Identities involving binomial-coefficients, Bernoulli- and Stirlingnumbers

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05 Stirling-matrices St_1 and St_2

Abstract: The Stirling-matrices occur as other initimate and basic relatives of the ZV-(Vandermonde) matrix. Variants transform powerseries to exponentialseries and conversely. Using finite sizes they or their scaled variants give rational approximations to logarithms and exponentials. The most striking property for me is, that they are eigenmatrices of the Bernoullian-matrix G_p, which sums geometric series to zeta-type series of any like powers to any finite number of terms.

Most of the formulae here are heuristic findings (although in the meantime I found most of the formulas in textbooks and online-references). The focus in my recent study was primarily at the binomial- and the Gp-matrix; but I expect to understand more details of these matrices when analyzing the Stirling-matrices intensely.

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1. Definitions/ Identities

1.1. Introduction

The following matrices are defined here:

(1.1.1.) *St_1: lower triangular matrix of Stirling-numbers 1'st kind*

The definition for Stirling-numbers of first kind is the expansion of coefficients of *x* after expansion of the product

St1

$$\begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 2 & -3 & 1 & . & . & . \\ -6 & 11 & -6 & 1 & . & . \\ 24 & -50 & 35 & -10 & 1 & . \\ -120 & 274 & -225 & 85 & -15 & 1 \end{bmatrix}$$

1 *for row r=0*
(x-1)(x-2)(x-3)...(x-r) *for a row r>0*

example:

$(x-1)(x-2) = 2 - 3*x + 1*x^2$ // coefficients of row 2
$(x-1)(x-2)(x-3) = -6 + 11*x - 6*x^2 + 1*x^3$ // coefficients of row 3

see [MW-St1],[AS-ST]

(1.1.2.) *St_2: lower triangular matrix of Stirling-numbers 2'nd kind*

St2

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 3 & 1 & . & . & . \\ 1 & 7 & 6 & 1 & . & . \\ 1 & 15 & 25 & 10 & 1 & . \\ 1 & 31 & 90 & 65 & 15 & 1 \end{bmatrix}$$

see [MW-St1],[AS-ST]

Shifted versions

Sometimes these matrices are defined with an additional leading row and column containing zeros except *1* at *[0,0]* (see en.wikipedia.org).

(1.1.3.) *$St_1^{(1)}$: St_1, downshifted one row/column*

This definition can be seen as extension of $\mathbf{St_1}$ by the *(x-0)*-factor:

St1(1)

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & -1 & 1 & . & . & . \\ 0 & 2 & -3 & 1 & . & . \\ 0 & -6 & 11 & -6 & 1 & . \\ 0 & 24 & -50 & 35 & -10 & 1 \end{bmatrix}$$

1 *for row r=0*
(x-0)(x-1)(x-2)(x-3)...(x-(r-1)) *for row r>0*

see [WIKI-St1],[AS-ST]

(1.1.4.) *$St_2^{(1)}$: St_2, downshifted one row/column*

St2(1)

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 1 & 1 & . & . & . \\ 0 & 1 & 3 & 1 & . & . \\ 0 & 1 & 7 & 6 & 1 & . \\ 0 & 1 & 15 & 25 & 10 & 1 \end{bmatrix}$$

see [WIKI-St2],[AS-ST]

factorial scaled versions fSt_1, fSt_1F, St_2F, fSt_2F

Of special interest ar also the factorial row- and row/column-scaled versions.

(**1.1.5.**) $fSt_1 := F^{-1} * St_1$

fSt1

1	.	.	.	.	.
-1	1	.	.	.	.
1	-3/2	1/2	.	.	.
-1	11/6	-1	1/6	.	.
1	-25/12	35/24	-5/12	1/24	.
-1	137/60	-15/8	17/24	-1/8	1/120

(**1.1.6.**) $fSt_1F := F^{-1} * St_1 * F$

fSt1F

1	.	.	.	.	.
-1	1	.	.	.	.
1	-3/2	1	.	.	.
-1	11/6	-2	1	.	.
1	-25/12	35/12	-5/2	1	.
-1	137/60	-15/4	17/4	-3	1

They perform the summing to logarithms, if the columns are used as coefficients of a powerseries. (see paragraphs below)

(**1.1.7.**) $St_2F := St_2 * F$

St2F

1	.	.	.	.	.
1	1	.	.	.	.
1	3	2	.	.	.
1	7	12	6	.	.
1	15	50	60	24	.
1	31	180	390	360	120

(**1.1.8.**) $fSt_2F := F^{-1} * St_2 * F$

fSt2F

1	.	.	.	.	.
1	1	.	.	.	.
1/2	3/2	1	.	.	.
1/6	7/6	2	1	.	.
1/24	5/8	25/12	5/2	1	.
1/120	31/120	3/2	13/4	3	1

They perform the summing to exponentials (inverse of the summing of ***fSt_1F***), if the columns are used as coefficients of a powerseries. (see paragraphs below)

1.2. Definition in terms of their reciprocity

The matrices ***St_1*** and ***St_2*** (as well as the shifted versions) are also defined by their mutual reciprocity (either ***St_2*** as reciprocal of ***St_1*** or opposite):

(1.2.1.) $St_2 = St_1^{-1}$

$$\sum_{k=0}^{r} St1[r,k] * St2[k,c] = \delta_{r,c}$$ *where δ is the Kronecker-delta*

$St_1 * St_2 = I$

St1

1	.	.	.	.	.
-1	1	.	.	.	.
2	-3	1	.	.	.
-6	11	-6	1	.	.
24	-50	35	-10	1	.
-120	274	-225	85	-15	1

*

St2

1	.	.	.	.	.
1	1	.	.	.	.
1	3	1	.	.	.
1	7	6	1	.	.
1	15	25	10	1	.
1	31	90	65	15	1

=

I

1	.	.	.	.	.
.	1	.	.	.	.
.	.	1	.	.	.
.	.	.	1	.	.
.	.	.	.	1	.
.	.	.	.	.	1

1.3. Recursive definitions

Recursive definitions include:

(1.3.1.)
$$St1[r,0] = (-1)^r * r!$$
$$St1[r+1,c] = (-1)^{r-c} * abs((r+1)* St1[r,c]) + abs(St1[r-1,c-1])$$

(1.3.2.)
$$St2[r,0] = 1$$
$$St2[r+1,c] = (c+1)* St2[r,c] + St2[r-1,c-1]$$

(additional remarks: see "details/derivations")

1.4. $St_2^{(1)}$ occurs also as matrix of coefficients of the derivatives of $e^{e\wedge x}$

If one computes the coefficients of the derivatives of $exp(e^x)$

$f:=\quad f(x) = exp(e^x)$
and $\quad z:=e^x$

Then

(1.4.1.)
$$\begin{array}{lllllll} f & = f *(& 1) & & & & \\ f' & = f *(& 0 & 1\,z) & & & \\ f'' & = f *(& 0 & 1\,z & 1\,z^2) & & \\ f''' & = f *(& 0 & 1\,z & 3\,z^2 & 1\,z^3) & \\ f^{(4)} & = f *(& 0 & 1\,z & 7\,z^2 & 6\,z^3 & 1\,z^4) \\ \text{... etc} & & & & & & \end{array}$$

St2(1)

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 1 & 1 & . & . & . \\ 0 & 1 & 3 & 1 & . & . \\ 0 & 1 & 7 & 6 & 1 & . \\ 0 & 1 & 15 & 25 & 10 & 1 \end{bmatrix}$$

which means in matrix-notation

(1.4.2.)
$$f * St_2^{(1)} * \quad V(e^x) = [\, f,\ f',\ f'',\ f''',\ f^{(4)}, ...]\sim$$
$$f * St_2 \quad * e^x\, V(e^x) = \quad [\, f',\ f'',\ f''',\ f^{(4)}, ...]\sim$$

or

(1.4.3.)
$$St_2^{(1)} * V(e^x) = [\, f,\ f',\ f'',\ f''',\ f^{(4)}, ...]\sim \ / f$$

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2. Simple relations to other vectors and matrices

2.1. Shifted versions by binomial-transformation

A binomial-transformation performs the shifting of $\mathbf{St_2}$ to $\mathbf{St_2^{(1)}}$ resp $\mathbf{St_1}$ to $\mathbf{St_1^{(1)}}$ and more generally a shifted binomial-transform $\mathbf{P^{(k)}}$ performs the shifting of $\mathbf{St_2^{(k)}}$ to $\mathbf{St_2^{(k+1)}}$ resp $\mathbf{St_1^{(k)}}$ to $\mathbf{St_1^{(k+1)}}$.

Shifting of St₂:

(2.1.1.) $P^{-1} * St_2 = St_2^{(1)}$

$P^{-1} * St_2 = St_2^{(1)}$

St2
$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 3 & 1 & . & . & . \\ 1 & 7 & 6 & 1 & . & . \\ 1 & 15 & 25 & 10 & 1 & . \\ 1 & 31 & 90 & 65 & 15 & 1 \end{bmatrix}$$

P^-1 * St2 = St2(1)
$$\begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ -1 & 3 & -3 & 1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ -1 & 5 & -10 & 10 & -5 & 1 \end{bmatrix} * St_2 = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 1 & 1 & . & . & . \\ 0 & 1 & 3 & 1 & . & . \\ 0 & 1 & 7 & 6 & 1 & . \\ 0 & 1 & 15 & 25 & 10 & 1 \end{bmatrix}$$

Shifting of St₁:

(2.1.2.) $St_1 * P = St_1^{(1)}$

$St_1 * P = St_1^{(1)}$

P
$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix}$$

St1 * P = St1(1)
$$\begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 2 & -3 & 1 & . & . & . \\ -6 & 11 & -6 & 1 & . & . \\ 24 & -50 & 35 & -10 & 1 & . \\ -120 & 274 & -225 & 85 & -15 & 1 \end{bmatrix} * P = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & -1 & 1 & . & . & . \\ 0 & 2 & -3 & 1 & . & . \\ 0 & -6 & 11 & -6 & 1 & . \\ 0 & 24 & -50 & 35 & -10 & 1 \end{bmatrix}$$

The shifting in the limit

A consequence of this is, that repeated application of equivalently shifted binomial-matrices approximate to the identity matrix:

(2.1.3.) $St_1 * P * P^{(1)} * P^{(2)} * \dots = I$

and that means, that the iterated product of shifted binomialmatrices approximate $\mathbf{St2}$.

(2.1.4.) $\Pi_{k=0..oo} P^{(k)} = St_2$

Example:

$P*P^{(1)}$ (01)
$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 3 & 1 & . & . & . \\ 1 & 7 & 5 & 1 & . & . \\ 1 & 15 & 17 & 7 & 1 & . \\ 1 & 31 & 49 & 31 & 9 & 1 \end{bmatrix}$$

$P*P^{(1)}*P^{(2)}$ (012)
$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 3 & 1 & . & . & . \\ 1 & 7 & 6 & 1 & . & . \\ 1 & 15 & 25 & 9 & 1 & . \\ 1 & 31 & 90 & 52 & 12 & 1 \end{bmatrix}$$

$P*P^{(1)}*P^{(2)}*P^{(3)}$ (0123)
$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 3 & 1 & . & . & . \\ 1 & 7 & 6 & 1 & . & . \\ 1 & 15 & 25 & 10 & 1 & . \\ 1 & 31 & 90 & 65 & 14 & 1 \end{bmatrix}$$

...

$P*P^{(1)}*P^{(2)} \dots = St_2$ (St2)
$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 3 & 1 & . & . & . \\ 1 & 7 & 6 & 1 & . & . \\ 1 & 15 & 25 & 10 & 1 & . \\ 1 & 31 & 90 & 65 & 15 & 1 \end{bmatrix}$$

Conversely, the infinite leftmultiplication-product of shifted inverses of binomialmatrices approximate $\mathbf{St_1}$.

(2.1.5.) $\Pi_{k=oo..0} P^{(k)-1} = St_1$

Example

$...*P^{(2)-1}*P^{(1)-1}*P^{-1} = St_1$

1	.	.	.	.	.
-1	1	.	.	.	.
2	-3	1	.	.	.
-6	11	-6	1	.	.
24	-50	35	-10	1	.
-120	274	-225	85	-15	1

...

$P^{(2)-1}*P^{(1)-1}*P^{-1}$

1	.	.	.	.	.
-1	1	.	.	.	.
2	-3	1	.	.	.
-6	11	-6	1	.	.
18	-39	29	-9	1	.
-54	135	-126	56	-12	1

$P^{(1)-1}*P^{-1}$

1	.	.	.	.	.
-1	1	.	.	.	.
2	-3	1	.	.	.
-4	8	-5	1	.	.
8	-20	18	-7	1	.
-16	48	-56	32	-9	1

2.2. Rowsums of St_1 and St_2, and rightmultiplication with a powerseries

The rowsums of $\mathbf{St_2}$ are known as Bell-numbers:

(2.2.1.) $St_2 * V(1) = B$

$$\sum_{c=0}^{r} St2_{r,c} = B_r$$

St_2:

1	.	.	.	.	.
1	1	.	.	.	.
1	3	1	.	.	.
1	7	6	1	.	.
1	15	25	10	1	.
1	31	90	65	15	1

* $V(1)$:

1
1
1
1
1
1

= B:

1
2
5
15
52
203

The rowsums of $\mathbf{St_1}$ (except or row $r=0$) are zero, which is obvious since the *first* column of $\mathbf{St_2}$ (= $\mathbf{St_1}^{-1}$) is just the 1-vector, as well as from the definition as product of $(x-1)(x-2)...$ when setting $x=1$.

(2.2.2.) $St_1 * V(1) = [1,0,0,0,...]$

$$\sum_{c=0}^{r} St1_{r,c} = 0 \qquad // \text{ for } r>0$$

Using the *second* column of $\mathbf{St_2}$ as weighting coefficients it may be of special interest, that this column equals numbers of the form 2^n-1.

Rightmultiplication by powerseries-vectors produce factorial scaled binomials.

Examples:

various weighted row-sums of St_1 *columns from St_2* *powerseries (or ZV~)*

$$
* \begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}
\begin{bmatrix}0\\1\\3\\7\\15\\31\end{bmatrix}
\begin{bmatrix}0\\0\\1\\6\\25\\90\end{bmatrix}
\begin{bmatrix}1\\2\\4\\8\\16\\32\end{bmatrix}
\begin{bmatrix}1\\3\\9\\27\\81\\243\end{bmatrix}
\begin{bmatrix}1\\4\\16\\64\\256\\1024\end{bmatrix}
$$

$$
\underbrace{\begin{bmatrix}1&.&.&.&.&.\\-1&1&.&.&.&.\\2&-3&1&.&.&.\\-6&11&-6&1&.&.\\24&-50&35&-10&1&.\\-120&274&-225&85&-15&1\end{bmatrix}}_{St1}
= \begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix}
\begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix}
\begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}
\begin{bmatrix}1\\1\\0\\0\\0\\0\end{bmatrix}
\begin{bmatrix}1\\2\\2\\0\\0\\0\end{bmatrix}
\begin{bmatrix}1\\3\\6\\6\\0\\0\end{bmatrix}
$$

ZV is a complete set of powerseries whose bases differ by *1*:

(**2.2.3.**) $St_1 * ZV\sim \ = {}^dF * P\sim$

$$
* \underbrace{\begin{bmatrix}1&1&1&1&1&1\\1&2&3&4&5&6\\1&4&9&16&25&36\\1&8&27&64&125&216\\1&16&81&256&625&1296\\1&32&243&1024&3125&7776\end{bmatrix}}_{ZV\sim}
$$

$$
\underbrace{\begin{bmatrix}1&.&.&.&.&.\\-1&1&.&.&.&.\\2&-3&1&.&.&.\\-6&11&-6&1&.&.\\24&-50&35&-10&1&.\\-120&274&-225&85&-15&1\end{bmatrix}}_{St1}
= \underbrace{\begin{bmatrix}1&1&1&1&1&1\\.&1&2&3&4&5\\.&.&2&6&12&20\\.&.&.&6&24&60\\.&.&.&.&24&120\\.&.&.&.&.&120\end{bmatrix}}_{!*P\sim}
$$

2.3. Columnsums (and leftmultiplication with powerseries) of St_2 and fSt_2F

The columnsums of St_2 are all divergent, so we see the summing vector as limit of a powerseries in *x*, when *x->1* :

(**2.3.1.**) $1/x*V(1/x)\sim \ * St_2 \ = S(1/x)\sim$

$$
* \underbrace{\begin{bmatrix}1&.&.&.&.&.\\1&1&.&.&.&.\\1&3&1&.&.&.\\1&7&6&1&.&.\\1&15&25&10&1&.\\1&31&90&65&15&1\end{bmatrix}}_{St2}
$$

$$
\begin{bmatrix}1/x & 1/x\hat{}2 & 1/x\hat{}3 & 1/x\hat{}4 & 1/x\hat{}5 & 1/x\hat{}6\end{bmatrix}
= \begin{bmatrix}s0 & s1 & s2 & s3 & s4 & s5\end{bmatrix}
$$

This can be summed (or is convergent in the leading columns) for *1/x<1*, since the progression in each column is of the order of a geometric series with quotient *q*, which increases with the column-number *c*, *x* must then at least equal *q*, or must be negative to allow Euler_p-summation of an appropriate order.

The following formula for $1/x<1$ seems to be valid for any entry s_c of **S~** in column c:

(2.3.2.)

$$\sum_{r=0}^{oo} \frac{St2_{r,c}}{x^{r+1}} = \prod_{k=1}^{c+1} \frac{1}{x-k} = s_c\left(\frac{1}{x}\right) \qquad \text{//where } x \neq 1,2,3,\ldots c$$

$$= \frac{1}{x-1} * \frac{1}{x-2} \cdots * \frac{1}{x-(c+1)}$$

or

$$\sum_{r=0}^{oo} St2_{r,c} x^{r+1} = \prod_{k=1}^{c+1} \frac{x}{1-kx} = s_c(x) \qquad \text{//where } 1/x \neq 1,2,3,\ldots c$$

(see also [AS-ST])

Note, that in *(2.3.2)* the productterms in the denominators are just the rowsums of $\mathbf{St_1} * \mathbf{V}(x)$:

(2.3.3.)

$$t_r(x) = (St_1 * V(x))_r$$

$$s_c\left(\frac{1}{x}\right) = \frac{1}{t_{c+1}(x)}$$

Example: $x=-2$, $S = [\,-2/3,\ 2^2/(3*5),\ -2^3/(3*5*7),\ 2^4/(3*5*7*9)\ldots]$
$= [-2/3,\quad 4/15,\quad -8/105,\quad 16/945,\ldots.]$

Note: if x equals a positive integer $1<=x<=c$, we have division by zero, and the value is not defined.

2.4. fSt_2F : column-sums

When the columns of the factorial scaled version of $\mathbf{St_2}$, $\mathbf{fSt_2F}$ are summed (leftmultiplied by the summing-vector $\mathbf{V}(1)$~), this equals:

(2.4.1.)

$$V(1)\sim * fSt_2F = e\, {}^dV(e-1)\sim$$

$$\lim_{r->oo} \sum_{r=0}^{oo} S2_{r,c} \frac{c!}{r!} = e*(e-1)^c \qquad \text{// for a fixed column } c$$

Example:

$V(1)\sim * fSt_2F = e\, V(e-1)\sim$

fSt2F:

1	.	.	.	.	.
1	1	.	.	.	.
1/2	3/2	1	.	.	.
1/6	7/6	2	1	.	.
1/24	5/8	25/12	5/2	1	.
1/120	31/120	3/2	13/4	3	1

lim r->oo [1 1 1 1 1 1] * fSt2F = [e] * [e1^0 e1^1 e1^2 e1^3 e1^4 e1^5] *)

*) e1 is written for e-1 = exp(1)-1

The obvious generalization of ***V****(1)* in *(2.4.1)* into a powerseries ***V****(x)* gives the transformation of a powerseries in *x* into a powerseries in e^x*-1* (although with a cofactor):

(2.4.2.) $V(x)\sim * fSt_2F \quad = e^x * V(e^x - 1)\sim$

$$\lim_{r\to oo} \sum_{r=0}^{oo} x^r * S2_{r,c} \frac{c!}{r!} = e^x * (e^x - 1)^c \qquad \text{//for a fixed column } c$$

(for a further smooting of the result see second next paragraph below)

Example:

$V(x) \sim * fSt_2F = e^x * V(e^x-1)$

	fSt2F		
1	.	.	.
1	1	.	.
1/2	3/2	1	.
1/6	7/6	2	1
1/24	5/8	25/12	5/2
1/120	31/120	3/2	13/4

[1 x x^2 x^3 x^4 x^5] * fSt2F = [e^x*]*[(e^x-1)^0 (e^x-1)^1 (e^x-1)^2 (e^x-1)^3]

Another variant: column-sums in zf St₂ FZ

A further variant can be given by introducing the zeta-series as additional cofactors.

Define the further scaled version of $\boldsymbol{St_2}$:

$zfSt_2FZ = {}^dZ(1) * fSt_2F * {}^dZ(-1)$

Then

(2.4.3.) $V(x)\sim * zfSt_2FZ \quad = (e^x-1) * V(e^x - 1)\sim$

$$\lim_{r\to oo} \sum_{r=0}^{oo} x^r * S2_{r,c} \frac{(c+1)!}{(r+1)!} = (e^x - 1)^{c+1}$$

is a transformation of a powerseries in *x* into one in $(e^x - 1)$.

Example:

$x\, V(x) \sim * zfSt_2FZ = (e^x - 1)\, V(e^x - 1))$

		zfSt2FZ			
1	.	.	.	.	.
1/2	1	.	.	.	.
1/6	1	1	.	.	.
1/24	7/12	3/2	1	.	.
1/120	1/4	5/4	2	1	.
1/720	31/360	3/4	13/6	5/2	1

[x x^2 x^3 x^4 x^5 x^6] * zfSt2FZ = [e1^1 e1^2 e1^3 e1^4 e1^5 e1^6]

where e1 means $e^x - 1$

Double-sum of fSt2, focusing the Bell-numbers

Eq. *(2.4.2)* with the factorial term *c!* rearranged to the rhs

$$
\begin{aligned}
V(x)\sim * fSt_2F &= e^x * V(e^x - 1)\sim \\
V(x)\sim * fSt_2F * F^{-1} &= e^x * V(e^x - 1)\sim * F^{-1} \\
V(x)\sim * fSt_2 &= e^x * V(e^x - 1)\sim * F^{-1}
\end{aligned}
$$

and then again be summed over then columns:

$$
\begin{aligned}
V(x)\sim * fSt_2 * V(1) &= e^x * V(e^x - 1)\sim * F^{-1} * V(1) \\
&= e^x * (1 + (e^x-1)^1/1! + (e^x-1)^2/2! + (e^x-1)^3/3! + ...) \\
&= e^x * exp(e^x - 1)
\end{aligned}
$$

and since the rowsums of **fSt_2** are the Bell-numbers scaled by the reciprocal factorial:

$$fSt_2 * V(1) = F^{-1} * (St_2 *V(1)) = F^{-1} * B$$

we get the result involving the Bell-numbers:

(2.4.4.)
$$\lim \sum_{c=0}^{oo} \sum_{r=0}^{oo} S2_{r,c} \frac{x^r}{r!} = \lim \sum_{r=0}^{oo} \frac{x^r}{r!} \sum_{c=0}^{oo} S2_{r,c} =$$
$$\lim \sum_{r=0}^{oo} x^r \frac{B_r}{r!} = e^x e^{e^x-1} = e^{e^x+(x-1)}$$

(2.4.5.) and
$$\sum_{r=0}^{oo} \frac{B_r}{r!} = e^e$$

Double-sum of zf St2

The double sum over all columns of **$zfSt_2$** is, after rearranging the *(c+1)!* -term to the rhs in *(2.4.4)*:

(2.4.6.)
$$\lim_{r->oo} \sum_{r=0}^{oo} x^r * S2_{r,c} \frac{1}{(r+1)!} = \frac{(e^x-1)^{c+1}}{(c+1)!} \qquad \text{// for a column } c$$

Summing over all columns:

(2.4.7.) *lhs-double-sum in (2.4.5)*

$$\lim \sum_{c=0}^{oo} \sum_{r=0}^{oo} x^r S2_{r,c} \frac{1}{(r+1)!} = \lim \sum_{r=0}^{oo} \frac{x^r}{(r+1)!} \sum_{c=0}^{oo} S2_{r,c}$$
$$= \lim \sum_{r=0}^{oo} x^r \frac{B_r}{(r+1)!}$$

(2.4.8.) *rhs-sum in (2.4.5)*

$$\sum_{c=0}^{oo} \frac{(e^x-1)^{c+1}}{(c+1)!} = exp(e^x-1)-1 = e^{e^x-1}-1$$

This gives

(2.4.9.)
$$V(x)\sim * zfST2 *V(1) = exp(e^x - 1) - 1$$

$$\lim \sum_{c=0}^{oo} \sum_{r=0}^{oo} x^r S2_{r,c} \frac{1}{(r+1)!} =$$
$$\lim \sum_{r=0}^{oo} x^r \frac{B_r}{(r+1)!} = e^{e^x-1}-1$$

and

(2.4.10.) $$\sum_{r=0}^{oo} \frac{B_r}{(r+1)!} = e^{e-1} - 1$$

2.5. Columnsums (and leftmultiplication with powerseries) of St_1 and fSt_1F

If the previous **St_2**-related formulae are varied extended by postmultiplication by **St_1** we get the inverse summation-expressions for the variants of **St_1**:

Define

$zfSt_1FZ = {}^dZ(1) * fSt_1F * {}^dZ(-1)$

Then

(2.5.1.)
$$x\,V(x)\sim * zfSt_1FZ = \log(1+x) * V(\log(1+x))\sim$$
$$\lim_{r->oo} \sum_{r=0}^{oo} x^r * S1_{r,c} \frac{(c+1)!}{(r+1)!} = \log(1+x)^{c+1}$$

transforms a powerseries in x into a powerseries of the logarithm of $(1+x)$.

Example:

$x\,V(x)\sim * zfSt_1FZ = \log(1+x)\,V(\log(1+x))$

zfSt1FZ

$$[\,x \quad x^2 \quad x^3 \quad x^4 \quad x^5 \quad x^6\,] * \begin{bmatrix} 1 & . & . & . & . & . \\ -1/2 & 1 & . & . & . & . \\ 1/3 & -1 & 1 & . & . & . \\ -1/4 & 11/12 & -3/2 & 1 & . & . \\ 1/5 & -5/6 & 7/4 & -2 & 1 & . \\ -1/6 & 137/180 & -15/8 & 17/6 & -5/2 & 1 \end{bmatrix} = [\,l1^1 \quad l1^2 \quad l1^3 \quad l1^4 \quad l1^5 \quad l1^6\,]$$

where l1 means log(1+x)

Proof:

(2.5.2.) $zfSt_1FZ = zfSt_2FZ^{-1}$

since

$${}^dZ(1)* F^{-1} * St_2 * F * {}^dZ(-1) \quad * \quad {}^dZ(1)* F^{-1} * St_1 * F * {}^dZ(-1)$$
$$= {}^dZ(1)* F^{-1} * St_2 \quad * St_1 * F * {}^dZ(-1)$$
$$= {}^dZ(1)* F^{-1} \quad * I * F * {}^dZ(-1)$$
$$= I$$

From this and right-multiplication of $(2.5.3.)$ by **$zfSt_1FZ$**

$$y\,V(y)\sim * zfSt_2FZ = (e^y-1) * V(e^y-1)\sim$$
$$y\,V(y)\sim * zfSt_2FZ * zfSt_1FZ = (e^y-1) * V(e^y-1)\sim \; * zfSt_1FZ$$
(2.5.4.) $$y\,V(y)\sim = (e^y-1) * V(e^y-1)\sim \; * zfSt_1FZ$$

Replacing e^y-1 by x and y by $\log(1+y)$ gives $(2.5.1)$.

---

3. Relations to other matrices / Summations

3.1. St_2 and St_1 as compositions of binomially -weighted sums of zetaseries

Composition of St_2

St_2 occurs in the matrix-multiplication $P^{-1} * ZV$ in its factorial scaled version St_2F

(3.1.1.) $P^{-1} * ZV = St_2F\sim$

$P^{-1} * ZV = St_2F\sim$

ZV

$$* \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 \\ 1 & 3 & 9 & 27 & 81 & 243 \\ 1 & 4 & 16 & 64 & 256 & 1024 \\ 1 & 5 & 25 & 125 & 625 & 3125 \\ 1 & 6 & 36 & 216 & 1296 & 7776 \end{bmatrix}$$

P^-1 St2F~

$$\begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ -1 & 3 & -3 & 1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ -1 & 5 & -10 & 10 & -5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 3 & 7 & 15 & 31 \\ . & . & 2 & 12 & 50 & 180 \\ . & . & . & 6 & 60 & 390 \\ . & . & . & . & 24 & 360 \\ . & . & . & . & . & 120 \end{bmatrix}$$

or, sometimes even given as definition of St_2 in the sense of a generation-function (see [AS-ST]):

(3.1.2.) $St_2\sim = F^{-1} * P^{-1} * ZV$

$St_2\sim = F^{-1} * P^{-1} * ZV$

ZV

$$* \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 \\ 1 & 3 & 9 & 27 & 81 & 243 \\ 1 & 4 & 16 & 64 & 256 & 1024 \\ 1 & 5 & 25 & 125 & 625 & 3125 \\ 1 & 6 & 36 & 216 & 1296 & 7776 \end{bmatrix}$$

! P^-1 St2~

$$diag(\begin{bmatrix} 1 \\ 1 \\ 1/2 \\ 1/6 \\ 1/24 \\ 1/120 \end{bmatrix}) * \begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ -1 & 3 & -3 & 1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ -1 & 5 & -10 & 10 & -5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 3 & 7 & 15 & 31 \\ . & . & 1 & 6 & 25 & 90 \\ . & . & . & 1 & 10 & 65 \\ . & . & . & . & 1 & 15 \\ . & . & . & . & . & 1 \end{bmatrix}$$

Composition of St_1

The converse is also true, at least for the case of any finite dimension. (for infinite dimension the inverse of the Vandermondematrix ZV is not defined)

For any finite size the following is valid:

(3.1.3.) $ZV^{-1} * P = fSt_1\sim$

Example, size = 6

$ZV^{-1} * P = fSt_1\sim$

P

$$* \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix}$$

ZV^-1 fSt1~

$$\begin{bmatrix} 6 & -15 & 20 & -15 & 6 & -1 \\ -87/10 & 117/4 & -127/3 & 33 & -27/2 & 137/60 \\ 29/6 & -461/24 & 31 & -307/12 & 65/6 & -15/8 \\ -31/24 & 137/24 & -121/12 & 107/12 & -95/24 & 17/24 \\ 1/6 & -19/24 & 3/2 & -17/12 & 2/3 & -1/8 \\ -1/120 & 1/24 & -1/12 & 1/12 & -1/24 & 1/120 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ . & 1 & -3/2 & 11/6 & -25/12 & 137/60 \\ . & . & 1/2 & -1 & 35/24 & -15/8 \\ . & . & . & 1/6 & -5/12 & 17/24 \\ . & . & . & . & 1/24 & -1/8 \\ . & . & . & . & . & 1/120 \end{bmatrix}$$

3.2. St_2 and St_1 form an eigensystem of the bernoullian matrix G_p

With the diagonal eigenvaluematrix of reciprocals of natural numbers ${}^{d}\mathbf{Z}(1)$ they form the eigensystem of the bernoullian-matrix $\mathbf{G}_p$. $\mathbf{G}_p$ is called "bernoulian" since it contains the bernoulli-numbers in its first column and is also a simple column-scaled version of the matrix ***BN***, which contains the coefficients of the Bernoulli-polynomials (see chapter Matrix $\mathbf{G}_p$ and $\mathbf{G}_m$ for more detailed discussion of this)

(3.2.1.) $St_2 * {}^{d}Z(1) * St_1 = G_P$

$$
* \begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 2 & -3 & 1 & . & . & . \\ -6 & 11 & -6 & 1 & . & . \\ 24 & -50 & 35 & -10 & 1 & . \\ -120 & 274 & -225 & 85 & -15 & 1 \end{bmatrix}
$$

$$
\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 3 & 1 & . & . & . \\ 1 & 7 & 6 & 1 & . & . \\ 1 & 15 & 25 & 10 & 1 & . \\ 1 & 31 & 90 & 65 & 15 & 1 \end{bmatrix} * diag(\begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \\ 1/5 \\ 1/6 \end{bmatrix}) = \begin{bmatrix} 1 & . & . & . & . & . \\ 1/2 & 1/2 & . & . & . & . \\ 1/6 & 1/2 & 1/3 & . & . & . \\ 0 & 1/4 & 1/2 & 1/4 & . & . \\ -1/30 & 0 & 1/3 & 1/2 & 1/5 & . \\ 0 & -1/12 & 0 & 5/12 & 1/2 & 1/6 \end{bmatrix}
$$

3.3. Some useful consequences arising from the eigensystem-decomposition of G_p

Since we have an eigensystem with a very simple eigenvalue-diagonal-matrix, multiplication of the Stirlingmatrices by $\mathbf{G_p}$ leaves them "nearly invariant" aside of the scaling of rows($\mathbf{St_1}$) and columns ($\mathbf{St_2}$) by the incremented row/columnnumber:

(3.3.1.) ${}^{d}Z(-1) * St_1 * G_P = St_1$

$$
* \begin{bmatrix} 1 & . & . & . & . & . \\ 1/2 & 1/2 & . & . & . & . \\ 1/6 & 1/2 & 1/3 & . & . & . \\ 0 & 1/4 & 1/2 & 1/4 & . & . \\ -1/30 & 0 & 1/3 & 1/2 & 1/5 & . \\ 0 & -1/12 & 0 & 5/12 & 1/2 & 1/6 \end{bmatrix}
$$

$$
diag(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}) * \begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 2 & -3 & 1 & . & . & . \\ -6 & 11 & -6 & 1 & . & . \\ 24 & -50 & 35 & -10 & 1 & . \\ -120 & 274 & -225 & 85 & -15 & 1 \end{bmatrix} = \begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 2 & -3 & 1 & . & . & . \\ -6 & 11 & -6 & 1 & . & . \\ 24 & -50 & 35 & -10 & 1 & . \\ -120 & 274 & -225 & 85 & -15 & 1 \end{bmatrix}
$$

and

(3.3.2.) $G_p * St_2 * {}^{d}Z(-1) = St_2$

$$
* \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 3 & 1 & . & . & . \\ 1 & 7 & 6 & 1 & . & . \\ 1 & 15 & 25 & 10 & 1 & . \\ 1 & 31 & 90 & 65 & 15 & 1 \end{bmatrix} * diag(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix})
$$

$$
\begin{bmatrix} 1 & . & . & . & . & . \\ 1/2 & 1/2 & . & . & . & . \\ 1/6 & 1/2 & 1/3 & . & . & . \\ 0 & 1/4 & 1/2 & 1/4 & . & . \\ -1/30 & 0 & 1/3 & 1/2 & 1/5 & . \\ 0 & -1/12 & 0 & 5/12 & 1/2 & 1/6 \end{bmatrix} = \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 3 & 1 & . & . & . \\ 1 & 7 & 6 & 1 & . & . \\ 1 & 15 & 25 & 10 & 1 & . \\ 1 & 31 & 90 & 65 & 15 & 1 \end{bmatrix}
$$

3.4. Summation of G_p and BN by leftmultiplication with the factorial-vector

From the definition of bernoulli-numbers the identity of the first column of the result is known; the others can be computed using derivatives:

(3.4.1.) $${}^r(F^{-1}) * G_p * {}^dZ(-1) = {}^r(F^{-1}) * e/(e-1)$$
$${}^r(F^{-1}) * BN = {}^r(F^{-1}) * e/(e-1)$$

$$\lim_{r\to oo} \begin{bmatrix} 1 & 1 & 1/2 & 1/6 & 1/24 & 1/120 \end{bmatrix} * \begin{bmatrix} 1 & . & . & . & . & . \\ 1/2 & 1/2 & . & . & . & . \\ 1/6 & 1/2 & 1/3 & . & . & . \\ 0 & 1/4 & 1/2 & 1/4 & . & . \\ -1/30 & 0 & 1/3 & 1/2 & 1/5 & . \\ 0 & -1/12 & 0 & 5/12 & 1/2 & 1/6 \end{bmatrix} * diag\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1/2 & 1/6 & 1/24 & 1/120 \end{bmatrix} * \begin{bmatrix} e/(e-1) \end{bmatrix}$$

Expressed in terms of ***BN***, the matrix of coefficients of the Bernoulli-polynomials this is

(3.4.2.) $${}^r(F^{-1}) * BN = {}^r(F^{-1}) * e/(e-1)$$

$$\lim_{r\to oo} \begin{bmatrix} 1 & 1 & 1/2 & 1/6 & 1/24 & 1/120 \end{bmatrix} * \begin{bmatrix} 1 & . & . & . & . & . \\ 1/2 & 1 & . & . & . & . \\ 1/6 & 1 & 1 & . & . & . \\ 0 & 1/2 & 3/2 & 1 & . & . \\ -1/30 & 0 & 1 & 2 & 1 & . \\ 0 & -1/6 & 0 & 5/3 & 5/2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1/2 & 1/6 & 1/24 & 1/120 \end{bmatrix} * \begin{bmatrix} e/(e-1) \end{bmatrix}$$

Rewriting the factorial scaling as similarity-scaled matrices:

(3.4.3.) $$V(1)\sim * (f\, G_p\, Z\, F) = e/(e-1) * V(1)\sim$$
$$V(1)\sim * (f\, BN\, F) = e/(e-1) * V(1)\sim$$

Generally:

(3.4.4.) $$V(x)\sim * (f\, G_p\, Z\, F) = x\, V(x)\sim *\; e^x/(e^x - 1)$$
$$V(x)\sim * (f\, BN\, F) = x\, V(x)\sim *\; e^x/(e^x - 1)$$

The reciprocal expression can also be written:

(3.4.5.) $$e/(e-1) * {}^r(F^{-1}) * BN^{-1} = {}^r(F^{-1})$$

$$\lim_{r\to oo} \begin{bmatrix} e/(e-1) \end{bmatrix} \begin{bmatrix} 1 & 1 & 1/2 & 1/6 & 1/24 & 1/120 \end{bmatrix} * \begin{bmatrix} 1 & . & . & . & . & . \\ -1/2 & 1 & . & . & . & . \\ 1/3 & -1 & 1 & . & . & . \\ -1/4 & 1 & -3/2 & 1 & . & . \\ 1/5 & -1 & 2 & -2 & 1 & . \\ -1/6 & 1 & -5/2 & 10/3 & -5/2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1/2 & 1/6 & 1/24 & 1/120 \end{bmatrix}$$

Generally:

(3.4.6.) $$x\, V(x)\sim * (f\, BN^{-1}\, F) = (e^x - 1)/e^x * V(x)\sim$$

and for *x=1* the special simple identity involving the factorial scaled ***BN***-similar matrix ***fBN F*** occurs:

(3.4.7.) *e/(e-1)* V(1)~ * f BN F = V(1)~*

$$* \begin{bmatrix} 1 & . & . & . & . & . \\ 1/2 & 1 & . & . & . & . \\ 1/12 & 1/2 & 1 & . & . & . \\ 0 & 1/12 & 1/2 & 1 & . & . \\ -1/720 & 0 & 1/12 & 1/2 & 1 & . \\ 0 & -1/720 & 0 & 1/12 & 1/2 & 1 \end{bmatrix}$$

$$\lim_{r\to oo} \begin{bmatrix} e/(e-1) \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

where the columns in ***f BN F*** are simple shifts of the first column.

4. Details and some discussions

5. Citations

[Adamchik] *http://www.cs.cmu.edu/~adamchik/articles/stirling.pdf* , Pg 8

In this example, Adamchik uses the unsigned version of Stirling-numbers 1'st kind

Let us begin with the simple example

$$\sum_{k=1}^{\infty} \begin{bmatrix} k \\ 2 \end{bmatrix} \frac{1}{k!\,k}$$

Using the integral representation (15) and changing the order of summation and integration, we get

$$\sum_{k=1}^{\infty} \begin{bmatrix} k \\ 2 \end{bmatrix} \frac{1}{k!\,k} = \int_0^1 \frac{\pi^2 t - 6\,\mathrm{Li}_2(t)}{6\,t\,(1-t)}\,dt = \zeta(3)$$

From this identity one would expect the pattern to remain unchanged and so that:

$$\mathrm{G}_{p,1} = \sum_{k=1}^{\infty} \begin{bmatrix} k \\ p \end{bmatrix} \frac{1}{k!\,k} = \zeta(p+1) \qquad (19)$$

In my matrix-notation it means:

$$V(1)\sim * {}^dZ(1)\sim * {}^dJ * fSt1 = [\ \zeta(1),\ \zeta(2),\ \zeta(3),\ldots]$$

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extensively studied in [7]. It was shown there, for example, that

$$\sum_{k=1}^{\infty} \begin{bmatrix} k \\ p \end{bmatrix} \frac{z^k}{k!\,k} = \zeta(p+1) + \sum_{k=0}^{p} \frac{(-1)^{k-1}}{k!} \mathrm{Li}_{p+1-k}(1-z)\log^k(1-z) \qquad (22)$$

$$V(z)\sim * {}^dZ(1)\sim * {}^dJ * fSt1 = [\ \zeta(1)+f(0,z),\ \zeta(2)+f(1,z),\ \zeta(3)+f(2,z),\ldots]$$

where f(c,z) denotes the rhs-sum in (22)

6. References

[Project-Index] http://go.helms-net.de/math/binomial/index

[Intro] http://go.helms-net.de/math/binomial/intro.pdf

[binomialmatrix] http://go.helms-net.de/math/binomial/01_1_binomialmatrix.pdf

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[Wiki-St1] http://en.wikipedia.org/wiki/Stirling_numbers_of_the_first_kind

[Wiki-St2] http://en.wikipedia.org/wiki/Stirling_numbers_of_the_second_kind

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On Stirling numbers and Euler sums (1996)
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Gottfried Helms