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A generalized Bernoulli-recursion/-identity

Abstract: Another recursion for the definition of Bernoulli-numbers is given. Different from the most common recursion-formula (see [mathworld], for instance) this formula allows to be parametrized. So a whole family of related number-sequences can be defined by this recursion.

Amazingly, these are some of the most basic sequences, like Bernoulli-numbers, "η"-numbers, reciprocal of natural numbers (zeta(1)-series) the geometric series with q=1/2 and even the constant series with a(n)=1/2. I end this article with some easy, but amazing identities for sums of Bernoulli-numbers, which I haven't seen yet

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(Foreword in first version:) "After writing this article, I found a remark about the identity, which I describe here, in [Z.W.SUN]. Zhi Wei Sun mentions: "in a book of von Ettinghausen published in 1827 the author obtained, that we can compute B_2n..." ... just in this way and ..."with the help of continued fractions, in 1995 M. Kaneko [] rediscovered this...".

But the approach here might focus still another type of generality, which I didn't see before. So I'll present the text here anyway; if I find some related material, I'll insert references."

Gottfried Helms *(Vers. 07. Mai. 07)*

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1. A generalized Bernoulli-identity

1.1. Intro

The following result should be seen in contrast to the *known recursive definitions* for Bernoulli-numbers b_k: (I'll use $b_1 = +½$ all over the following text).

Common recursive-definition:

(1.1.1.) $1 = b_0$

and then by recursion:

$$0 = \sum_{k=0}^{n-1} ((-1)^k * binomial(n,k) * b_k) \quad // for\ n \ge 2$$

Examples:

(1.1.2.)

$$\begin{aligned}
1 &= 1\,b_0 \\
0 &= 1\,b_0 - 2\,b_1 \\
0 &= 1\,b_0 - 3\,b_1 + 3\,b_2 \\
0 &= 1\,b_0 - 4\,b_1 + 6\,b_2 - 4\,b_3 \\
&\dots
\end{aligned}$$

or

$$\begin{aligned}
1 &= 1\,b_0 \\
b_1 &= 1\,b_0 - 1\,b_1 \\
b_2 &= 1\,b_0 - 2\,b_1 + 1\,b_2 \\
b_3 &= 1\,b_0 - 3\,b_1 + 3\,b_2 - 1\,b_3 \\
b_4 &= 1\,b_0 - 4\,b_1 + 6\,b_2 - 4\,b_3 + 1\,b_4 \\
&\dots
\end{aligned}$$

which adresses the ***leading section*** of the first Bernoulli-numbers $b_0 .. b_{n-1}$

1.2. A generalized recursion formula

I just found another recursion formula, which adresses sections of Bernoulli-numbers of indexes $n .. 2n$.

(1.2.1.)

$$0 = \sum_{k=n}^{2n} (-1)^k\ binomial(n, k-n) * b_{k-1} * k$$

Examples:

(1.2.2.)

$$\begin{aligned}
0 &= 1\,b_0 *1 - 1\,b_1*2 \\
0 &= 1\,b_1 *2 - 2\,b_2*3 + 1\,b_3*4 \\
0 &= 1\,b_2 *3 - 3\,b_3*4 + 3\,b_4*5 - 1\,b_5*6 \\
0 &= 1\,b_3 *4 - 4\,b_4*5 + 6\,b_5*6 \quad 4\,b_6*7 + 1*b_7*8 \\
0 &= 1\,b_4 *5 - 5\,b_5*6 + 10\,b_6*7 - 10\,b_7*8 + 5*b_8*9 + 1*b_9*10 \\
&\dots.
\end{aligned}$$

This formula has also relevance in a more general context: not only the sequence of Bernoulli-numbers obeys this relations, but also some meaningful other sequences. The interesting aspect is, that these are all very basic number-theoretic-sequences.

The following example-sequences satisfy the above system of equations; from the problem it will become clear, that there are even infinitely many different sequences possible.

Notational remark: To distuingish the general case from the specific case of Bernoulli-numbers I use the letter *ß* for the general case and *b* for the Bernoulli-numbers:

Sequences as possible solutions of the eigenvector-problem for P_J

(1.2.3.)

$ß_0$	$ß_1$	$ß_2$	$ß_3$	$ß_4$	.....
1	1/2	0	- 1/4	0	... // multiples of Eta-function at negative or 0 exponent
1	1/2	1/6	0	-1/30	... // Bernoulli-numbers
1	1/2	1/4	1/8	1/16	... // geometric series
1	1/2	1/3	1/4	1/5	.. // harmonic series
1	1/2	1/2	1/2	1/2	.. // constant series
binomial-series require scaling of $ß_0$:					
2	1	1	1	1	.. // binomial(r,0) series = 2* constant series
1	1/2	1/3	1/4	1/5	.. // binomial(r,1) series = harmonic-series
1/3	1/6	1/10	1/15	1/21	... // binomial(r,2) series
1/10	1/20	1/35	1/56	1/84	// binomial(r,3) series

(and it seems as all binomial-series starting at the appropriate index fit this scheme)

1.3. Matrix-formulae: notation

To explain the "generalism" of this scheme, the reader has to be introduced into the concept of eigenvectors of the signed Binomial/"Pascal"-matrix $\mathbf{P_J}$.

The used matrix-*toolbox* contains the following vectors and matrices, with the following conventions:

- vectors are primarily assumed as column-vectors
- the transpose-symbol *"~"* is used (as in the openly available number-theoretic computerprogram Pari/GP) .
- the indices *r*,*c* for rows and columns are always assumed as beginning at zero
- the superscript prefix ^{d} is added, if a vector is used as a diagonal matrix
- matrices are generally assumed as lower triangular matrices.

Basic vectors are

Vandermonde vector of consecutive powers of a general parameter x
$V(x) = [1 , x , x^2 , x^3 ,...] \sim$
$^dV(x)$ *ist use as diagonal-matrix*
a vector containing the s'th powers of reciprocals of natural numbers; helps to work with Dirichlet/Zeta-series:
$Z(s) = [1, 1/2^s , 1/3^s , 1/4^s ,...] \sim$
$^dZ(s)$ *ist use as diagonal-matrix*
vector of factorials and reciprocals of factorials
$F = diag(0! , 1! , 2! , 3! , 4! ,...)$
$f = diag(1 , 1 , 1/2! , 1/3! , 1/4! ,...)$
vector of Bernoulli-numbers
$B = [b_0 , b_1, b_2,...]$ *where* b_k *are the k'th bernoulli-numbers, and* $b_1 = +1/2$

I also use for convenience $\mathbf{J}$ and $\mathbf{I}$ for the vectors resp diagonalmatrices

Identity-matrix $I = {}^dV(1) = diag(1,1,1,...)$
alt.Identity $J = {}^dV(-1) = diag(1,-1,1,-1,...)$

The binomial-matrix:

P: the matrix of binomial-coefficients

$$P = \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix}$$

$P_J = P * J$

the binomial-matrix having columns with alternating signs

$$P_J = \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & -1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ 1 & -3 & 3 & -1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix}$$

Eigenvector-relations:

(1.3.1.)
$P_J * V(½) = V(½) * 1$
saying $V(½)$ *is an eigenvector of* P_J
$P_j * B = B * 1$
$P_j * Z(1) = Z(1) * 1$

$$\begin{array}{cccc} & V(½) & B & Z(1) \\ & * \begin{bmatrix} 1 \\ 1/2 \\ 1/4 \\ 1/8 \\ 1/16 \\ 1/32 \end{bmatrix} & * \begin{bmatrix} 1 \\ 1/2 \\ 1/6 \\ 0 \\ -1/30 \\ 0 \end{bmatrix} & * \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \\ 1/5 \\ 1/6 \end{bmatrix} \\ P_j = \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & -1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ 1 & -3 & 3 & -1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix} & = \begin{bmatrix} 1 \\ 1/2 \\ 1/4 \\ 1/8 \\ 1/16 \\ 1/32 \end{bmatrix} * 1 & = \begin{bmatrix} 1 \\ 1/2 \\ 1/6 \\ 0 \\ -1/30 \\ 0 \end{bmatrix} * 1 & = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \\ 1/5 \\ 1/6 \end{bmatrix} * 1 \end{array}$$

Many of the special properties of the Bernoulli-numbers can be related to the property of being an eigenvector of the binomial-matrix. All the above mentioned sequences are eigenvectors of $\boldsymbol{P_J}$, and thus a generalized approach to unify these sequences in a common generation scheme is at hand.

1.4. Eigenvector-approach for finding the formula

To solve an eigenvector problem using the eigenvalue *1* requires solving the matrix-equation:

$$P * X = X * \lambda \qquad // \text{ where } \lambda \text{ is an eigenvalue}$$
$$P * X - \lambda X = 0$$
$$(P - \lambda I) X = 0$$

(1.4.1.) $(P_J - 1*I) * X = 0$
(use eigenvalue $\lambda = 1$)

$$\begin{bmatrix} 0 & . & . & . & . & . \\ 1 & -2 & . & . & . & . \\ 1 & -2 & 0 & . & . & . \\ 1 & -3 & 3 & -2 & . & . \\ 1 & -4 & 6 & -4 & 0 & . \\ 1 & -5 & 10 & -10 & 5 & -2 \end{bmatrix} * \begin{bmatrix} x0 \\ x1 \\ x2 \\ x3 \\ x4 \\ x5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This can be done by rowwise gaussian elimination. One gets:

(1.4.2.)
$G_3 :=$ Gaussian row-elimination on $(P_J - I)$

$$\begin{bmatrix} 0 & . & . & . & . & . \\ 1 & -2 & . & . & . & . \\ 0 & 0 & 0 & . & . & . \\ 0 & -1 & 3 & -2 & . & . \\ 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 1 & -4 & 5 & -2 \end{bmatrix} * \begin{bmatrix} x0 \\ x1 \\ x2 \\ x3 \\ x4 \\ x5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

One special property of the binomialmatrix is, that now each second row is zero due to linear dependencies of rows, which occurs regularly in the process of elimination.

A rescaling of rows and columns of $\boldsymbol{G_3}$ exhibits then a straightforward pattern of coefficients; the $\boldsymbol{X}$-vector must be rescaled accordingly and is rewritten here as $\boldsymbol{Y}$:

(1.4.3.)

$$\begin{aligned} G_2 * Y &= {}^dZ(-1) * G_3 * {}^dZ(1) * {}^dZ(-1) * X \\ &= [{}^dZ(-1) * G_3 * {}^dZ(1)] * [{}^dZ(-1) * X] \\ &= G_2 * Y \\ &= 0 \end{aligned}$$

$$\begin{bmatrix} 0 & . & . & . & . & . \\ 2 & -2 & . & . & . & . \\ 0 & 0 & 0 & . & . & . \\ 0 & -2 & 4 & -2 & . & . \\ 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 2 & -6 & 6 & -2 \end{bmatrix} * \begin{bmatrix} y0 \\ y1 \\ y2 \\ y3 \\ y4 \\ y5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which again can be improved by rescaling with the factor *1/2* (this row rescaling is irrelevant for the solution in ***Y*** and thus also irrelevant in ***X***):

(1.4.4.)

$$G_1 * Y = 1/2 * G_2 * Y = 0$$

$$\begin{bmatrix} 0 & . & . & . & . & . \\ 1 & -1 & . & . & . & . \\ 0 & 0 & 0 & . & . & . \\ 0 & -1 & 2 & -1 & . & . \\ 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 1 & -3 & 3 & -1 \end{bmatrix} * \begin{bmatrix} y0 \\ y1 \\ y2 \\ y3 \\ y4 \\ y5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Discarding the empty rows gives the final formula:

(1.4.5.)

$$G * Y = 0$$

$$\begin{bmatrix} 1 & -1 & . & . & . & . & . & . & . & . & . & . \\ . & -1 & 2 & -1 & . & . & . & . & . & . & . & . \\ . & . & 1 & -3 & 3 & -1 & . & . & . & . & . & . \\ . & . & . & -1 & 4 & -6 & 4 & -1 & . & . & . & . \\ . & . & . & . & 1 & -5 & 10 & -10 & 5 & -1 & . & . \\ . & . & . & . & . & -1 & 6 & -15 & 20 & -15 & 6 & -1 \end{bmatrix} * \begin{bmatrix} y0 \\ y1 \\ y2 \\ y3 \\ y4 \\ y5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

To find an eigenvector of P_J one has to solve $G * Y = 0$ for ***Y*** and then to rescale

(1.4.6.) $X = {}^{d}Z(1) * Y,$

to obtain

$$X = [x_0, x_1, x_2, x_3, ...] \sim$$

as one possible eigenvector.

This can be done by recursion and is expressed by formula *(1.2.1)*.

1.5. One degree of freedom for a pair of selection

The specific property of the binomial-matrix P_J is, that on each row we have ***two*** new coefficients, which allows one degree of freedom for the choice of one value, the ratio or the pair cosine/sine (defining for instance the slope of a vector) of an angle-parameter phi. (The length of this vector is determined by the previous result of the recursion).

The formula is then, again using $ß$ for a general solution:

(1.5.1.)

$$\begin{aligned}
0 &= 1\,ß_0*1 - 1\,ß_1*2 \\
0 &= 1\,ß_1*2 - 2\,ß_2*3 + 1\,ß_3*4 \\
0 &= 1\,ß_2*3 - 3\,ß_3*4 + 3\,ß_4*5 - 1\,ß_5*6 \\
0 &= 1\,ß_3*4 - 4\,ß_4*5 + 6\,ß_5*6 - 4\,ß_6*7 + 1\,ß_7*8 \\
0 &= 1\,ß_4*5 - 5\,ß_5*6 + 10\,ß_6*7 - 10\,ß_7*8 + 5\,ß_8*9 - 1\,ß_9*10 \\
0 &= \ldots
\end{aligned}$$

Here $ß_0$ is free, but the relation with $ß_1$ is fixed. We set for all solutions $ß_0 = b_0 = 1$ and thus $ß_1 = b_1 = + ½$.

To display the cosine/sine-appeal more visible this can be rewritten with the pairs of

coefficients (c_k , s_k) :

(1.5.2.)

$$\begin{aligned}
0 &= 1\,c_0*1 - 1\,s_0*2 \\
0 &= 1\,s_0*2 - 2\,c_1*3 + 1\,s_1*4 \\
0 &= 1\,c_1*3 - 3\,s_1*4 + 3\,c_2*5 - 1\,s_2*6 \\
0 &= 1\,s_1*4 - 4\,c_2*5 + 6\,s_2*6 - 4\,c_3*7 + 1\,s_3*8 \\
0 &= 1\,c_2*5 - 5\,s_2*6 + 10\,c_3*7 - 10\,s_3*8 + 5\,c_4*9 - 1\,s_4*10
\end{aligned}$$

If I insert in a recursive computation values for (c_k , s_k) I can freely choose either

* one of the two required values or
* their ratio / the angle.

Recall the list of possible eigenvectors :

$ß_0$	$ß_1$	$ß_2$	$ß_3$	$ß_4$	.....	
1	1/2	0	- 1/4	0	...	// multiples of Eta-function at negative or 0 exponent
1	1/2	1/6	0	-1/30	...	// Bernoulli-numbers
1	1/2	1/4	1/8	1/16	...	// geometric series
1	1/2	1/3	1/4	1/5	..	// harmonic series
1	1/2	1/2	1/2	1/2	..	// constant series
binomial-series require scaling of $ß_0$:						
2	1	1	1	1	..	// binomial(r,0) series = 2* constant series
1	1/2	1/3	1/4	1/5	..	// binomial(r,1) series = harmonic-series
1/3	1/6	1/10	1/15	1/21	...	// binomial(r,2) series
1/10	1/20	1/35	1/56	1/84		// binomial(r,3) series

(and it seems as all binomial-series starting at the appropriate index fit this scheme)

This gives

* the Bernoulli-numbers, when setting $s_k = 0$ for all $k>0$; this means I choose the angle $0°$ for all pairs,
* multiple of eta-function $(=2*\eta(-k))$ when all $c_k = 0$; this means I choose $90° = pi/2$ for all pairs.
* If I set the multiples of 2 from s_1 for all s_k I get the harmonic series, (ratios *1/2,3/4,5/6,...)*
* if I set all powers of 4 for c_k I get the geometric series; (ratios all $1/2$)
* and even I can set all $s_k, c_k = ½$ *for* $k>0$. (ratios all 1)
* and the reciprocals of all binomial-series seem to fit, if started at an appropriate index (ratios for *binomial(k,2) = 2/4, 4/6, 6/8,...*for *binomial(k,3) = 3/6, 5/8, 7/10,...*)

This is the meaning of the "generalized Bernoulli-recursion" and it exhibits a very special relation between these sets of basic number-theoretical numbers:

For any eigenvector of $\mathbf{P_J}$ *(including the vector of Bernoulli-numbers) the recursive identity holds:*

(1.5.3.)

$$ß_0 = 1$$

Recursion:

$$0 = \sum_{k=n}^{2n} (-1)^k \text{binomial}(n, k-n) * ß_{k-1} * k$$

where the whole set of ß's may be scaled by one appropriate multiple to adapt to the above indicated sets.

2. Some courious identities as consequences

One can find some new(?) identities based on this system of equations. Here I replace, for readers convenience the coefficients (c_k, s_k) which are constantly scaled by their index by (C_k, S_k) and determine the column-sums:

(2.1.1.)

```
0 = 1 C0  - 1 S0
0 =        1 S0  - 2 C1 +  1 S1
0 =               1 C1  - 3 S1 + 3 C2  - 1 S2
0 =                      1 S1  - 4 C2  + 6 S2  -  4 C3 +   1 S3
0 =                             1 C2  - 5 S2  + 10 C3  - 10 S3  + 5 C4  - 1 S4
  ...
Sums = -------------------------------------------------------------------------------
0 =  1 C0 + 0 S0  -1 C1  -  1 S1  + 0 C2  +  1 S2  +  1 C3  +  0 S3  -  1 C4   -1 S4 ...// generally

0 =  1 b0 *1     -1 b2*3        ...         + 1 b6*7         -1 b8*9     ....   // bernoulli
```

then all possible solutions for eigenvectors -as far as convergence is given - should satisfy the identity, which is composed from the column-sums:

(2.1.2.)

$$0 = \sum_{k=0}^{oo} \frac{2}{sqrt(3)} \cos\left(\frac{\pi}{2} * \frac{2k-1}{3}\right) * ß_k * (k+1) \qquad // ß_1 = +½$$

where the constant term $2/sqrt(3)$ may be omitted, and is here introduced only to get the $cos()$-coefficients scaled to $(1,0,-1)$.[1]

If the coefficients $ß_k$ contain the Bernoulli-numbers b_k, this formula can be remarkably simplified, since all $b_{2k+1} = s_{k+1} = 0$ (for $k>0$) and also each third of the remaining terms is zeroed by the vanishing column-sum:

(2.1.3.)

$$0 = \sum_{k=0}^{oo} \left(b_{6k} * (6k+1) - b_{6k+2} * (6k+3) \right)$$

However, note, that in the case of Bernoulli-numbers (as well as with the $\eta(-n)$-values) this sum is strongly ***divergent*** and requires for instance Borel-summation to get a value assigned to. The same technique is required in the next example.

Note also, that this formula gives also an asymptotic growth-rate for the Bernoulli-numbers (which I didn't explicate here though).

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[1] I don't have a proof, that indeed the column-sums have only $(1,0,-1)$ as cofactors and in the suggested periodicity. However in OEIS, Paul Barry *[see: A010892]* mentions just this $(1,0,-1)$-identity of the binomial-coefficients, as they occur in the column-sums , as well as the Fibonacci-sum assumption in the next example.

* **A second courious identity:**

Switching signs in each second row this can also be related to the Fibonacci-numbers:

(2.1.4.)

$$
\begin{array}{lllllllllllll}
0 = 1\,C_0 & -1\,S_0 & & & & & & & & & & \\
0 = & -1\,S_0 & +2\,C_1 & -1\,S_1 & & & & & & & & \\
0 = & & 1\,C_1 & -3\,S_1 & +3\,C_2 & -1\,S_2 & & & & & & \\
0 = & & & -1\,S_1 & +4\,C_2 & -6\,S_2 & +4\,C_3 & -1\,S_3 & & & & \\
0 = & & & & 1\,C_2 & -5\,S_2 & +10\,C_3 & -10\,S_3 & +5\,C_4 & -1\,S_4 & & \\
\ldots & & & & & & & & & & & \\
\text{Sums} = & \text{-----} & & & & & & & & & & \\
0 = 1\,C_0 & -2\,S_0 & +3\,C_1 & -5\,S_1 & +8\,C_2 & -13\,S_2 & +21\,C_3 & -34\,S_3 & +55\,C_4 & -89\,S_4 \ldots & // \text{ generally} \\
 & & & & & & & & & & \\
0 = 1\,b_0 *1 & -2b_1 *2 & +3\,b_2 *3 & & +8\,b_4 *5 & & +21\,b_6 *7 & & +55\,b_8 *9 & \ldots. & // \text{ bernoulli}
\end{array}
$$

then all possible solutions for eigenvectors - as far as convergence is given - shall satisfy the following identity over column-sums, which involves the Fibonacci-numbers Fib_k:[2]

(2.1.5.)

$$0 = \sum_{k=0}^{oo} (-1)^k \; Fib_{k+1} \; * ß_k *(k+1) \qquad // \; ß_1 = +½$$

Again, if the coefficients $ß_k$ contain the Bernoulli-numbers b_k , the formula reduces to the zero-sum

(2.1.6.)

$$0 = -4b_1 + \sum_{k=0}^{oo} \left(\; Fib_{2\,k+1}\, b_{2\,k} \; *(2\,k+1) \; \right)$$

which however again needs techniques of divergent summation, to arrive at this value.

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[2] *(see [A010892] again)*

* Convergent examples of identities

The above formulae do not converge for all eigenvectors, for instance when using Bernoulli-numbers. The weirdest problem is here, that finite approximations discard increasing numbers of terms with an overall growing weight.

However, to get an impression of the validity of these formulae one can adapt them to get convergent series. For brevity I express this in terms of matrix-multiplications.

The above formula says:

$$V(1)\sim * G* {}^dZ(-1) * X = 0$$

where ***X*** is one of the inserted eigenvectors, here we consider the vector ***B*** of Bernoulli-numbers.

To have a convergent example one could row-scale the matrix ***G***, to get compositions of decreasing weight for higher indexes, for instance by inverse-factorial (${}^dF^{-1}$) weighting of rows of ***G***:

$$V(1)\sim * {}^dF^{-1} * G* {}^dZ(-1) * B = 0$$
$$T\sim \quad * B = 0$$

This gives for ***T*** the vector

$$T = [\ 1,\ 0,\ -3/2,\ -1/3,\ 7/8,\ 3/10,\ -43/144,\ -113/840 ,\dots]\sim$$

with still diverging terms of ***T***~ * ^{d}***B*** :

$$T\sim * {}^dB = [1, 0, -0.25, 0, -0.029167, 0, -0.0071,\dots 0.08698, 0, 0.1916, 0, 0.4367, 0, 1.0280, 0, 2.4947, 0, 6.2291, 0, 15.968, 0,\dots \text{<diverging>}]$$

and the partial sums are also diverging - the factorial scaling alone does not suffice.

A bit sharper (possibly the weakest sufficient) rowscaling seems to be:

$$T\sim \quad = V(1/2)\sim * {}^dF^{-1} * G * {}^dZ(-1)$$
$$= [\ 1,\ -1,\ -21/8,\ 7/12,\ 565/384,\ -49/640, \dots]$$

The limit of zero is again expected by formula

$$\lim \quad \Sigma_{n\to oo}\ T_n * b_n = 0$$

$$0 = 1 * b_0 - 1 * b_1 - \frac{21}{8} b_2 + \frac{565}{384} b_4 - \frac{18977}{46080} b_6 + \frac{85217}{1146880} b_8 \dots$$

explicitely

$$0 = 1 - \frac{1}{2} - \frac{7}{16} - \frac{113}{2304} - \frac{2711}{276480} \dots$$

The partial sums (by Euler-summation) are then (rounded to 4 decimals)

$$S_n = [0.5,\ 0.625,\ 0.5703, 0.4570,\ 0.3422,\ 0.2466,\ 0.1739,\ 0.1212, \dots]$$

with the (assumed) limit:

$$\lim S_{n\to oo} = 0 .$$

A sharper (and definitely sufficient) scaling uses the squares of the inverse factorials:

$T\sim \quad = V(1)\sim *^dF^{-2} * G * {}^dZ(-1)$
$\quad = [\ 1\ \ 0\ \ -21/4\ \ 10/9\ \ 205/64\ \ -331/600\ ...]$

and then the limit of zero is expected by the formula:

$\lim \quad \Sigma_{n->oo}\ T_n * b_n = 0$

$$0 = 1 * b_0 - \frac{21}{4} b_2 + \frac{205}{64} b_4 - \frac{68341}{103680} b_6 + \frac{68111}{1032192} b_8 \ldots$$

explicitely

$$0 = 1 - \frac{7}{8} - \frac{41}{384} - \frac{9763}{622080} - \frac{68111}{30965760} \ldots$$

The partial sums are then

$S_n = [1, 0.125, 0.0182292, 0.002535, \ ...]$

with the now much more suggestive approximation to the expected limit

$\lim S_{n->oo} = 0$.

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* **A general remark on constructing identities:**

The occurence of two such zero-sum-identities suggest another generalization.

The least we can say is, that we can construct

* infinitely many

* and nearly arbitrary

such identities just by composition of sums/differences of rows. Moreover, if the divergent summation-method, which can be applied, is regular, we can add many more other compositions to eventually arrive at values different from the zero-sum.

The set of possible selections however may be principally limited in that not all combinations of, say the Bernoulli-number multiples, can be constructed this way to get the weighted sum equal zero. I didn't look deeply in this question, so I'll leave it with this remark.

2.2. $H = G^{-1}$; the reciprocal of G

It is possible to describe a reciprocal of ***G***. Computing stepwise the inverses of the finite top-left submatrices of ***G*** gives the following matrix ***H***.

(2.2.1.) $H = G^{-1}$

Example

$H = G^{-1} =$

1	-1	2	-5	14	-42	132	-429	1430	-4862
.	-1	2	-5	14	-42	132	-429	1430	-4862
.	.	1	-3	9	-28	90	-297	1001	-3432
.	.	.	-1	4	-14	48	-165	572	-2002
.	.	.	.	1	-5	20	-75	275	-1001
.	.	.	.	.	-1	6	-27	110	-429
.	.	.	.	.	.	1	-7	35	-154
.	.	.	.	.	.	.	-1	8	-44
.	.	.	.	.	.	.	.	1	-9
.	.	.	.	.	.	.	.	.	-1

The rows of ***H*** are described in OEIS in various entries; row *0* is known as "Catalan-numbers" and the subsequent rows *r* seem most consistently be described as "r'th convolution" of that row. *http://www.research.att.com/~njas/sequences/A033184* with the description of entries in row *n*,column *m*:

*a(n,m) = (m+1)*binomial(2*n - m , n - m) / (n+1)*

The (divergent) sums of the first top rows, approximated by Euler-summation give

0.6180339887493705
-0.3819660112452572
0.2360679774723745
-0.1458980336327082
0.09016994333424594
-0.05572808874184361
0.03444185038347767
-0.02128622818153165
0.01315559990577021
-0.008130583574793153

which seems to be the powerseries of the golden ratio *φ=(sqrt(5)-1)/2* and the resulting vector of rowsums is then (proposed):

(2.2.2.) $(J*H) * E = \varphi * V(\varphi)$ *(Euler-summation)*

$$(-1)^r \sum_{c=0}^{oo} H_{r,c} = \varphi^{1+r}$$ *for a fixed row r*

Generalized one finds (empirically) for an arbitrary *a* as rowsum in row *r* of ***H***

define $y = x/4, \quad z=½*((x+1)^{½} - 1)$

(2.2.3.) $(J * H) * y\, V(y) = z\, V(z)$

$$(-1)^r \sum_{c=0}^{oo} (\frac{x}{4})^{1+r} H_{r,c} = (\frac{1}{2}(\sqrt{x+1} - 1))^{1+r}$$

2.3. H and G as eigenmatrix of a known Riordan array

G and G^{-1} (= **H**) form an eigensystem of another known matrix (where the columns are scaled by the column-number to get integer entries):

$Y = G * {}^dZ(1) * G^{-1} * {}^dZ(-1)$

$$Y = \begin{bmatrix} 1 & 1 & 3 & 10 & 35 & 126 \\ . & 1 & 1 & 3 & 10 & 35 \\ . & . & 1 & 1 & 3 & 10 \\ . & . & . & 1 & 1 & 3 \\ . & . & . & . & 1 & 1 \\ . & . & . & . & . & 1 \end{bmatrix}$$

The matrix is known to OEIS, see: [A106268]

2.4. Using G again: binomial sums of the golden ratio

One more identity can be derived, given that proposition (2.2.3) holds. Leftmultiply the equation with $(J*H)^{-1}$ (= $G*J$) then we have a summing identity involving binomial-coefficients and powers of the golden ratio:

(2.4.1.) $G * J * \varphi * V(\varphi) = E$

(2.4.2.) $$\sum_{c=0}^{oo} G_{r,c} * (-1)^c * \varphi^{1+c} = 1 \qquad or \qquad \varphi^{r+1} \sum_{c=0}^{r+1} \binom{r+1}{c} * \varphi^c = 1$$

which can be generalized to the following simple equation (in the conventional notation for a fixed row *r* in **G**):

define $y = x^{½} / 2$

(2.4.3.) $G*J*\ (y - ½)\ V(y - ½) = (y^2 - ¼)\ V(y^2 - ¼)$

$$(y - \frac{1}{2})^{r+1} * \sum_{c=0}^{r+1} \binom{r+1}{c} (y - \frac{1}{2})^c = (y^2 - \frac{1}{4})^{r+1}$$

and obviously agrees with the common binomial-transform.

Proof:

(2.4.4.) *for a fixed row r:*

$$(y - \frac{1}{2})^{r+1} \sum_{c=0}^{r+1} \binom{r+1}{c} * (y - \frac{1}{2})^c \qquad = (y - \frac{1}{2})^{r+1} (y + \frac{1}{2})^{r+1} \qquad = (y^2 - \frac{1}{4})^{r+1}$$

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3. References

[Project-Index] http://go.helms-net.de/math/binomial/index.htm

[Z.W.SUN] Zhi Wei Sun : SOME CURIOUS RESULTS ON BERNOULLI AND EULER POLYNOMIALS
A talk given at Institut Camille Jordan, Universit´e Claude Bernard Lyon-I (Jan. 13, 2005), and University of Wisconsin at Madison (April 4, 2006).
Online at Zhi Wei Sun: http://pweb.nju.edu.cn/zwsun

[A010892] http://www.research.att.com/~njas/sequences/A010892

A010892	Inverse of 6th cyclotomic polynomial. A period 6 sequence.
	1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1
COMMENT	Any sequence b(n) satisfying the recurrence b(n)=b(n-1)-b(n-2) can be written as b(n) = b(0) * a(n) + (b(1)-b(0)) * a(n-1) . (...)
REFERENCES	Paul Barry, A Catalan Transform and Related Transformations on Integer Sequences, Journal of Integer Sequences, Vol. 8 (2005), Article 05.4.5.
LINKS	Ralph E. Griswold, Shaft Sequences Index entries for sequences related to Chebyshev polynomials.
FORMULA	G.f.: 1/(1-x+x^2); a(n)=a(n-1)-a(n-2), a(0)=1, a(1)=1; a(n)= S(n, 1) = U(n, 1/2) (Chebyshev U(n, x) polynomials). a(n)=sum{k=0..n, C(k, n-k)(-1)^(n-k) }. - Paul Barry (pbarry(AT)wit.ie), Sep 13 2003 a(n)=sum{k=0..floor(n/2), C(n-k, k)(-1)^k} - Paul Barry (pbarry(AT)wit.ie), Jul 28 2004

[A106268] http://www.research.att.com/~njas/sequences/A106268
for more comments see also
http://www.research.att.com/~njas/sequences/A001700

A001700	C(2n+1, n+1): number of ways to put n+1 indistinguishable balls into n+1 distinguishable boxes = number of (n+1)-st degree monomials in n+1 variables = number of monotone maps from 1..n+1 to 1..n+1. (Formerly M2848 N1144)
	1, 3, 10, 35, 126, 462, 1716, 6435, 24310, 92378, 352716, 1352078, 5200300, 20058300, 77558760, 300540195, 1166803110, 4537567650, 17672631900, 68923264410, 269128937220, 1052049481860, 4116715363800, 16123801841550 (list; graph; listen)
COMMENT	To show for example that C(2n+1, n+1) is the number of monotone maps from 1..n+1 to 1..n+1, notice that we can describe such a map by a nondecreasing sequence of length n+1 with entries from 1 to n+1. The number k of increases in this sequence is anywhere from 0 to n. We can specify these increases by throwing k balls into n+1 boxes, so the total is Sum_{k=0..n} C((n+1)+k-1, k) = C(2n+1, n+1). (...)

Eigenvectors of Pj = "invariant sequences under binomial-transformation"

Sun, Zhi-Hong;
Invariant sequences under binomial-transform,
Fibonacci Quarterly 39, No 4, Pg 324-333
http://www.hytc.cn/xsjl/szh/iis.pdf

Zentralblatt MATH Database 1931 – 2007

0987.05013

Sun, Zhi-Hong
Invariant sequences under binomial transformation. (English)
Fibonacci Q. 39, No.4, 324-333 (2001). [ISSN 0015-0517]
http://www.sdstate.edu/ wcsc/http/fibhome.html

A sequence, which satisfies the first relation, is called invariant sequence (IS), and a sequence, which satisfies the second, is called inverse invariant sequence (IIS). The author notes that $\{\alpha_n\} \in$ ISS if and only if $\alpha_0 = 0$ and either $\{\alpha_{n+1}/(n+1)\}$ or $\{n\alpha_{n-1}\} \in$ IS. He also notes that $\{1/2^n\}$, $\{[nF_{n-1}\}$, $\{L_n\}$ and $\{(-1)^n B_n\}$ are invariant sequences, where F_n, L_n, and B_n denote the Fibonacci, Lucas and Bernoulli numbers.

Gottfried Helms, 09.02.2007, minor corrections and updates 05.02.2009