|5 & 1/s\verb|^|6 \end{bmatrix} \quad \begin{bmatrix} 1/(s-t)\verb|^|1 & 1/(s-t)\verb|^|2 & 1/(s-t)\verb|^|3 & 1/(s-t)\verb|^|4 \end{bmatrix}$$

If *s=t+1* (or *s-t=1*) in *(2.2.5)* then the resulting vector is always the unit-vector ***V****(1)~* , which will be of relevance in the summation article.

The entries of the result can be found by computing the derivatives, see "details /proofs/leftmultiplication".

2.2.6. Toeplitz shaped coefficients version ; left-multiplication with powerseries

Assume that the binomial-matrix $\mathbf{P}_b$ has also additional coefficients $b_0,b_1,b_2,...$, though in the triangular shifted version as it is in the Toeplitz-manner *Toeplitz(B)*, see the following picture. (This case shall occur at different chapters of these articles. The simple case is then, if all $b_0,b_1,b_2,...$, equal *1* and $\mathbf{P}_b$ equals the ordinary Pascalmatrix):

Example:

(2.2.7.) $\quad V(x)\sim * \mathbf{P}_b = \mathbf{Y}\sim$

$$\begin{array}{r} * \begin{bmatrix} b0 & . & . & . & . & . \\ b1 & b0 & . & . & . & . \\ b2 & 2{*}b1 & b0 & . & . & . \\ b3 & 3{*}b2 & 3{*}b1 & b0 & . & . \\ b4 & 4{*}b3 & 6{*}b2 & 4{*}b1 & b0 & . \\ b5 & 5{*}b4 & 10{*}b3 & 10{*}b2 & 5{*}b1 & b0 \end{bmatrix} \mathrm{Pb} \\ \begin{bmatrix} 1 & x & x^{\wedge}2 & x^{\wedge}3 & x^{\wedge}4 & x^{\wedge}5 \end{bmatrix} * \quad = \begin{bmatrix} y0 & y1 & y2 & y3 & y4 & y5 \end{bmatrix} \end{array}$$

The sum of each y_c in column c:

(2.2.8.)
$$y_c = \sum_{k=0}^{oo} b_{k-c} \binom{k}{c} x^k = \frac{1}{c!} \sum_{k=0}^{oo} b_{k-c} \frac{k!}{(k-c)!} x^k$$

Define first the function based on the multiplication with the first column ($c=0$):

$$f := f(x) = \sum_{k=0}^{oo} b_k x^k$$

Then the binomial-cofactors describe derivatives of $f(x)$. The row-shifting must be compensated by multiplying of appropriate powers of x, so that we get:

(2.2.9.)
$$[\, y_0, y_1, y_2, ...] = [\, f, \frac{x}{1!}(xf)', \frac{x^2}{1!}(x^2 f)'', ...]$$

The expansion of derivatives in this expressions involves again the binomial-coefficients, thus the matrix $\mathbf{P}$ again. Provided that the sums $f(x)$ are convergent for the desired x, then, in matrix-notation the result for $\mathbf{Y}\sim$ is:

(2.2.10.) $\quad \mathbf{Y}\sim \quad = \mathbf{V}(x)\sim * \mathbf{P}_b$
$\quad\quad = \mathbf{V}(x)\sim * {}^d[\, f/0!,\ f'/1!,\ f''/2!,\ f'''/3!,...] * \mathbf{P}\sim * {}^d\mathbf{V}(x))$

2.2.11. left-multiplication with harmonic/zeta-series

The left-multiplication with the harmonic series $\mathbf{Z}(1)\sim$ involves *zeta(1)* in column 1 of the multiplication and a value would not be assignable. From known results of divergent summation values for the matrixmultiplication $\mathbf{Z}(s)\sim *\mathbf{P}$ can be assigned, even if the exponent in the zeta-seriesvector is not "nice", although it involves a bit more effort.

Zeta-series with higher exponents can conventionally be used as left-multiplicators; in the first column of the result we have then the ordinary zeta-values.

2.3. Relation to other matrices

2.3.1. Triangular unit matrix

In connection with ***DR***, the triangular unit matrix, ***P*** can be shifted. Also has the inverse of ***DR*** entries of ***P***.

Examples:

$$DR * P \qquad \begin{bmatrix} 1 & . & . & . & . & . \\ 2 & 1 & . & . & . & . \\ 3 & 3 & 1 & . & . & . \\ 4 & 6 & 4 & 1 & . & . \\ 5 & 10 & 10 & 5 & 1 & . \\ 6 & 15 & 20 & 15 & 6 & 1 \end{bmatrix}$$

$$DR^{-1} * P \qquad \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 1 & 1 & . & . & . \\ 0 & 1 & 2 & 1 & . & . \\ 0 & 1 & 3 & 3 & 1 & . \\ 0 & 1 & 4 & 6 & 4 & 1 \end{bmatrix}$$

$$DR^{-3} \qquad \begin{bmatrix} 1 & . & . & . & . & . \\ -3 & 1 & . & . & . & . \\ 3 & -3 & 1 & . & . & . \\ -1 & 3 & -3 & 1 & . & . \\ . & -1 & 3 & -3 & 1 & . \\ . & . & -1 & 3 & -3 & 1 \end{bmatrix}$$

The k-shifted versions multiplied together form in the limit the Stirling-matrices $\boldsymbol{St_1}$ and $\boldsymbol{St_2}$, so the Stirling numbers can be completely definded by multiplication of binomial-coefficients.

2.3.2. Vandermonde-matrix and Stirling matrix

The binomial-matrix has an intimate relation to the Vandermonde matrix ***ZV*** (due to the implementation of the binomial-theorem. Together with the matrix $\boldsymbol{St_2}$ of Stirlingnumbers *2*'nd kind and a diagonal factorial matrix it can be seen as LU-component of ***ZV***:

$$P *^d F * St_2\sim = ZV$$

Based on this relation an approach to compute the inverse of the Vandermonde-matrix (which unfortunately is varying with the size of finite dimension) is shown, which uses the easily computable inverses of ***P*** and $\boldsymbol{St_2}$:

$$ZV^{-1} = St_2^{-1} \sim *^d F^{-1} * P^{-1}$$

This is discussed in more details in Inverse Vandermonde matrix.

2.3.3. (Divergent) Euler-summation of any order

The method of Euler to sum divergent series is based on properties of the binomial-matrix and its specific operation on powerseries. Higher orders of the Euler-summation method can easily be described when the Pascal-matrix is introduced as an operator, which is allowed to get higher integer powers, to best fit the summing needs for divergent series of equivalent high orders. Since the binomial-matrix can also take fractional and even complex-powers, the Euler-summation can then be applied using the best fractional or even of complex order.

This is discussed in the Summation-article.

3. Proofs and details

3.1. Integer and complex powers as powerseries-operation

Theorem:

(3.1.1.) $P^s = {}^dV(s) * P * {}^dV(1/s) = P ☼ Toeplitz(s)$

(3.1.2.) $P^s := P^s_{r,c} = binomial(r,c)*s^{r-c}$ // = 0 if c>r

Proof :

According to the binomial-theorem it is

(3.1.3.) $P * V(1) = V(2)$
(3.1.4.) $P * V(2) = P*P * V(1) = P^2 * V(1)$

and in the iteration

(3.1.5.) $P^n * V(1) = V(n+1)$

The entries in each row of $\mathbf{P}^n$ need to be such, that the binomial-transformation of $\mathbf{V}(1)$ to $\mathbf{V}(n+1)$ occurs.

The binomial-theorem for the last transformation reads for a single row r:

(3.1.6.) $(n+1)^r = n^r (bi(r,0)*1 + n^{r-1} bi(r,1)*1 + n^{r-2} bi(r,2)*1 + ... n\ bi(r,r-1)*1 + bi(r,r)*1$

This can be rewritten

(3.1.7.) $n^r*(1+1/n)^r = n^r ((bi(r,0)* n^0 + bi(r,1)* n^{-1} + bi(r,2)*n^{-2} + ...+ bi(r,r-1)*n^{r-1} + bi(r,r)*n^{-r})$

whose single terms are the coefficients of a row of a matrix, which performs the binomial-transform of $\mathbf{V}(1)$ to $\mathbf{V}(n+1)$. Each of these rows has a row-scaling factor n^r according to the row-index r, which agrees to a premultiplication with a diagonalmatrix ${}^d\mathbf{V}(n)$ containing n^r in the r'th row.

Also it has a column-specific coefficent n^{-c} which agrees with a postmultipliciation of the vector of the binomial-coefficients for this row with a diagonal-matrix ${}^d\mathbf{V}(1/n)$ containing n^{-c} in its c'th column. The latter is the same for each row r and thus independent of the row-index r.

Thus (2.1.7) , written for each row as a complete matrix, where the summands form the coefficients, is nothing else than

$${}^dV(n) * P * {}^dV(1/n)$$

and if the summation of each row (by postmultiplication with the summing-vector $\mathbf{V}(1)$)

$${}^dV(n) * P * {}^dV(1/n) * V(1) = V(n+1)$$

and at the same time it is from (2.1.5)

$$P^n * V(1) = V(n+1)$$

then also

(3.1.8.) ${}^dV(n) * P * {}^dV(1/n) = P^n$

and this is also the Hadamard-product of $\mathbf{P}$ with the Toeplitz-matrix $\mathbf{T}(n) = \mathbf{V}(n) * \mathbf{V}(1/n)\sim$

(3.1.9.) $P^n = {}^dV(n) * P * {}^dV(1/n) = P ☼ Toeplitz(n)$

Since the binomial-theorem does not only provide the terms of $(n+1)^r$ for integer n but for all complex s, (2.1.7) is valid for all complex parameters s, with the trivial exception of $s=0$:

(3.1.10.) $P^s = {}^dV(s) * P * {}^dV(1/s) = P ☼ Toeplitz(s)$

(3.1.11.) $P^s := P^s_{r,c} = binomial(r,c)*s^{r-c}$ // = 0 if c>r

and the theorem is proved.

3.2. The matrix-exponential

Theorem:

P is the matrix-exponential of a matrix L, whose entries are the sequence of natural numbers in the first principal subdiagonal

$$P = \exp(L)$$

where

$$L = \mathrm{subdiag}_{(1)}([1,2,3,4,...])$$

$$L = \begin{bmatrix} . & . & . & . & . & . \\ 1 & . & . & . & . & . \\ . & 2 & . & . & . & . \\ . & . & 3 & . & . & . \\ . & . & . & 4 & . & . \\ . & . & . & . & 5 & . \end{bmatrix}$$

Proof:

A formal proof you may find in [*matexp*]. For the current case it may be instructive to see a heuristic.

The series-expansion for the (scalar) exponential power-series is valid also for matrices. With a logarithm-matrix ***L*** this reads then as:

(3.2.1.) $\exp(L) = L^0/0! + L^1/1! + L^2/2! + L^3/3! + \dots$

The assumption is, that ***L*** is the subdiagonal-matrix containing the sequence of natural numbers in the first principal subdiagonal.

For a matrix ***L*** with arbitrary coefficients *a,b,c,d,...* in that subdiagonal the powerseries looks like

$$L^0/0! = \frac{\begin{bmatrix} 1 & . & . & . & . \\ . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 1 \end{bmatrix}}{0!} \quad L^1/1! = \frac{\begin{bmatrix} . & . & . & . & . \\ a & . & . & . & . \\ . & b & . & . & . \\ . & . & c & . & . \\ . & . & . & d & . \end{bmatrix}}{1!} \quad L^2/2! = \frac{\begin{bmatrix} . & . & . & . & . \\ . & . & . & . & . \\ a*b & . & . & . & . \\ . & c*b & . & . & . \\ . & . & c*d & . & . \end{bmatrix}}{2!}$$

$$L^3/3! = \frac{\begin{bmatrix} . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a*c*b & . & . & . & . \\ . & c*b*d & . & . & . \end{bmatrix}}{3!} \quad L^4/4! = \frac{\begin{bmatrix} . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a*c*b*d & . & . & . & . \end{bmatrix}}{4!}$$

In each of the occuring subdiagonals the entries are products of neighboured entries of the original matrix L.

The sequence of terms of this series is finite for finite dimension *d* of the logarithm-matrix, since a subdiagonal-matrix (which is ***L***) is nilpotent to the exponent *d*.

(3.2.2.) The sum is

$$\begin{bmatrix} 1 & . & . & . & . \\ a/1! & 1 & . & . & . \\ ab/2! & b/1! & 1 & . & . \\ abc/3! & bc/2! & c/1! & 1 & . \\ abcd/4! & bcd/3! & cd/2! & d/1! & 1 \end{bmatrix}$$

If *a=1, b=2,c=3,...*and the factorials are replaced by this it is symbolically

$$\begin{bmatrix} 1 & . & . & . & . \\ a/a & 1 & . & . & . \\ ab/ab & b/a & 1 & . & . \\ abc/abc & bc/ab & c/a & 1 & . \\ abcd/abcd & bcd/abc & cd/ab & d/a & 1 \end{bmatrix}$$

and in numbers

$$\begin{bmatrix} 1 & . & . & . & . \\ 1/1 & 1 & . & . & . \\ 1*2/(1*2) & 2/1 & 1 & . & . \\ 1*2*3/(1*2*3) & 2*3/(1*2) & 3/1 & 1 & . \\ 1*2*3*4/(1*2*3*4) & 2*3*4/(1*2*3) & 3*4/(1*2) & 4/1 & 1 \end{bmatrix}$$

rewritten

$$\begin{bmatrix} 1 & . & . & . & . \\ 1/1 & 1 & . & . & . \\ 2!/2! & 2!/(1!\ 1!) & 1 & . & . \\ 3!/3! & 3!/(1!\ 2!) & 3!/(2!\ 1!) & 1 & . \\ 4!/4! & 4!/(1!\ 3!) & 4!/(2!\ 2!) & 4!/(3!\ 1!) & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & . & . & . & . \\ 1 & 1 & . & . & . \\ 1 & 2 & 1 & . & . \\ 1 & 3!/2! & 3!/2! & 1 & . \\ 1 & 4!/3! & 4!/(2!\ 2!) & 4!/3! & 1 \end{bmatrix}$$

which comes out to reflect the definition of binomials.

(End of Proof)

3.3. Complex powers of P

Theorem:

(3.3.1.)	$P^{s} = {}^{d}V(s) * P * {}^{d}V(1/s) = P ☼ Toeplitz(s)$
(3.3.2.)	$P^{s} := P^{s}_{r,c} = binomial(r,c)*s^{r-c}$ // = 0 if c>r

Proof :

A power of an exponential is a multiple of the logarithm.

So if we use the logarithm ***L*** and multiply it by an arbitrary complex scalar factor s we have initially

$$\begin{bmatrix} . & . & . & . & . \\ s & . & . & . & . \\ . & 2*s & . & . & . \\ . & . & 3*s & . & . \\ . & . & . & 4*s & . \end{bmatrix}$$

and each *r'th* power of this introduces the *r'th* power of s in the *r'th* subdiagonal

The matrix exponential is then

(3.3.3.) $P^{s} = P ☼ Toeplitz(s)$
$= V(s)\sim * P * V(s)^{-1}$

$$\begin{bmatrix} 1 & . & . & . & . \\ s\ *1 & 1 & . & . & . \\ s\text{^}2*1 & s\ *2 & 1 & . & . \\ s\text{^}3*1 & s\text{^}2*3 & s\ *3 & 1 & . \\ s\text{^}4*1 & s\text{^}3*4 & s\text{^}2*6 & s\ *4 & 1 \end{bmatrix}$$

which is a Hadamardproduct of ***P*** with the triangular Toeplitzmatrix ***T****(s)*

(3.3.4.) $P^{s} := P^{s}_{r,c} = binomial(r,c)*s^{r-c}$ // = 0 if c>r

(End of proof)

3.4. Leftmultiplication with powerseries

For the leftmultiplication with a powerseries ***V****(x)~* the fact can be used, that the multiplication with a column *c* of ***P*** is just the derivative multiplied by *x*.

With a rowvector ***V****(x)* the entries of the result ***Y*** in

(3.4.1.) $V(x)\sim * P = Y\sim$

can be described using a function

(3.4.2.) $f(x) := \Sigma_{r=0..oo}\ bi(r,c) * x^{r}$

Then

$$\begin{aligned} y_0 &= f(x) &&= (1 + x + x^2 + x^3 + ...) \\ y_1 &= f'(x) * x &&= (1 + 2x + 3x^2 + ...)\, x \\ y_2 &= f''(x) * x^2/2! &&= (2 + 6x + 12x^2 + ...)\, x^2/2! \\ &... \end{aligned}$$

(3.4.3.) $y_c = f^{(c)}(x) * x^c/c! = (c! + ...)\, x^c/c!$

...

(3.4.4.) $Y\sim = [\ f(x)/0!\ ,\ f'(x)\, x/1!\ ,\ f''(x)\ x^2/2!\ , ...\ ]$

Inserting the values for the derivatives gives

$$Y\sim = [1/(1-x), \quad 1/(1-x)^2\, x/1!\ , 2!/(1-x)^3\, x^2/2!, \dots , c!/(1-x)^{c+1}\, x^c/c!, \dots \quad]$$
$$= 1/(1-x)\ [1, \quad x/(1-x), \quad (x/(1-x))^2\ , \quad \dots , (x/(1-x))^c, \quad \dots \quad]$$
(3.4.5.) $$= 1/(1-x) * V(x/(1-x)) \sim$$

and using $z = x/(1-x)$:

$$V(x)\sim \quad * P = 1/(1-x) * V(x/(1-x)) \sim$$
(3.4.6.) $$x\ V(x)\sim * P = z * V(z) \sim$$

To keep this sums convergent $abs(x)$ must be limited to $abs(x)<1$. The divergent summability for $x<= -1$ can be shown by Euler-summation (see chapter summation) using this toolbox of matrices only; the complete analytic continuation for any complex $x<>1$ cannot be adressed here.

Using integer powers of ***P*** means iteration:

$$x\ V(x)\sim * P * P = z * V(z) \sim * P = z/(1-z) * V(z/(1-z)) \sim \ = x/(1-2x)\ V(x/(1-2x))$$
(3.4.7.) $$x\ V(x)\sim * P^n = \ x/(1-nx)\ V(\ x/(1 - nx))$$

4. Loose ends

4.1. Variations using the matrix-logarithm

4.1.1. a column-shift: finding a cyclotomic expression

using *exp(**L**+1)* means shifting of ***P***

L =Sd(1,Z(-1)+1)

$$\begin{bmatrix} . & . & . & . & . & . \\ 2 & . & . & . & . & . \\ . & 3 & . & . & . & . \\ . & . & 4 & . & . & . \\ . & . & . & 5 & . & . \\ . & . & . & . & 6 & . \end{bmatrix}$$

Pc = exp(L)

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 2 & 1 & . & . & . & . \\ 3 & 3 & 1 & . & . & . \\ 4 & 6 & 4 & 1 & . & . \\ 5 & 10 & 10 & 5 & 1 & . \\ 6 & 15 & 20 & 15 & 6 & 1 \end{bmatrix}$$

and provides values of the related cyclotomic polynomial for any power *n>0*

(4.1.1.1.) *Pc* V(x-1) =(x* V(x) - 1*V(1))/ (x-1)*

(4.1.1.2.) *entry in row r :=* $(x^{r+1} - 1)/(x - 1)$

$$* \begin{bmatrix} 1 \\ 3 \\ 9 \\ 27 \\ 81 \\ 243 \end{bmatrix} \text{ or } * \begin{bmatrix} x\text{^}0 \\ x\text{^}1 \\ x\text{^}2 \\ x\text{^}3 \\ x\text{^}4 \\ x\text{^}5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 2 & 1 & . & . & . & . \\ 3 & 3 & 1 & . & . & . \\ 4 & 6 & 4 & 1 & . & . \\ 5 & 10 & 10 & 5 & 1 & . \\ 6 & 15 & 20 & 15 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 21 \\ 85 \\ 341 \\ 1365 \end{bmatrix} \text{ or} = \begin{bmatrix} (x\text{^}1-1)/(x - 1) \\ (x\text{^}2-1)/(x - 1) \\ (x\text{^}3-1)/(x - 1) \\ (x\text{^}4-1)/(x - 1) \\ (x\text{^}5-1)/(x - 1) \\ (x\text{^}6-1)/(x - 1) \end{bmatrix}$$

5. References

[Project-Index] *http://go.helms-net.de/math/binomial/index*

[Intro] *http://go.helms-net.de/math/binomial/00_0_intro.pdf*

[binomialmatrix] *http://go.helms-net.de/math/binomial/01_1_binomialmatrix.pdf*

[signed binomial] *http://go.helms-net.de/math/binomial/01_2_signedbinomialmatrix.pdf*

[Gaussmatrix] *http://go.helms-net.de/math/binomial/04_1_gaussmatrix.pdf*

[Stirlingmatrix] *http://go.helms-net.de/math/binomial/05_1_stirlingmatrix.pdf*

[Hasse] *http://go.helms-net.de/math/binomial/01_x_recihasse.pdf*

[Vandermonde] *http://go.helms-net.de/math/binomial/10_3_InverseVandermonde1.pdf*

[SumLikePow] (Sums of like powers)
http://go.helms-net.de/math/binomial/04_3_SummingOfLikePowers.pdf

[GenBernRec] (Generalized Bernoulli-recursion)
http://go.helms-net.de/math/binomial/02_2_GeneralizedBernoulliRecursion.pdf

Projekt ***Bernoulli-numbers***, first versions of the above, contain a ***first rough exploratory*** course but are already cover most topics and contain also the basic material about G_p and G_m which is still missing in the above list:

[Bernoulli] *http://go.helms-net.de/math/binomial/bernoulli_en.pdf*

[Summation] *http://go.helms-net.de/math/binomial/pmatrix.pdf*

[Matexp] Matrixexponential Alan Edelman & Gilbert Strang, MIT
http://web.mit.edu/18.06/www/pascal-work.pdf

[Laguerre] Laguerrematrix
http://mathworld.wolfram.com/LaguerrePolynomial.html

[Roots] "Polynomials From Pascal's Triangle" Mathforum at Drexel
http://mathpages.com/home/kmath304.htm

[Toeplitzmatrix] Toeplitz-matrices Wikipedia
http://en.wikipedia.org/wiki/Toeplitz_matrix

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