Identities involving binomial-coefficients,
Bernoulli- and Stirlingnumbers

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The Vandermonde-Matrix

An approach to define a reciprocal

Abstract: The Vandermonde-matrix plays a special rôle in this collection of articles. It is the only matrix, which is not triangular, and care has to be taken when using it in matrix-formulae assuming infinite matrix-dimension. For instance, a reciprocal is not easily defineable (if this would be meaningful at all). On the other hand many binomial-formulae are targetting this matrix, say in applaying the binomial-theorem to powerseries or computing sums of like powers by bernoulli-polynomials.

Here I present an approach to define a reciprocal of the Vandermonde-matrix as an asymptotic of a powerseries-construction. The vandermonde-matrix itself as well as its reciprocal in this approach are seen as a limit for x->1 in this powerseries.

The limit-problem is then similar to find/define a limit for the powerseries $1+x+x^2+x^3+\dots = 1/(1-x)$ for x->1. While a definite answer for this limit-case ***could not yet*** *be given, a step forward could be made, since the estimate is now* ***size-independent*** *and dependent on the closeness of the used limit x->1 instead .*

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Project-Index	*http://go.helms-net.de/math/binomial/index*
Intro/notation	*http://go.helms-net.de/math/binomial/intro.pdf*

1. Definition and basic properties of the Vandermondematrix

1.1. Definition

The Vandermonde-matrix ***ZV*** is defined as collection of powers of natural numbers:

(1.1.1.) $ZV := ZV_{r,c} = (r+1)^c$ *//assuming zero-based row-and column-indices r and c*

Example:

$$ZV = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 \\ 1 & 3 & 9 & 27 & 81 & 243 \\ 1 & 4 & 16 & 64 & 256 & 1024 \\ 1 & 5 & 25 & 125 & 625 & 3125 \\ 1 & 6 & 36 & 216 & 1296 & 7776 \end{bmatrix}$$

It can also been written either as collection of powerseries-vectors

(1.1.2.) $ZV = [\ V(1)\sim, V(2)\sim, V(3)\sim, \ldots\ ]$

or as collection of zeta-vectors

(1.1.3.) $ZV = [Z(0), Z(-1), Z(-2), \ldots]$

1.2. A couple of basic properties

Neither the row- nor the columnsums are convergent.

*** Column-sums**

By interpreting the columns as zeta-series one may assign the appropriate zeta-values to its column-sums. Define the powerseries-vector

$$V(x) = [1, x, x^2, x^3, \ldots]\sim$$
$$E = V(1)$$

then it seems, that we could use:

(1.2.1.) $V(1)\sim * ZV = ZT_0\sim = [\zeta(0), \zeta(-1), \zeta(-2), \ldots]$

Example:

$V(1)\sim * ZV = [\zeta(0), \zeta(-1), \zeta(-2), \ldots]$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 \\ 1 & 3 & 9 & 27 & 81 & 243 \\ 1 & 4 & 16 & 64 & 256 & 1024 \\ 1 & 5 & 25 & 125 & 625 & 3125 \\ 1 & 6 & 36 & 216 & 1296 & 7776 \end{bmatrix} = \begin{bmatrix} Z(0) & Z(-1) & Z(-2) & Z(-3) & Z(-4) & Z(-5) \end{bmatrix}$$

but this guess seems to introduce subtle inconsistencies. See the note below.

*** Row-sums**

For the row-sums one may assign values according to the analytic continuation for powerseries V(x), where $x <= -1$ and $x >= 1$, according to the formula $s = 1/(1-x)$ where the row-sums were then

(1.2.2.) $ZV * V(1) = S = [??, -1/1, -1/2, \ldots]\sim$ *// with S_0 undefined.*

Example:

$ZV * V(1) = [??, -1/1, -1/2,...]$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 \\ 1 & 3 & 9 & 27 & 81 & 243 \\ 1 & 4 & 16 & 64 & 256 & 1024 \\ 1 & 5 & 25 & 125 & 625 & 3125 \\ 1 & 6 & 36 & 216 & 1296 & 7776 \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} s0=? \\ -1 \\ -1/2 \\ -1/3 \\ -1/4 \\ -1/5 \end{bmatrix}$$

(Extended rowsums occur in a paragraph below in connection with rightmultiplication by the ***P***-matrix).

Note: The assumptions, especially in *(1.2.1)*, must be confirmed to be meaningful in this matrix-context. The assumption of ***ZV***-columns representing zeta-series (and assignment of zeta-values to its column-sums) seems to lead to subtle inconsistencies. These already occur, if only the sum of column-sums are equalled to the sum of rowsums. These sums should agree, if this guess should be consistent in the present matrix-context.

So it is possibly a better definition of ***ZV*** as a composition of the binomial- and Stirlingmatrix instead as a collection of zeta-vectors $Z(-x) = [1^x, 2^x, 3^x , ...]$. This will be shown in the next paragraph.

1.3. a couple of basic matrix-relations

* leftmultiplication by matrix P^{-1} gives $St_2F\sim$

Best known is possibly the property, that the forward-differences of like powers disappear in a binomial-transformation.

Let ${}^dF$ be the diagonal-matrix of factorials *diag([0!,1!,2!,...])*, St_2F the factorial-scaled version of St_2, the lower-triangular matrix of Stirling-Numbers 2'nd kind then

define $St_2F := St_2 * {}^dF$

(1.3.1.) $P^{-1} * ZV = St_2F \sim$

Example:

$P^{-1} * ZV = St_2F \sim$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ -1 & 3 & -3 & 1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ -1 & 5 & -10 & 10 & -5 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 \\ 1 & 3 & 9 & 27 & 81 & 243 \\ 1 & 4 & 16 & 64 & 256 & 1024 \\ 1 & 5 & 25 & 125 & 625 & 3125 \\ 1 & 6 & 36 & 216 & 1296 & 7776 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 3 & 7 & 15 & 31 \\ . & . & 2 & 12 & 50 & 180 \\ . & . & . & 6 & 60 & 390 \\ . & . & . & . & 24 & 360 \\ . & . & . & . & . & 120 \end{bmatrix}$$

This formula can be rewritten to give a definition of ***ZV***:

(1.3.2.) $ZV = P * St_2F \sim$

Example:

$$ZV = P * St_2F \sim$$

$$* \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 3 & 7 & 15 & 31 \\ . & . & 2 & 12 & 50 & 180 \\ . & . & . & 6 & 60 & 390 \\ . & . & . & . & 24 & 360 \\ . & . & . & . & . & 120 \end{bmatrix}$$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 \\ 1 & 3 & 9 & 27 & 81 & 243 \\ 1 & 4 & 16 & 64 & 256 & 1024 \\ 1 & 5 & 25 & 125 & 625 & 3125 \\ 1 & 6 & 36 & 216 & 1296 & 7776 \end{bmatrix}$$

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* rightmultiplication by matrix fSt_1 gives a logarithmic Vandermonde-variant

define: $fSt_1F := {}^dF^{-1} * St_1 * {}^dF = fSt_2F^{-1}$
${}^dZ_1(-x) = diag((1+1)^x, (1+2)^x, (1+3)^x, \dots))$

we have also:

(1.3.3.) $ZV * fSt_1F = {}^dZ_1(1) * LZV$

Example:

$$ZV * fSt_1 \sim = {}^dZ_1(1) * LZV$$

$$* \begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 1 & -3/2 & 1 & . & . & . \\ -1 & 11/6 & -2 & 1 & . & . \\ 1 & -25/12 & 35/12 & -5/2 & 1 & . \\ -1 & 137/60 & -15/4 & 17/4 & -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 \\ 1 & 3 & 9 & 27 & 81 & 243 \\ 1 & 4 & 16 & 64 & 256 & 1024 \\ 1 & 5 & 25 & 125 & 625 & 3125 \\ 1 & 6 & 36 & 216 & 1296 & 7776 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/4 \\ 1/5 \\ 1/6 \\ 1/7 \end{bmatrix} \begin{bmatrix} l2^\wedge 0 & l2^\wedge 1 & l2^\wedge 2 & l2^\wedge 3 & l2^\wedge 4 & l2^\wedge 5 \\ l3^\wedge 0 & l3^\wedge 1 & l3^\wedge 2 & l3^\wedge 3 & l3^\wedge 4 & l3^\wedge 5 \\ l4^\wedge 0 & l4^\wedge 1 & l4^\wedge 2 & l4^\wedge 3 & l4^\wedge 4 & l4^\wedge 5 \\ l5^\wedge 0 & l5^\wedge 1 & l5^\wedge 2 & l5^\wedge 3 & l5^\wedge 4 & l5^\wedge 5 \\ l6^\wedge 0 & l6^\wedge 1 & l6^\wedge 2 & l6^\wedge 3 & l6^\wedge 4 & l6^\wedge 5 \\ l7^\wedge 0 & l7^\wedge 1 & l7^\wedge 2 & l7^\wedge 3 & l7^\wedge 4 & l7^\wedge 5 \end{bmatrix}$$

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* rightmultiplication by Matrix P ($= P_k(1)$)

Rightmultiplication of ***ZV*** with ***P*** (or a power of ***P***) means to assume divergent summation of powerseries $V(x)\sim$ with $x>=1$ by binomials. With elementary derivation for the convergent cases $|x|<1$ can be found, that the result of a binomial-summation of powerseries is for a column c:

(1.3.4.) $$\lim \frac{1}{x} \sum_{r=0}^{oo} \binom{r}{c} * \frac{1}{x^r} = \frac{1}{x-1} * \frac{1}{(x-1)^c}$$ *for a given column c*

and in matrix-notation for each row $V(r+1)\sim$ of ***ZV***, targetting all columns of ***P*** in a whole :

(1.3.5.) $$\lim_{rows \to oo} \frac{1}{x} \mathbf{V}\left(\frac{1}{x}\right) \sim * \mathbf{P} = \frac{1}{x-1} \mathbf{V}\left(\frac{1}{x-1}\right) \sim$$

(see project-article [binomialmatrix])

Example:

*lim 1/x*V(1/x)~ * P = 1/(x-1)*V(1/(x-1)) ~*

P:

1	.	.	.	.	.
1	1	.	.	.	.
1	2	1	.	.	.
1	3	3	1	.	.
1	4	6	4	1	.
1	5	10	10	5	1

1/2	1/4	1/8	1/16	1/32	1/64
1/3	1/9	1/27	1/81	1/243	1/729
1/4	1/16	1/64	1/256	1/1024	1/4096
1/5	1/25	1/125	1/625	1/3125	1/15625
1/6	1/36	1/216	1/1296	1/7776	1/46656
1/7	1/49	1/343	1/2401	1/16807	1/117649

1	1	1	1	1	1
1/2	1/4	1/8	1/16	1/32	1/64
1/3	1/9	1/27	1/81	1/243	1/729
1/4	1/16	1/64	1/256	1/1024	1/4096
1/5	1/25	1/125	1/625	1/3125	1/15625
1/6	1/36	1/216	1/1296	1/7776	1/46656

The known analytic continuation of this powerseries summation extends the domain of *x* to complex values, allowing the general formula for any complex *s* except *s=1*:

(1.3.6.) $$\lim_{rows \to oo} \frac{1}{s} V\left(\frac{1}{s}\right) \sim * P = \frac{1}{s-1} V\left(\frac{1}{s-1}\right) \sim$$ *// for all complex s except s=1*

This allows to apply this relation to the matrix-product ***ZV*** * ***P***, where the first row must remain undefined yet for each row *r* setting *t=r+1*:

(1.3.7.) $$\lim_{rows \to oo} V(t) \sim * P = -\frac{1}{t} * \frac{t}{t-1} V\left(-\frac{t}{t-1}\right) \sim$$ *// for all complex t except t=1*

Example: (for brevity extracting the row-scaling in the result, using "☼" for Hadamard-multiplication)

*ZV * P = diag(Q) * M*

P:

1	.	.	.	.	.
1	1	.	.	.	.
1	2	1	.	.	.
1	3	3	1	.	.
1	4	6	4	1	.
1	5	10	10	5	1

ZV: *

1	1	1	1	1	1
1	2	4	8	16	32
1	3	9	27	81	243
1	4	16	64	256	1024
1	5	25	125	625	3125
1	6	36	216	1296	7776

= Q:

??
-1
-1/2
-1/3
-1/4
-1/5

☼ M:

??	??	??	??	??	??
1	-2	4	-8	16	-32
1	-3/2	9/4	-27/8	81/16	-243/32
1	-4/3	16/9	-64/27	256/81	-1024/243
1	-5/4	25/16	-125/64	625/256	-3125/1024
1	-6/5	36/25	-216/125	1296/625	-7776/3125

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* using matrix P_0 (= $P_k(0)$)

In the chapter about the binomial-matrix I introduced two hierarchies of Pascal-like matrices. The one variant, which is of interest here is P_0.

The Pascalmatrix ***P*** (= P_1) can be seen as matrixexponential of a matrix containing the natural numbers in the first subdiagonal. Viewing this as a version using exponent *1* for these entries (thus "$P_k(1)$" or short "P_1"), the "lower order" version with exponent *0* (thus all entries in the first subdiagonal are *1*) is called $P_k(0)$ or simply P_0. (Remember that the version with exponent *2* is known as scaled Laguerre-matrix).

The matrix P_0 can be defined in two ways:

(1.3.8.) *define* $P_0 = P_k(0) := P_{0\ r,c} = 1/(r-c)!$ *// = 0 if c>r*

(1.3.9.) *or define* $P_0 = {}^dF^{-1} * P * {}^dF$

Example

$P0 = {}^dF^{-1} * P * {}^dF$

P0

1	.	.	.	.	.
1	1	.	.	.	.
1/2	1	1	.	.	.
1/6	1/2	1	1	.	.
1/24	1/6	1/2	1	1	.
1/120	1/24	1/6	1/2	1	1

* Rightmultiplication

Rightmultiplication of ***ZV*** with a column of P_0 transforms the geometric series of each row in ***ZV*** into the related exponential-series, scaled by the beginning entry in the row of ***ZV,*** so by

(1.3.10.) $ZV * P_0 = e*{}^dV(e) * ZV$

we have a nearly invariant transformation, which only performs rowscaling of ***ZV***. (This also means, we have formally an eigensystem and which shall shortly be discussed later in the context of the reciprocal of ***ZV***).

Example:(using "☼" for Hadamard-multiplication)

P0

1	.	.	.	.	.
1	1	.	.	.	.
1/2	1	1	.	.	.
1/6	1/2	1	1	.	.
1/24	1/6	1/2	1	1	.
1/120	1/24	1/6	1/2	1	1

ZV

1	1	1	1	1	1
1	2	4	8	16	32
1	3	9	27	81	243
1	4	16	64	256	1024
1	5	25	125	625	3125
1	6	36	216	1296	7776

=

e
e^2
e^3
e^4
e^5
e^6

☼ ZV

1	1	1	1	1	1
1	2	4	8	16	32
1	3	9	27	81	243
1	4	16	64	256	1024
1	5	25	125	625	3125
1	6	36	216	1296	7776

* Leftmultiplication

Leftmultiplication of ***ZV*** with P_0 seems tobe of less interest. As a notable property for the chapter on divergent summation however it may be observed, that we have an asymptotic for the *r*'th row of the result as *r-> inf* as

$P_0 * ZV = M$

(1.3.11.) $\lim_{r \to oo} M_r = e * V(r)$

or $\lim_{r \to oo} m_{r,c} = e * r^c$

Example:(using "☼" for Hadamard-multiplication)

$P_0 * ZV = M$

ZV

1	1	1	1	1	1
1	2	4	8	16	32
1	3	9	27	81	243
1	4	16	64	256	1024
1	5	25	125	625	3125
1	6	36	216	1296	7776

P0 *

1	.	.	.	.	.
1	1	.	.	.	.
1/2	1	1	.	.	.
1/6	1/2	1	1	.	.
1/24	1/6	1/2	1	1	.
1/120	1/24	1/6	1/2	1	1

=

?	?	?	?	?	?
?	?	?	?	?	?
?	?	?	?	?	?
?	?	?	?	?	?
?	?	?	?	?	?
?	?	?	?	?	?

asymptotic for row r: = [e] ☼ [1 r r^2 r^3 r^4 r^5]

The operation, which is asymptotically performed is then a one-row shift of the *r+1*'th row in ***ZV*** to the *r*'th row in the result, and it may be of interest, that the expansion of the asymptotic behaviour is for a fixed column *c* of ***ZV*** for *r->inf* :

(1.3.12.)
$$\lim_{r \to inf} \sum_{k=0}^{r} \frac{(k+1)^c}{(r-k)!} = exp(1) * r^c$$

1.4. The problem of inconsistency assuming ZV as matrix of zeta-series

If the above matrixoperations are assumed as consistent, then - being linear transformations- the same operations on the column-sums of ***ZV*** instead of its elements should be valid.

Assume for instance, ***ZV*** column-summed by a leftmultiplication with a powerseries vector ***V***(x), where *x* is chosen to get summation convergent; for instance, we may choose $x = ½$. Then we had

(1.4.1.) $½ V(½)\sim * ZV \quad * fSt_1\sim \quad = ½ V(½)\sim * P \;= V(1)\sim$

where the first partial product, using associativity, is

(1.4.2.) $½ V(½)\sim * ZV \qquad = T\sim \quad = [1,2,6,26,150,...]$

and inserting this in *(1.4.1)*

(1.4.3.) $T \qquad * fSt_1\sim \quad = ½* V(½)\sim \; * P$
(1.4.4.) $V(1)\sim \qquad = \qquad V(1)\sim$

Example: *evaluation of lhs*

$½*V(½)\sim * \; ZV * fSt_1\sim = T\sim * \; fSt_1\sim = V(1)\sim$

$$* \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ . & 1 & -3/2 & 11/6 & -25/12 & 137/60 \\ . & . & 1/2 & -1 & 35/24 & -15/8 \\ . & . & . & 1/6 & -5/12 & 17/24 \\ . & . & . & . & 1/24 & -1/8 \\ . & . & . & . & . & 1/120 \end{bmatrix}$$

fSt1~

$$* \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 \\ 1 & 3 & 9 & 27 & 81 & 243 \\ 1 & 4 & 16 & 64 & 256 & 1024 \\ 1 & 5 & 25 & 125 & 625 & 3125 \\ 1 & 6 & 36 & 216 & 1296 & 7776 \end{bmatrix}$$

ZV

$$\begin{bmatrix} 1/2 & 1/4 & 1/8 & 1/16 & 1/32 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 6 & 26 & 150 & 1082 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

evaluation of rhs

$½*V(½)\sim * P = V(1)\sim$

$$* \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 1/4 & 1/8 & 1/16 & 1/32 & 1/64 \end{bmatrix} \qquad = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

For the convergent case , $|x|<1$, this relation holds generally, and, for instance,

define $y = 1/x$
(1.4.5.) $1/y\; V(1/y)\sim * P = 1/(y-1)\; V(1/(y-1))$

For the divergent case, when $x<=-1$, values can be assigned via Euler-summation, and they agree with the formula *(1.4.5)*.

(1.4.6.) $1/y\; V(1/y)\sim * P = 1/(y-1)\; V(1/(y-1)) \qquad // \text{for } 1/y \;< 1$

So for the convergent or the oscillating divergent case the operation seems regular insofar as we could proceed in *(1.4.3)* using the associativity and confirm that the Stirling-transform of the sums over columns of ***ZV*** agrees with the expanded operation on elements of each column, such that in

(1.4.7.) $½ V(½)\sim * ZV \quad * fSt_1\sim \quad = 1* V(1)\sim$
(1.4.8.) $½ V(½)\sim * \qquad P \qquad = 1* V(1)\sim$
(1.4.9.) $T\sim \qquad * fSt_1\sim \quad = 1* V(1)\sim$

we could use each way of associativity according either to *(1.4.8)* or to *(1.4.9)*.

Example:

$T\sim * \ fSt_1\sim = V(1)\sim$

fSt1~

1	-1	1	-1	1	-1
.	1	-3/2	11/6	-25/12	137/60
.	.	1/2	-1	35/24	-15/8
.	.	.	1/6	-5/12	17/24
.	.	.	.	1/24	-1/8
.	.	.	.	.	1/120

$$* \quad [\,1 \quad 2 \quad 6 \quad 26 \quad 150 \quad 1082\,] = [\,1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1\,]$$

But what if $x\to 1$? If we would set the guessed zeta-values as sums of the columns of ***ZV***:

assumed $V(1)\sim * ZV = T\sim \ = [\zeta(0),\zeta(-1),\zeta(-2),...] = [-1/2, \ -1/12, \ 0, \ \ 1/120, \ ...]$

into ***T***, then the summation by $\mathbf{fSt_1}\sim$ would lead to curious results:

define $R\sim \quad := T\sim * fSt_1\sim = [-1/2, 5/12, -3/8, ...]$

and it seems absurd, that this would agree with any thinkable natural setting for the evaluation in the other way of associativity

$$V(1)\sim \quad * P \quad = [\ ??, ??, ??\ , ??, ...]$$

which for all cases except the undefined case $x=1$ is known to contain ascending powers of a constant parameter in the result.

2. An approach to define the reciprocal

2.1. Approximation by finite dimensioned submatrices

A reciprocal of **ZV** is - to say the least- very difficult to define.

To mention the most obvious problem, the entries of the inverses of finite submatrices are not constant with the dimension of the submatrices, and also diverge with the size of the submatrix:

Example: inverses of small dimensions:

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \quad \begin{bmatrix} 3 & -3 & 1 \\ -5/2 & 4 & -3/2 \\ 1/2 & -1 & 1/2 \end{bmatrix} \quad \begin{bmatrix} 4 & -6 & 4 & -1 \\ -13/3 & 19/2 & -7 & 11/6 \\ 3/2 & -4 & 7/2 & -1 \\ -1/6 & 1/2 & -1/2 & 1/6 \end{bmatrix} \quad \begin{bmatrix} 5 & -10 & 10 & -5 & 1 \\ -77/12 & 107/6 & -39/2 & 61/6 & -25/12 \\ 71/24 & -59/6 & 49/4 & -41/6 & 35/24 \\ -7/12 & 13/6 & -3 & 11/6 & -5/12 \\ 1/24 & -1/6 & 1/4 & -1/6 & 1/24 \end{bmatrix}$$

The structure of the first row $r=0$ is obvious; the last row of a dimension n:

define $Q(n) := ZV_{[0..n,0..n]}^{-1}$

then for the rows r=0..n

(2.1.1.) $Q(n)_{0,c} = (-1)^c \text{ binomial}(n+1,c+1)$

... $= ???$

(2.1.2.) $Q(n)_{n,c} = (-1)^{n+c} \text{ binomial}(n,c)/n!$

The last column contains the coefficients of the Stirling-numbers first kind, scaled by the reciprocal of the factorial:

(2.1.3.) $Q(n)_{r,n} = fSt_1\sim_{r,n}$

Example: Here the first versions of **Q**(n) * n! *to compare the last column with* $\mathbf{St}_1\sim$ *:*

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \quad \begin{bmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix} \quad \begin{bmatrix} 24 & -36 & 24 & -6 \\ -26 & 57 & -42 & 11 \\ 9 & -24 & 21 & -6 \\ -1 & 3 & -3 & 1 \end{bmatrix} \quad \begin{bmatrix} 120 & -240 & 240 & -120 & 24 \\ -154 & 428 & -468 & 244 & -50 \\ 71 & -236 & 294 & -164 & 35 \\ -14 & 52 & -72 & 44 & -10 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix}$$

Interestingly the eigenvalues of that submatrices have the amazing property being summable to integer values in all multiplicative combinations.

define λ_k = *eigenvalue k in dimension n*

then

$\Sigma \lambda_k = \Sigma \lambda_k \lambda_j = \Sigma \lambda_k \lambda_j \lambda_i = ... = \lambda_0 * \lambda_1 * ... * \lambda_n = 0 \pmod 1$ *// is integer*

with the number-triangle me (matrix of eigenvalue-compositions):

$$ME = \begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 1 & -3 & 1 & . & . & . \\ -2 & 15 & -12 & 1 & . & . \\ 12 & -206 & 318 & -76 & 1 & . \\ -288 & 10644 & -29654 & 13712 & -701 & 1 \end{bmatrix}$$

where the coefficients of a row r occur from the above indicated combinations of eigenvalues of dimension r , if the product $\Pi(x - \lambda_k)$ in x of all eigenvalues l_k is expanded then

$\Pi(x - \lambda_k) = ME_r * V(x)$ *// for a certain row r, according to the dimension n*

and also the sum of like powers are integral:

$\Sigma l_k^j = 0 \pmod 1$ *// is integer for all 0<=j <=n*

The iterative approach of approximation by increasing the dimension obviously indicates great difficulties, and it suggests a matrix, which can only be described by divergent summation of each entry, if at all a meaning can be assigned this way.

So another approach is needed.

2.2. LU-decomposition of ZV into P and St2F

ZV can -according to formula *(1.3.2)* - be LU-decomposed into the two components ***P*** and the factorial scaled Stirling-matrix of 2'nd kind $\mathbf{St_2F}$.

Recalling formula *(1.3.2)*:

(2.2.1.) $ZV = P * St_2F \sim$ *// recalling (1.3.2)*

Example:

$ZV = P * St_2F \sim$

$$
* \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 3 & 7 & 15 & 31 \\ . & . & 2 & 12 & 50 & 180 \\ . & . & . & 6 & 60 & 390 \\ . & . & . & . & 24 & 360 \\ . & . & . & . & . & 120 \end{bmatrix}
$$

$$
\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 \\ 1 & 3 & 9 & 27 & 81 & 243 \\ 1 & 4 & 16 & 64 & 256 & 1024 \\ 1 & 5 & 25 & 125 & 625 & 3125 \\ 1 & 6 & 36 & 216 & 1296 & 7776 \end{bmatrix}
$$

2.3. Reciprocal by inverses of the LU-components

Now, since for both LU-components ***P*** and $\mathbf{St_2F}$ a reciprocal can be found, the reciprocal of ***ZV*** (call it ***W)*** can at least formally be written. Recall $\mathbf{St_1}$ the lower triangular matrix of 1'st kind (which is the inverse/reciprocal of $\mathbf{St_2}$), and $\mathbf{fSt_1}$ its row-scaled version $^d\mathbf{F}^{-1} * \mathbf{St_1}$:

define $fSt_1 = {}^dF^{-1} * St_1 = St_2F^{-1}$

then (formally)

(2.3.1.) $W := ZV^{-1} = fSt_1 \sim * P^{-1}$

* Finite submatrices

This approach is also valid for finite submatrices.

Examples with finite sizes: (the matrix function ve(matrix,size) extracts the top-left submatrix of size)

$ve(fSt1,2)\sim*ve(P^{-1},2)$ $ve(fSt1,3)\sim*ve(P^{-1},3)$ $ve(fSt1,4)\sim*ve(P^{-1},4)$ $ve(fSt1,5)\sim*ve(P^{-1},5)$

$$
\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \quad \begin{bmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix} \quad \begin{bmatrix} 24 & -36 & 24 & -6 \\ -26 & 57 & -42 & 11 \\ 9 & -24 & 21 & -6 \\ -1 & 3 & -3 & 1 \end{bmatrix} \quad \begin{bmatrix} 120 & -240 & 240 & -120 & 24 \\ -154 & 428 & -468 & 244 & -50 \\ 71 & -236 & 294 & -164 & 35 \\ -14 & 52 & -72 & 44 & -10 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix}
$$

*** asymptotics with infinite size**

Example

$$W = ZV^{-1} = fSt_1\sim * P^{-1}$$

PInv

$$* \begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ -1 & 3 & -3 & 1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ -1 & 5 & -10 & 10 & -5 & 1 \end{bmatrix}$$

ZV^

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ . & 1 & -3/2 & 11/6 & -25/12 & 137/60 \\ . & . & 1/2 & -1 & 35/24 & -15/8 \\ . & . & . & 1/6 & -5/12 & 17/24 \\ . & . & . & . & 1/24 & -1/8 \\ . & . & . & . & . & 1/120 \end{bmatrix} = \begin{bmatrix} ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \end{bmatrix}$$

where all entries are sums of divergent series.

Unfortunately they are also of difficult type, since they involve only like signs. So simple Euler-summation or Borel-summation would not be applicable, because their range of convergence for powerseries in x is $x<1$ in the complex plane.

The first column of the result may -again- be guessed as *zeta(0)*, but it is obvious from the previous chapter, that such a guess must be related to the current context.

So to proceed a formal description of the entries should be found, which -for instance- is expressed in terms of powerseries of a variable x, where the result can then be assumed as the limit when x->1.

2.4. first definition of the entries of $W = ZV^{-1}$

Let ***W*** denote the sought reciprocal of ***ZV*** and wrc its row/column-indexed entries, r/c zero-based.

define $W := w_{r,c}$

Example:

(2.4.1.) *definition of W and its entries* $w_{r,c}$

P^-1

$$* \begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ -1 & 3 & -3 & 1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ -1 & 5 & -10 & 10 & -5 & 1 \end{bmatrix}$$

fSt1~

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ . & 1 & -3/2 & 11/6 & -25/12 & 137/60 \\ . & . & 1/2 & -1 & 35/24 & -15/8 \\ . & . & . & 1/6 & -5/12 & 17/24 \\ . & . & . & . & 1/24 & -1/8 \\ . & . & . & . & . & 1/120 \end{bmatrix} = \begin{bmatrix} w0.0 & w0.1 & w0.2 & w0.3 & w0.4 & w0.5 \\ w1.0 & w1.1 & w1.2 & w1.3 & w1.4 & w1.5 \\ w2.0 & w2.1 & w2.2 & w2.3 & w2.4 & w2.5 \\ w3.0 & w3.1 & w3.2 & w3.3 & w3.4 & w3.5 \\ w4.0 & w4.1 & w4.2 & w4.3 & w4.4 & w4.5 \\ w5.0 & w5.1 & w5.2 & w5.3 & w5.4 & w5.5 \end{bmatrix}$$

Except of one hint, namely an analytical expression for the rowsums rows in $fSt_1\sim$ agreeing to the first column of the result, nothing is known about the entries $w_{r,c}$.

This direct approach has thus to be extended, and a representation for ***W*** as a limit involving a powerseries-expression is sought. Obviously this implies also an equivalent powerseries-variant for ***ZV*** itself.

2.5. Expressing ZV and its reciprocal W as limits of a set of powerseries

To find any meaningful finite formal description for the entries of ***W*** by evaluation of the values and to be able to possibly recognize functional rules, a variable powerseries-parameter $\mathbf{V}(x) = [1, x, x^2, x^3, ...]$ is introduced in the construction of ***ZV*** as well as in its reciprocal, thus defining $\mathbf{ZV}(x)$ and $\mathbf{W}(1/x)$ as parameter-dependent matrices in the following way:

definition $ZV(x) := P * (x\ {}^dV(x)) * St_2F\sim$

then $\lim_{x\to 1} ZV(x) = ZV$

Example:

$P * x\ {}^dV(x) * St_2F\sim\ = ZV(x)$

$\lim_{x\to 1} ZV(x) = ZV$

$$* \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 3 & 7 & 15 & 31 \\ . & . & 2 & 12 & 50 & 180 \\ . & . & . & 6 & 60 & 390 \\ . & . & . & . & 24 & 360 \\ . & . & . & . & . & 120 \end{bmatrix}$$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix} * diag(\begin{bmatrix} x\verb|^|1 \\ x\verb|^|2 \\ x\verb|^|3 \\ x\verb|^|4 \\ x\verb|^|5 \\ x\verb|^|6 \end{bmatrix}) = \begin{bmatrix} z0.0 & z0.1 & z0.2 & z0.3 & z0.4 & z0.5 \\ z1.0 & z1.1 & z1.2 & z1.3 & z1.4 & z1.5 \\ z2.0 & z2.1 & z2.2 & z2.3 & z2.4 & z2.5 \\ z3.0 & z3.1 & z3.2 & z3.3 & z3.4 & z3.5 \\ z4.0 & z4.1 & z4.2 & z4.3 & z4.4 & z4.5 \\ z5.0 & z5.1 & z5.2 & z5.3 & z5.4 & z5.5 \end{bmatrix}$$

Example: with x = 4 *we get the parametrized Vandermondematrix:*

$P * 4\ {}^dV(4) * St_2F\sim\ = ZV(4)$

$$* \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 3 & 7 & 15 & 31 \\ . & . & 2 & 12 & 50 & 180 \\ . & . & . & 6 & 60 & 390 \\ . & . & . & . & 24 & 360 \\ . & . & . & . & . & 120 \end{bmatrix}$$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix} * diag(\begin{bmatrix} 4\verb|^|1 \\ 4\verb|^|2 \\ 4\verb|^|3 \\ 4\verb|^|4 \\ 4\verb|^|5 \\ 4\verb|^|6 \end{bmatrix}) = \begin{bmatrix} 4 & 4 & 4 & 4 & 4 & 4 \\ 4 & 20 & 52 & 116 & 244 & 500 \\ 4 & 36 & 228 & 996 & 3684 & 12516 \\ 4 & 52 & 532 & 4180 & 25684 & 135892 \\ 4 & 68 & 964 & 11204 & 106180 & 839108 \\ 4 & 84 & 1524 & 23604 & 309684 & 3450804 \end{bmatrix}$$

and the (finite dimensioned with *rows=64 / columns=64*) inverse gives

$$ZV(4)^{-1} \sim \begin{bmatrix} 0.333333333333 & -0.111111111111 & 0.0370370370370 & -0.0123456790123 \\ -0.0958940241506 & 0.143075785828 & -0.0662104471278 & 0.0261853753801 \\ 0.0137934958017 & -0.0365625066508 & 0.0360334665216 & -0.0193678718547 \\ -0.00132271381953 & 0.00503873654041 & -0.00777332995526 & 0.00659482848749 \\ 0.0000951302632156 & -0.000472614694248 & 0.000997327654817 & -0.00119614587997 \\ -0.00000547345425495 & 0.0000335345724902 & -0.0000899473065381 & 0.000140796619381 \end{bmatrix}$$

which agrees with the computation according to the common matrix-inversion-formula (where $y = 1/x$):

$$(P * x\ {}^dV(x) * St_2F\sim)^{-1} = fSt_1\sim * y\ {}^dV(y) * P^{-1} = W(y)$$

W(1/4) ~

0.333333333333	-0.111111111111	0.0370370370370	-0.0123456790123
-0.0958940241506	0.143075785828	-0.0662104471278	0.0261853753801
0.0137934958017	-0.0365625066508	0.0360334665216	-0.0193678718547
-0.00132271381953	0.00503873654041	-0.00777332995526	0.00659482848749
0.0000951302632156	-0.000472614694248	0.000997327654817	-0.00119614587997
-0.00000547345425495	0.0000335345724902	-0.0000899473065381	0.000140796619381

2.6. Definition of the entries of the powerseries-parametrized matrix W(y)

As in the example before, a powerseries-vector ${}^dV(y)$ (where $y=1/x$) as a diagonal-multiplicator in the middle of the term is introduced.

The matrix-product looks like:

(2.6.1.) *definition* $W(y) = fSt_1\sim * \; y \; {}^dV(y) \sim * P^{-1}$

$\lim_{y \to 1} W(y) = ZV^{-1}$

Example:

$W(y) = fSt1\sim * y \; {}^dV(y) \sim * P^{-1}$

P^{-1}:

1	.	.	.	.	.
-1	1	.	.	.	.
1	-2	1	.	.	.
-1	3	-3	1	.	.
1	-4	6	-4	1	.
-1	5	-10	10	-5	1

$fSt1\sim$:

1	-1	1	-1	1	-1
.	1	-3/2	11/6	-25/12	137/60
.	.	1/2	-1	35/24	-15/8
.	.	.	1/6	-5/12	17/24
.	.	.	.	1/24	-1/8
.	.	.	.	.	1/120

*diag([y^1, y^2, y^3, y^4, y^5, y^6]) * P^{-1} =

w0.0	w0.1	w0.2	w0.3	w0.4	w0.5
w1.0	w1.1	w1.2	w1.3	w1.4	w1.5
w2.0	w2.1	w2.2	w2.3	w2.4	w2.5
w3.0	w3.1	w3.2	w3.3	w3.4	w3.5
w4.0	w4.1	w4.2	w4.3	w4.4	w4.5
w5.0	w5.1	w5.2	w5.3	w5.4	w5.5

2.7. A formal description for the entries of W(y)

In the previous formula *(2.6.1)* y can be selected to produce convergent (or Euler-summable) series and thus conventionally approximatible values for $w(y)_{r,c}$. The reciprocal of the vandermonde matrix, ***W***, may then finally be estimated as the limit of the matrix-product *(2.6.1)*, when $y \to 1$.

I succeeded to find a very plausible decomposition of the values of $w(y)_{r,c}$ which agrees well with the numerical approximations for various y and it comes out, that the final result involves logarithms of negative integer arguments.

The formal reciprocal is given using logarithm of $(1-y)$ and a finite summative function $b()$.

Conjecture:

define y *be the free parameter of the powerseries V(y),*
define $\lambda = \log(1 - y)$;

define $$b(r,c,\mu) = \sum_{k=0}^{c} fSt1_{r,k} * \frac{c!}{(c-k)!} * \mu^k$$

then

(2.7.1.) $$W(y) := w(\lambda)_{r,c} = \left(\frac{y}{1-y}\right)^{c+1} * \frac{b(c,r,\lambda^{-1})}{r!} * \lambda^r$$

2.8. Approximation-Examples

Real arguments near y=1

Example: using x = 2, y=1/2:

(2.8.1.) $ZV(2) * W(2^{-1}) = I + eps$

ZV(2)

2.00000000000	2.00000000000	2.00000000000	2.00000000000
2.00000000000	6.00000000000	14.0000000000	30.0000000000
2.00000000000	10.0000000000	42.0000000000	154.000000000
2.00000000000	14.0000000000	86.0000000000	470.000000000
2.00000000000	18.0000000000	146.000000000	1074.00000000
2.00000000000	22.0000000000	222.000000000	2062.00000000

$W(2^{-1})$ *

1.00000000000	-1.00000000000	1.00000000000	-1.00000000000
-0.693147180560	1.69314718056	-2.19314718056	2.52648051389
0.240226506959	-0.933373687519	1.77994727780	-2.51099633799
-0.0555041086648	0.295730615624	-0.762417459383	1.35573321865
0.00961812910763	-0.0651222377725	0.212987545584	-0.467126698712
-0.00133335581464	0.0109514849223	-0.0435126038085	0.114508452337

ID+eps =

1.00000000000	9.45795227343E-98	-4.39273588791E-96	1.35410703507E-94
-3.70019302171E-80	1.00000000000	-1.60318665165E-76	4.94117050698E-75
-1.36326561851E-68	1.27175939540E-66	1.00000000000	1.81960812384E-63
-2.67436249524E-60	2.49446859443E-58	-1.15799825057E-56	1.00000000000
-8.43461919779E-54	7.86607074801E-52	-3.65107786178E-50	1.12475804734E-48
-1.95121321612E-48	1.81942000654E-46	-8.44366328530E-45	2.60077293368E-43

Example using x=1.1, y=1/1.1

(2.8.2.) $ZV(1.1) * W(1.1^{-1}) = I + eps$

ZV(1.1)

1.1000000	1.1000000	1.1000000	1.1000000
1.1000000	2.3100000	4.7300000	9.5700000
1.1000000	3.5200000	11.022000	34.012000
1.1000000	4.7300000	19.976000	83.210600
1.1000000	5.9400000	31.592000	165.95040
1.1000000	7.1500000	45.870000	291.01600

$W(1.1^{-1})$ *

10.000000	-100.00000	1000.0000	-10000.000
-23.978953	339.78953	-3897.8953	42312.286
28.749509	-527.28461	6971.7938	-82710.922
-22.979437	517.28946	-7809.3176	101332.49
13.775571	-367.55008	6261.9481	-88650.539
-6.6064752	203.82046	-3875.9550	59632.710

ID+eps =

1.0000000	4.7147545E-62	-6.6650223E-60	6.4174187E-58
-3.3379104E-45	1.0000000	-1.3045930E-40	1.2555099E-38
-6.6069868E-34	1.8267264E-31	1.0000000	2.4816715E-27
-6.9701286E-26	1.9262831E-23	-2.7192965E-21	1.0000000
-1.1832630E-19	3.2687484E-17	-4.6125467E-15	4.4332739E-13
-1.4746696E-14	4.0721748E-12	-0.0000000000057440626	0.000000055187220

Example using x=1.01, y = 1/1.01

(2.8.3.) $ZV(1.01) * W(1.01^{-1}) = I + eps$

ZV(1.01)

1.0100000	1.0100000	1.0100000	1.0100000
1.0100000	2.0301000	4.0703000	8.1507000
1.0100000	3.0502000	9.1912020	27.655012
1.0100000	4.0703000	16.372706	65.766560
1.0100000	5.0904000	25.614812	128.72897
1.0100000	6.1105000	36.917520	222.78586

$W(1.01^{-1})$ *

100.00000	-10000.000	1000000.0	-1.0000000E8
-461.51205	56151.205	-6115120.5	6.4484539E8
1064.9669	-152647.89	18072349.	-2.0110723E9
-1638.3168	270328.37	-34665231.	4.0689348E9
1890.2574	-352857.42	48802160.	-6.0357238E9
-1744.7531	363501.05	-53992976.	7.0260363E9

ID+eps =

1.0000000	3.5531732E-42	-2.7804968E-39	1.5257319E-36
-4.1791753E-26	1.0000000	-4.8494028E-20	2.6589100E-17
-7.3952843E-15	1.0966646E-11	0.99999999	0.0000046948993
-0.00000069908473	0.0010359932	-0.80893820	443.95850
-1.0655779	1578.1584	-1231569.9	6.7401233E8
-119450.30	1.7681403E8	-1.3791138E11	7.5438665E13

Complex argument near y=1

*Example: x= 1/(1+0.1*I); y = 1 + 0.1*I*

(2.8.4.) $ZV(1+0.1*I) * W((1+0.1*I)^{-1}) = ID + eps$

*ZV(1+0.1*I)*

0.99009901-0.099009901*I	0.99009901-0.099009901*I	0.99009901-0.099009901*I
0.99009901-0.099009901*I	1.9605921-0.29506911*I	3.9015783-0.68718753*I
0.99009901-0.099009901*I	2.9310852-0.49112832*I	8.6960024-1.8557781*I
0.99009901-0.099009901*I	3.9015783-0.68718753*I	15.373371-3.6047815*I
0.99009901-0.099009901*I	4.8720714-0.88324674*I	23.933685-5.9341979*I
0.99009901-0.099009901*I	5.8425645-1.0793060*I	34.376944-8.8440271*I

*W((1+0.1*I)$^{-1}$)* *

-1.0000000+10.000000*I	99.000000+20.000000*I	299.00000-970.00000*I
18.010548-21.455055*I	-295.54000-221.56054*I	-2660.6454+3218.8394*I
-37.586171+10.555593*I	264.50976+587.97784*I	7399.8608-3424.0394*I
34.375348+11.578356*I	82.189149-718.59242*I	-10175.879-511.92405*I
-15.241231-20.164205*I	-367.04219+464.42322*I	7829.0576+4905.0870*I
0.68407444+14.074133*I	358.30869-125.01472*I	-3030.4335-5775.5241*I

ID+eps =

1.0000000-1.9606581E-60*I	-5.6102204E-58-6.5859270E-59*I	2.2704687E-56+6.5071878E-56*I
1.9020210E-41-3.7351337E-41*I	1.0000000+4.9349377E-41*I	5.3960313E-37+1.0952663E-36*I
2.5801312E-30-7.0485553E-30*I	-1.7780570E-27+2.2391555E-28*I	1.0000000+1.8290773E-25*I
1.6471733E-22-6.9691125E-22*I	-1.6582321E-19+4.1335278E-20*I	1.3373881E-17+1.5956811E-17*I
1.2553865E-16-1.0918731E-15*I	-2.4517073E-13+9.2871182E-14*I	2.3244961E-11+2.1884343E-11*I
-2.5411865E-13-1.2380213E-10*I	-0.000000026200027+0.000000013599542*I	0.000028854382+0.000021412953*I

*Example: x= 1/(1+0.01*I); y = 1 + 0.01*I*

(2.8.5.) $ZV(1+0.01*I) * W((1+0.01*I)^{-1}) = I + eps$

*ZV(1+0.1*I)*

0.99990001-0.0099990001*I	0.99990001-0.0099990001*I	0.99990001-0.0099990001*I
0.99990001-0.0099990001*I	1.9996001-0.029995001*I	3.9990002-0.069987002*I
0.99990001-0.0099990001*I	2.9993001-0.049991001*I	8.9969006-0.18995501*I
0.99990001-0.0099990001*I	3.9990002-0.069987002*I	15.993601-0.36990302*I
0.99990001-0.0099990001*I	4.9987002-0.089983002*I	24.989102-0.60983103*I
0.99990001-0.0099990001*I	5.9984003-0.10997900*I	35.983404-0.90973905*I

*W((1+0.1*I)$^{-1}$)* *

-1.0000000+100.00000*I	9999.0000+200.00000*I	29999.000-999700.00*I
161.68480-458.94622*I	-55731.937-16827.427*I	-1753474.1+6056216.3*I
-732.74854+929.77578*I	137977.77+90832.056*I	10090421.-16485128.*I
1611.6400-1043.5923*I	-194992.42-236412.23*I	-28446807.+26116302.*I
-2265.2869+568.59052*I	157341.36+389304.88*I	51005937.-24976246.*I
2265.0343+187.96965*I	-33531.765-453412.74*I	-64918721.+10572179.*I

ID+eps =

1.0000000+3.9138930E-44*I	2.8974033E-41-9.4018745E-41*I	-7.3055096E-38-1.7315413E-39*I
1.0193184E-24+6.4647895E-25*I	1.0000000-1.6382252E-21*I	-1.2610963E-18+8.2606618E-21*I
1.8148998E-13+1.0821641E-13*I	7.0996242E-11-0.0000000000028864160*I	0.99999978+0.000000075787965*I
0.000017223105+0.0000096808900*I	0.0059380343-0.027131523*I	-20.545882+1.2384409*I
26.305720+13.968064*I	7961.0186-41078.258*I	-30880981.+2609820.8*I
2950412.0+1482395.1*I	7.7897476E8-4.5701479E9*I	-3.4122717E12+3.6639110E11*I

It may be of interest, that with *real(x) = 1* and *imag(x)<>0*, the reciprocal has the first element exactly with real-part of *1*, systematically different from the iterative approximation-approach with increasing finite size.

For the top-left element $w(y)_{0,0}$ we have a zero-denominator for the limit-case *y=1* , so I still cannot assign a meaningful value to this element. For subsequent entries in this column however it may be possible, that the multiplication with powers of $\lim_{y\to 1} \log(1-y)$ allow to assign meaningful values to these entries.

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Gottfried Helms, 06.02.2007