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10-4 A Powertower-sum

Abstract: A formula for the summing of infinitely increasing powertower-expressions is given. It focuses the two-parameter problem $S(s,x) = x - s^x + s^{s^x} - s^{s^{s^x}} \dots$

The solution is an infinite polynomial in x, whose coefficients $a_s[k]$ are dependent of the chosen base-parameter s. The range for admissible s is principally $e^{-e} < s < r$, where $r > e^{1/e}$ (but I've not yet determined whether an upper bound exists).

This implies, that s exceeds the known upper bound for the parameter s for the single ***infinite*** *powertower ${}^{inf}s = s^{s^{\cdots}}$ of the range $e^{-e} < s < e^{1/e}$ where, if s is outside, the infinite powertower-expression diverges. So possibly from here also another analytical continuation for the infinite powertower can be derived. On the other hand, when approching the lower bound the approximation seems to become irreparably bad and at the lower bound monotonic increasing divergence in the parameters $a_s[k]$ occurs.*

It is an experimental approach and needs further confirmation and characterizing of its radius of convergence. For the known convergent cases the results apparently agree with conventional summation; a rigorous proof for the consistency/regularity of its assignment of non-convergent cases needs further study.

Keywords: infinite matrix, geometric series, tetration, infinite powertower, alternating series summation, divergent summation.

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Contents:

1. Statement of the problem and basic considerations *3*
1.1. A powertower problem / Intro 3
1.2. Transformation of a powerseries into an exponential series 4
1.3. Iterated application 4
2. Matrix B, its powers and summability of its alternating sums of powers *6*
2.1. The structure of Matrix B 6
2.2. A parametrization for B 7
2.3. Investigation of a convergent case B_s where s=sqrt(2) 8
2.3.1. Powers of B_s 8
2.3.2. Alternating sum of powers of B_s 8
2.3.3. Using the shortcut-formula for geometric series 9
2.4. Investigation of a divergent case B_s (= "B_e") where s=exp(1) 10
2.4.1. Powers of B_e 10
2.4.2. Direct alternating sum of powers of B_e 10
2.4.3. Using the shortcut-formula for the alternating geometric series 11
2.5. Sum-values derived at example parameters s 12
2.6. Preliminary resume 13
3. Some details *14*
3.1. Convergence in the columns of M_e 14
3.2. Approximating inverses of $(I+B_s)$ by its L-D-U-decomposition: 14
3.2.3. The components 15
Rational arithmetic 15
Float arithmetic: 15
3.2.4. Asymptotics for $M_{inf} = B1_{inf}^{-1} = R^{-1} * D^{-1} * L^{-1}$ 16
3.2.5. Remarkable simplifications for the computation 17
3.3. Another check for a conventionally convergent parameter /Example 18
3.3.1. Check appropriateness of B_{test} : for a single step of transformation 18
3.3.2. Now check results of infinitely iterated use of B_{test} by use of M_{test} 19
3.3.3. The result of the above operations is now: 19
4. Citations and suggestions for further reading *20*
4.1. Powertower: encyclopedia... 20
4.1.1. Wikipedia 20
4.1.2. Mathworld 21
4.2. Some Critical Points of the.... 22
4.3. Some tetration identities 23
5. References *24*

1. Statement of the problem and basic considerations

1.1. A powertower problem / Intro

(1.1.1.) $$S(s) = 1 - s + s^s - s^{s^s} + s^{s^{s^s}} - \ldots + \ldots$$

(1.1.2.) $$S(s,x) = x - s^x + s^{s^x} - s^{s^{s^x}} + s^{s^{s^{s^x}}} - \ldots + \ldots$$

Recently I found a possible path how to deal with such series, even for s and x, for which the expression for S is divergent. This however depends on proofs for convergence (or at least regular summability) of intermediate result.

For a start, a current approximation for $S(e,1)$ was:

$S(\exp(1),1) = 0.24696086896[0...$ // divergent summation

Although for divergent series a value is not defined in general, to some divergent series a meaningful and consistent value can be assigned by Cesaro-, Euler-, or Borel-summation. To other series these summation-methods do not suffice to "limit" them in terms of their partial sums.

Now the actual case is a very badly diverging series, and for parameters s, where moreover s is outside the permitted range for infinite powertowers of $e^{-e} < s < e^{1/e}$ it should be even more astonishing, that a value can be assigned at all. The described method cannot exceed the lower bound e^{-e} , where it produces monotonic increasing divergent series of terms, which would have to be summed, but the upper bound of $e^{1/e}$ for the base-parameter s of an infinite powertower apparently can be exceeded.

Operations of this type of iterated exponentiation are usually called "*powertower*","*tetration*" and several results even concerning the infinitely iterated version were found beginning with some analyses by LAMBERT and EULER, for instance the assignement of a value as $2 = \text{sqrt}(2)^{\text{sqrt}(2)^{\text{sqrt}(2)^{\cdots}}}$ I myself stumbled only by chance into the concept of tetration.

I am currently studying repeated application of matrix-formulae in the context of binomialmatrices, power- and zetaseries, which for instance leads to a simple formulation and solution for the "*summing of like powers problem*". [1](see article in the project index). The method of iterated application of a matrixoperation, and to find a meaningful solution for the idea of infinite iterated application (as it is basically done with the elementary shorthand formula for the evaluation of the geometric series for scalar parameters), gave a solution in terms of zeta-values much similar to the bernoulli-polynomials.

In the context of this, I realized, that another matrix operation yields the exponential-series when fed by a powerseries, and iterated application would analoguously give iterated exponentials - with the perspective to have a solution for the infinite iteration as well - by the same means, which showed to be successful with the powerseries and the zeta-values/ bernoulli-numbers.

The result of the method is, that for parameters s and x in the formula

$$S(s,x) = x - s^x + s^{s^x} - s^{s^{s^x}} + s^{s^{s^{s^x}}} - \ldots + \ldots$$

one gets a polynomial in x, where the coefficients $a_{s,[k]}$ are dependend on s:

$$S(s,x) = \text{limit} \quad a_{s,[0]} + a_{s,[1]}x + a_{s,[2]}x^2 + a_{s,[3]}x^3 + \ldots.$$

which for $x=1$ reduces finally to the simple sum:

$$S(s,1) = - a_{s,[0]} = - \text{limit} \ \Sigma \ b_{s,[k]}$$

If the sequence of coefficients $b_{s,[k]}$ or $a_{s,[k]}$ diverge, but oscillate in sign, which happens near the lower bound $s = e^{-e} + eps$ they may still be Euler-summable, - but I could not yet work out an estimation for their rate of growth, so such attempted results are questionable.

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[1] http://go.helms-net.de/math/binomial_new/04_3_SummingOfLikePowers.pdf

1.2. Transformation of a powerseries into an exponential series

Assume, I have the coefficients of the powerseries of x in a vector

(1.2.1.) $V(x) = column([\ 1, x, x^2 , x^3 , ...])$

Now assume another vector A_0, containing

(1.2.2.) $A_0 = column([1/0!, 1/1!, 1/2!, ...])$

then the rowvector $V(x)\sim$ by A_0 is

(1.2.3.) $V(x)\sim * A_0 = exp(x)$

Now I have another vector A_1, and this performs

(1.2.4.) $V(x) \sim * A_1 = exp(x)^2$

and assume, I have a set of vectors A_0, A_1,... collected in a matrix A, giving

(1.2.5.) $V(x)\sim * A = [\ exp(x) , exp(x)^2 , exp(x)^3 , ...]$

or more precisely a matrix B (like A with an additional left column) which performs

(1.2.6.) $V(x)\sim * B = [\ 1,\ exp(x) , exp(x)^2 , exp(x)^3 ,...]$

then I can say:

(1.2.7.) $V(x) \sim * B = V(y)\sim$ // where $y = exp(x)$

and B transforms a powerseries in x in one of $exp(x)$, providing the characteristic for the result y to represent a powerseries again.

1.3. Iterated application

Now, if I consider iterated application, then I have

(1.3.1.)
$$V(x)\sim * B = V(exp(x))\sim$$
$$V(exp(x))\sim * B = V(exp(exp(x)))\sim$$
$$V(exp(exp(x)))\sim * B = V(exp(exp(exp(x))))\sim$$

... and so on.

I may write this in terms of powers of B:

(1.3.2.)
$$V(x)\sim * B^0 = V(x)\sim$$
$$V(x)\sim * B^1 = V(exp(x))\sim$$
$$V(x)\sim * B^2 = V(exp(exp(x)))\sim$$
$$V(x)\sim * B^3 = V(exp(exp(exp(x))))\sim$$
...
$$V(x)\sim * B^k = V(...)\sim$$

...

B is not triangular, so its powers are not defined for infinite dimension - but it seems, that this problem is manageable: if no conventional convergence occurs, then at least in terms of Euler-summation, see the next chapter.

The alternating sum of theses values could then be written in terms of a geometric series of B:

(1.3.3.)
$$V(x)\sim * (B^0 - B^1 + B^2 - B^3 + B^4 +... - ...)$$
$$= V(x)\sim - V(exp(x))\sim + V(exp(exp(x))) - ...$$

which is badly diverging in each entry of the result rowvector besides that of column 0, where we get the alternating series $1-1+1-1+...$ which may be evaluated by Eulersummation to the value ½.

On the other hand, for a geometric series the shorthand formula is also valid for matrices,

(1.3.4.) $(I - B + B^2 - B^3 + B^4 -+...) * (I + B) = I$
(1.3.5.) $(I - B + B^2 - B^3 + B^4 -+...) = (I + B)^{-1} = M$

writing ***M*** for the reciprocal term

(1.3.6.) $M = (I + B)^{-1}$

if the powers can be used **and** the parenthese is invertible.

Excurs to the zeta-problem

The same idea was apparently successfully applied to the Pascal-matrix ***P***, which implements the binomial-theorem for a powerseries ***V(x)*** by

$$P*V(x) = V(x+1)$$

and

$$1/x\ V(1/x)\sim * P^{-1} = 1/(x+1) * V(1/x+1)$$

The infinite iterated application and alternative summation

$$(P^0 - P^1 + P^2 - ...)\ V(1) = H = [\eta(0), \eta(-1) , \eta(-2), ...] \qquad // \text{ where } \eta \text{ is the eta-function}$$

can be performed by Euler-summation of the components or by the inversion of $I + P$:

$$ETA = (I + P)^{-1}$$

and gives a matrix ***ETA*** similar to that of the matrix of coefficients of bernoulli-polynomials (which contain ζ-values with binomial scaling) and performs the same summation as expected by

$$ETA * V(1) = H$$

with ***H*** containing the $\eta(-r)$-values in row r. The same concept can, if the poles at $\zeta(1)$ are appropriately handled, be applied to the ζ-function (only we cannot use the matrix-inversion) and leads the to a matrix ***ZETA***, which contains the coefficients of the integrals of the Bernoulli-polynomials, and indeed performs the non-alternating summation of like powers to ζ-values (and also to finite sums of like powers in the same sense, as the difference of two Hurwitz-zeta-terms provides it with $\zeta(s,1) - \zeta(s,a)$)

The current approach is simply an analogy for the matrix ***B*** instead of ***P***, where in the eta/zeta-case we have a triangular matrix ***P***, which behaves much simpler than the infinite square-matrix ***B***.

Heuristically for finite dimensions d, the matrices $_dM$ seems to stabilize with higher d, and a convergent for $_dM$ with d->inf seems reasonable, so practically I assume the finite version of $_dM$ with $d=64$ as sufficient precise, where I calculate with float-precision of about *200* digits (Pari/Gp), and final approximations to ~ *12* significant digits for display.

But this needs more consideration and a proof, however for the beginning a qualitative consideration may suffice.

2. Matrix B, its powers and summability of its alternating sums of powers

2.1. The structure of Matrix B

The structure of matrix ***B*** is very simple: it is essentially a vandermonde matrix-premultiplied by inverse of factorials:

$$B = F^{-1} * VZ$$

where

(2.1.1.) $VZ =$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 4 & 9 & 16 & 25 \\ 0 & 1 & 8 & 27 & 64 & 125 \\ 0 & 1 & 16 & 81 & 256 & 625 \\ 0 & 1 & 32 & 243 & 1024 & 3125 \end{bmatrix}$$

(2.1.2.) $F^{-1} = diag([1/0! , 1/1! , 1/2!, ...])$

$$diag(\begin{bmatrix} 1 \\ 1 \\ 1/2 \\ 1/6 \\ 1/24 \\ 1/120 \end{bmatrix})$$

(2.1.3.) $B = F^{-1} * VZ$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1/2 & 2 & 9/2 & 8 & 25/2 \\ 0 & 1/6 & 4/3 & 9/2 & 32/3 & 125/6 \\ 0 & 1/24 & 2/3 & 27/8 & 32/3 & 625/24 \\ 0 & 1/120 & 4/15 & 81/40 & 128/15 & 625/24 \end{bmatrix}$$

The columns in ***B*** provide sequences which absolutely converge to zero in each column, since the premultiplication by inverse factorials limit and later dominate the increasing powers of natural numbers.

So the premultiplication by a powerseries as in *(1.3.1)* yields the following with trivially bounded values in all columns of the result:

Example:

(2.1.4.) $V(1)\sim \quad * B = V(exp(1))\sim$

$$\begin{bmatrix} 1.0000000 & 1.0000000 & 1.0000000 & 1.0000000 & 1.0000000 \\ 0 & 1.0000000 & 2.0000000 & 3.0000000 & 4.0000000 \\ 0 & 0.50000000 & 2.0000000 & 4.5000000 & 8.0000000 \\ 0 & 0.16666667 & 1.3333333 & 4.5000000 & 10.666667 \\ 0 & 0.041666667 & 0.66666667 & 3.3750000 & 10.666667 \\ 0 & 0.0083333333 & 0.26666667 & 2.0250000 & 8.5333333 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} * \begin{bmatrix} 1.0000000 & 2.7182818 & 7.3890561 & 20.085537 & 54.598150 \end{bmatrix}$$

and in fact we have the transformation of a powerseries-vector in *1* in one of *exp(1)*:

(2.1.5.) $$[1,\ 1,\ 1,\ 1,\ 1,\ ....] * B = [\ 1,\ e\ ,\ e^2\ ,\ e^3\ ,...]$$

The same is valid for each power of ***B***, although the size of the entries increases quickly.

2.2. A parametrization for B

To check this for convergent case first note, that ***B*** is easily parametrizable, in the sense, that for the current ***B*** we have formally

(2.2.1.) $V(x)\sim * B = V(e^x)\sim$

and the result in the second column $col=1$ of the repeated iteration represents using ***M*** then

(2.2.2.) $S(e,x) = x - exp(x) + exp(exp(x)) - \dots$

Since the columns in ***B*** are simply exponential-series in $1 = log(e)$, we may premultiply ***B*** with a power-series-vector of $log(s)$ to get the parametrized version $\boldsymbol{B}_s$ of ***B***:

(2.2.3.)
$$B_s = diag(V(log(s))) * B$$
$$V(x)\sim * B_s = V(s^x)\sim$$

and in the second column of the result we get then

by $V(x)\sim * M_s = V(x)\sim - V(s^x)\sim + V(s^{s^x})\sim - \dots$

(2.2.4.) $S(s,x) = V(x)\sim * M_s [,1]$
$$= x - s^x + s^{s^x} - \dots$$

2.3. Powers of B

B, of infinite dimension, is not triangular, so the entries of powers of ***B*** must be checked for convergence or regular summability of the involved series of terms. Although the heuristics seem all good and trustworthy, I can currently only make crude guesses.

Possibly I have a successful path for a more rigorous proof for convergence of the series, which construc the entries of each finite power of ***B***.

The first nontrivial column in each power of ***B*** is the second column, ***B****[,1]*. It seems, that the entries of the ***k***'th power of ***B*** equal the entries of the first column of the k-fold iterated exponential of the Pascal-matrix $\boldsymbol{P}_k(0)$, which is described in more detail in the article "The binomial matrix". Here it may suffice to say, that it is the factorial similarity scaling of ***P*** itself:

$$P_0 := P_k(0) = F^{-1} * P * F .$$

Then we seem to have

$$(P_0)[,0] = B[,1]$$
$$exp(P_0)[,0] = B^2[,1]$$
$$exp(exp(P_0))[,0] = B^3[,1]$$
$$\dots$$

and due to the property that in

$$e * exp(P_0 - I) = lim\ exp(P_0)^{(*)}$$
$$e^e * exp(exp(P_0) - e*I) = lim\ exp(exp(P_0))$$
$$\dots$$

the $(\boldsymbol{P}_0 - \boldsymbol{I})$ expression as well as its iteratives are nilpotent, we have the tool to determine the entries in question by finite number of constants and exponentials, and thus have a proof for the boundness of entries in the powers of ***B***. (The subsequent columns in $\boldsymbol{B}^k$ are scaled versions of that second column)

(*) This is not defined in the context of formal powerseries; but it is easy to show, that this is the meaningful limit.

2.4. Investigation of a convergent case B_s where s=sqrt(2)

2.4.1. Powers of B_s

First let's look at the entries of some powers of B_s: we must make sure, that a multiplication $B_s * B_s$ does not lead to infinite values even when thought in infinite dimension.

Examples:

(2.4.1.1.) $B_s =$

1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0	0.34657359	0.69314718	1.0397208	1.3862944
0	0.060056627	0.24022651	0.54050964	0.96090603
0	0.0069380136	0.055504109	0.18732637	0.44403287
0	0.00060113307	0.0096181291	0.048691779	0.15389007
0	0.000041667369	0.0013333558	0.010125171	0.042667386

(2.4.1.2.) B_s^2

1.0000000	1.4142136	2.0000000	2.8284271	4.0000000
0	0.16986579	0.48045301	1.0191948	1.9218121
0	0.039637065	0.14096494	0.36024119	0.79469485
0	0.0073445635	0.034239525	0.10610000	0.27241216
0	0.0012340172	0.0075566049	0.028086966	0.082998500
0	0.00019671138	0.0015578523	0.0068655348	0.023145742

(2.4.1.3.) B_s^3

1.0000000	1.6325269	2.6651441	4.3509196	7.1029933
0	0.096108488	0.31379939	0.76842892	1.6726412
0	0.025255285	0.091696706	0.24716510	0.58723993
0	0.0055312589	0.022914352	0.068887824	0.17968884
0	0.0011365493	0.0054119260	0.018117978	0.051636431
0	0.00022652708	0.0012374742	0.0045866245	0.014194954

The entries of the powers of B_s occur as matrix-multiplication of a row with a column.

The rows are zeta-like-sequences, while the columns are powerseries dominated by reciprocal factorials. The infinite series of the elementwise multiplications is therefore again dominated by the reciprocal of the factorials and is thus convergent to a finite value. This is valid for all entries of B_s^2 .

If B_s^2 is postmultiplied by another instance of B_s to make the third power B_s^3, the growthrate in the rows of B_s^2 is crucial. But the quotient of subsequent entries in row *0* is constant and in all other rows decreasing asymptotically to a constant.

The quotient along the columns is always decreasing to zero. So the growthrate of all occuring terms of one row/col-matrixmultiplication is again eventually negative, their absoulute value decreasing to zero and the sum therefore bounded.

So the instantiation of powers of B_s seems to be justified.

2.4.2. Alternating sum of powers of B_s

This may not be so with the alternating sums of the powers of B_s. I could not examine that question in detail yet. But assumed, that the growthrate of the entries is not too fast, the alternating sum of powers of B can be approximated by Euler-summation. With this I get the approximate infinite alternating sum of powers of B_s using *24* powers beginning with B_s^0 as

(2.4.2.1.) $AS(B_s^k)\sim$

0.500000	-0.362302	-0.201330	-0.0223437	0.161780	0.326291
0	0.763164	-0.402347	-0.467973	-0.400108	-0.163850
0	-0.0357842	0.845601	-0.326531	-0.506912	-0.628752
0	-0.00282394	-0.0348962	0.877567	-0.278810	-0.496530
0	0.00000282467	-0.00522550	-0.0314687	0.897804	-0.242763
0	0.0000371427	-0.000481460	-0.00599433	-0.0281836	0.912071

where in the current case we are interested in the sum of the first and the second column, which represent the alternating sums of $S_0 = 1 - 1 + 1 - 1 + 1 - \ldots$ and $S_1 = 1 - s + s^s - s^{s^s} + \ldots$ respectively.

The series, formed by the column-entries, is finite for the first columns and seem to converge even conventionally for the second column, so likely no further divergent summation technique is required. The sum of the second column using dimension *dim=32* is

(2.4.2.2.) $S_1 \sim 0.362301547355$

which agrees with the direct computation via Eulersummation of the first *32* terms:

(2.4.2.3.) $S_{1\_direct} \sim 0.362301547267$

in the first *10* decimal places.

2.4.3. Using the shortcut-formula for geometric series

The shortcut-formula for the alternating geometric series of $\mathbf{B}_s$ is:

(2.4.3.1.) $M_s = (I + B_s)^{-1}$

Using this I get:

Example:

(2.4.3.2.) $M_s = (I + B_s)^{-1}$

$$\begin{bmatrix} 0.500000 & -0.362302 & -0.201330 & -0.0223437 & 0.161780 & 0.326291 \\ 0 & 0.763164 & -0.402347 & -0.467973 & -0.400108 & -0.163850 \\ 0 & -0.0357842 & 0.845601 & -0.326531 & -0.506912 & -0.628752 \\ 0 & -0.00282394 & -0.0348962 & 0.877567 & -0.278810 & -0.496530 \\ 0 & 0.00000282473 & -0.00522550 & -0.0314687 & 0.897804 & -0.242763 \\ 0 & 0.0000371427 & -0.000481460 & -0.00599433 & -0.0281836 & 0.912071 \end{bmatrix}$$

which agrees perfectly with the Euler-summed result of the individual power-matrices.

2.5. Investigation of a divergent case B_s (= "B_e") where s=exp(1)

2.5.1. Powers of B_e

Again let's look at the entries of powers of B_e first: we must make sure, that a multiplication $B_e * B_e$ does not lead to infinite values even when thought in infinite dimension.

Examples:

(2.5.1.1.) B_e

1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0	1.0000000	2.0000000	3.0000000	4.0000000
0	0.50000000	2.0000000	4.5000000	8.0000000
0	0.16666667	1.3333333	4.5000000	10.666667
0	0.041666667	0.66666667	3.3750000	10.666667
0	0.0083333333	0.26666667	2.0250000	8.5333333

(2.5.1.2.) B_e^2

1.0000000	2.7182818	7.3890561	20.085537	54.598150
0	2.7182818	14.778112	60.256611	218.39260
0	2.7182818	22.167168	120.51322	545.98150
0	2.2652349	27.093206	190.81260	1055.5642
0	1.6989261	28.940470	258.60129	1719.8417
0	1.1779221	27.955262	312.33010	2469.6563

(2.5.1.3.) B_e^3

1.0000000	15.154262	229.65162	3480.1546	52727.132
0	41.193556	1248.5144	28378.949	573020.98
0	97.181401	4642.3097	144073.96	3683878.7
0	197.03389	13978.036	569359.75	17932111.
0	361.71232	36638.052	1908892.8	72607251.
0	616.80568	86776.445	5674940.6	2.5654727E8

The entries of the powers of B_e occur as matrix-multiplication of a row with a column.

The rows are of the order of zeta-sequences, while the columns are powerseries dominated by reciprocal factorials. The infinite series of the elementwise multiplications is therefore again dominated by the reciprocal of the factorials and is thus convergent to a finite value. This is valid for all entries of B_e^2 .

If B_e^2 is postmultiplied by another instance of B_e to make the third power B_e^3, the growthrate in the rows of B_e^2 is again crucial. But the quotient of subsequent entries in row 0 is constant and in all other rows decreasing asymptotically to a constant. The quotient along the columns is always decreasing to zero. So the growthrate of all occuring terms of one row/col-matrixmultiplication is again eventually negative, their absoulute value decreasing to zero and the sum therefore bounded.

So also the instantiation of powers of B_e seems to be justified.

2.5.2. Direct alternating sum of powers of B_e

This is not so with the ***alternating sum*** of the powers of B_e. The partial sums of the respective entries of a cell $B_e[row,col]$ over all powers of B_e^k diverge strongly and of the alternating sums diverge with oscillating values. I don't have an estimation yet but since the column sums of each partial nonalternating as well as of the alternating partial sum of $B_e^0 + B_e^1 + B_e^2 + B_e^3 + ... + B_e^k$ or $B_e^0 - B_e^1 + B_e^2 - B_e^3 + ... +- B_e^k$ diverge so at least the partial sum of at least one entry in that column must diverge with a comparable growth rate.

So I assume we don't have an instrument to sum the alternating series of powers of B_e by for instance Euler-summation.

2.5.3. Using the shortcut-formula for the alternating geometric series

The shortcut-formula for the alternating geometric series of $\boldsymbol{B_e}$ is:

(2.5.3.1.) $$M_e = (I + B_e)^{-1}$$

Using this I get for dimension *dim=32* by direct matrixinversion (Pari, precision 200 digits) :

Example:

(2.5.3.2.) $M_e = (I + B_e)^{-1}$

0.500000	-0.246961	-0.0362797	0.0630371	0.0437329	-0.0240343
0	0.603300	-0.385138	-0.120447	0.119791	0.134898
0	-0.116309	0.616864	-0.423212	-0.162428	0.147104
0	-0.000135474	-0.160405	0.599596	-0.439921	-0.169515
0	0.00811708	-0.0135945	-0.180596	0.589798	-0.446615
0	0.0000743439	0.0143853	-0.0212457	-0.192412	0.583987

which appears as a clear convergent, when dimensions are successively increased from *dim=2* to *dim=32*. (see appendix 3.1 for a longer display of the first six columns)

This is a much interesting and surprising result, since the most interesting aspect is the apparently convergent series, formed by the entries of column *col=1*.

The result of the summing of column *1* according to

(2.5.3.3.) $$S_1 = S(e,1) = V(1)\sim * M_e[,1] = 1 - e + e^e - e^{e^e} + \dots$$

is then by direct summation

(2.5.3.4.) $$S_1 = 0.246960868960\dots$$

In chap 3 I study the inversion of $(\boldsymbol{I} + \boldsymbol{B_s})$ in more detail and it seems, that this method provides reasonable values even for the parameters s, which are outside the conventional domain for infinite powertowers (valid s are in $e^{-e} < s < e^{1/e}$) and hence also exceed the upper bound for the possibility of evaluating the alternating sums of finite powertowers by direct or Euler-summation.

2.6. Sum-values derived at example parameters s

s	1 - s + s^s - s^s^s + ...	S(s,1)
5	1 - 5 + 5^5 - 5^5^5 +	0.1976203324895644
pi ~ 3.14	1 - pi + pi^pi - pi^pi^pi + ...	0.2324670739643584
e ~ 2.71	1 - e + e^e - e^e^e + ...	0.2469608978527557
2	1 - 2 + 2^2 - 2^2^2 + ...	0.2874086990716591
phi+1 (= 1.6180339887498948..)		0.3278048495149163
sqrt(2)		0.3623015472668659
phi (= 0.6180339887498948..)		0.7861534997409138
1/2		0.9382530028218765

The following are questionable, since at *s=1/e* occurs a singularity in the method. This singularity, however is removable, since approximations from below and above to the same finite value are possible:

using 1/e ~ 0.3678794411714423

1/e + 0.01		1.172056144413609
1/e		--- division by zero ---
1/e-0.01		1.223179007248831

The following approach the non removable singularity at $s=e^{-e}$ and the growthrate of the internal coefficients is not yet known to me

<0.135 (e^{-2})		oscillating divergence Eulersummation doubtful
0.0659 (e^{-e})		infinite sum of positive terms

Arbitrary extensions like the following are specifically questionable, at least as the Euler-sum method is not well configured for complex summation (if at all), although the absolute values of the summation-terms diverge "not too strong"

The implicite value *log(-1/2)* is used by the Pari-convention as ~ *-0.69314718 + 3.141592*I*

s = -1/2	1 - s + s^s - s^s^s + ..	0.37550 - 1.11870*I

2.7. Preliminary resume

With some more tests it seems, that the range for the parameter s can be extended to positive reals beyond the point of conventional divergence for the infinite powertower at $s = e^{1/e}$. Actually it seems, that the characteristic of M_s is even more "friendly" with increasing s in the currently checked region.

On the other hand in this method occurs additional divergence in the computation of M_s at the point $s = e^{-1}$. However, looking at the matrix-multiplications in detail, it appears, that the occuring singularity can be removed and indeed a value to $S(e^{-1}, 1)$ can be assigned. This is investigated in the other manuscript, see footnote[2].

Values $s <= e^{-e}$ cannot be computed, since the coefficients, which must be summed, are all positive and increasing in value. With $e^{-e} < s < e^{-2}$ the coefficients alternate in sign, and may be summable by Euler-summation - but I don't have an idea about the characteristics of the growthrate of the absolute value of the coefficients, so here more analysis is needed.

One may try to approximate sums for negative and complex values of s as well, using the principal branch for logarithms of negative or complex parameters. The characteristics of M_s , or more precisely the growth of the occuring coefficients for such parameters are not checked yet.

For another crosscheck of this general idea, the equivalent, but somehow inverse, can be checked easily by low order summation-techniques:

$$T(x) = x - \log(1+x) + \log(1+\log(1+x)) - \log(1+\log(1+\log(1+x))) + \ldots - \ldots$$

Here one needs to employ the factorial scaled matrix of stirling-numbers of first kind to construct $\boldsymbol{B}$ and $\boldsymbol{M}$. I crosschecked the validity of the results according to the method described here against normal Euler-summation of the approximants and found consistent results.

---

[2] Critical point for summation with this method at $s=\exp(-1)$
http://go.helms-net.de/math/binomial_new/PowertowerproblemDocSummation.htm

3. Some details

3.1. Convergence in the columns of M_e

Convergence of entries in the first few columns of M_e (*dim=32*)

5.00000E-1	-2.46961E-1	-3.62797E-2	6.30371E-2	4.37329E-2
0	6.03300E-1	-3.85138E-1	-1.20447E-1	1.19791E-1
0	-1.16309E-1	6.16864E-1	-4.23212E-1	-1.62428E-1
0	-1.35474E-4	-1.60405E-1	5.99596E-1	-4.39921E-1
0	8.11708E-3	-1.35945E-2	-1.80596E-1	5.89798E-1
0	7.43439E-5	1.43853E-2	-2.12457E-2	-1.92412E-1
0	-1.26966E-3	3.18326E-3	1.78006E-2	-2.57173E-2
0	-1.09363E-4	-2.38631E-3	5.91995E-3	2.01878E-2
0	2.61058E-4	-9.63832E-4	-2.92047E-3	8.00773E-3
0	5.91370E-5	4.42293E-4	-1.83595E-3	-3.24131E-3
0	-5.85365E-5	3.14910E-4	4.65930E-4	-2.57602E-3
0	-2.55524E-5	-7.10386E-5	5.84660E-4	4.30421E-4
0	1.25530E-5	-1.02944E-4	-3.39947E-5	8.20276E-4
0	9.89643E-6	3.35249E-6	-1.82969E-4	2.00739E-5
0	-2.13223E-6	3.25130E-5	-2.21403E-5	-2.51292E-4
0	-3.54489E-6	4.62470E-6	5.48865E-5	-5.13756E-5
0	7.81980E-8	-9.77157E-6	1.67513E-5	7.28874E-5
0	1.19274E-6	-3.08966E-6	-1.56418E-5	2.88186E-5
0	1.58090E-7	2.81110E-6	-8.06091E-6	-2.00867E-5
0	-3.85341E-7	1.39493E-6	4.31988E-6	-1.22833E-5
0	-9.89836E-8	-8.10720E-7	3.28345E-6	5.52830E-6
0	1.24876E-7	-5.26577E-7	-1.26214E-6	4.53948E-6
0	4.06472E-8	2.61593E-7	-1.21592E-6	-1.74066E-6
0	-4.34017E-8	1.68258E-7	4.66272E-7	-1.46904E-6
0	-1.16631E-8	-1.04270E-7	4.03209E-7	7.16833E-7
0	1.66801E-8	-3.35843E-8	-2.37014E-7	3.33624E-7
0	5.48538E-11	4.34439E-8	-8.17423E-8	-3.22897E-7
0	-5.75288E-9	-1.24774E-8	1.26476E-7	6.39121E-8
0	3.27399E-9	-4.51967E-10	-5.79396E-8	1.98179E-8
0	-8.98018E-10	1.07772E-9	1.41196E-8	-1.25323E-8
0	1.29329E-10	-2.46353E-10	-1.86407E-9	2.48309E-9
0	-7.90413E-12	1.93349E-11	1.05974E-10	-1.83600E-10

3.2. Approximating inverses of $(I+B_s)$ by its L-D-U-decomposition:

Call $(\boldsymbol{I} + \boldsymbol{B}) = \boldsymbol{B1}$, then I compute the inverse of $\boldsymbol{B1}$ by its *L-D-U*-decomposition, according to

(3.2.1.) $$B1 = L * D * R$$

where $\boldsymbol{L}$ is a lower triangular matrix, $\boldsymbol{D}$ is a diagonal matrix and $\boldsymbol{R}$ is a upper triangular matrix and the diagonals of $\boldsymbol{L}$ and $\boldsymbol{R}$ are normed to 1. Then

(3.2.2.) $$M = B1^{-1} = R^{-1} * D^{-1} * L^{-1}$$

The remarkable point is here, that $\boldsymbol{L}$,$\boldsymbol{D}$,$\boldsymbol{R}$ have constant entries for increasing dimension and thus their inverses $\boldsymbol{L}^{-1}$, $\boldsymbol{D}^{-1}$, $\boldsymbol{R}^{-1}$. The change in an entry in $\boldsymbol{M}$, which take place by increasing the dimension, is then due to one additional term, which is occuring from the single vector-product of the newly added row and column.

This is also especially useful if this multiplication is studied for the limit of infinite dimension or for points, where intermediate singularities occur in the multiplication (if one of the elements of $\boldsymbol{D}$ approaches zero, for instance)

In the following the decomposition for $(\boldsymbol{I} + \boldsymbol{B}_e)$ and $(\boldsymbol{I}+\boldsymbol{B}_e)^{-1}$ are given

3.2.3. The components

Rational arithmetic

(3.2.3.1.) $tst = (I + B_e)$

2	1	1	1	1	1
0	2	2	3	4	5
0	1/2	3	9/2	8	25/2
0	1/6	4/3	11/2	32/3	125/6
0	1/24	2/3	27/8	35/3	625/24
0	1/120	4/15	81/40	128/15	649/24

L-D-U-components

$tst = L * D * R$

(3.2.3.2.) $L =$

1	.	.	.	.	.
0	1	.	.	.	.
0	1/4	1	.	.	.
0	1/12	7/15	1	.	.
0	1/48	1/4	19/28	1	.
0	1/240	31/300	13/28	103/115	1

(3.2.3.3.) $D = diag()$

2	2	5/2	7/2	529/105	2021/276

(3.2.3.4.) $R =$

1	1/2	1/2	1/2	1/2	1/2
.	1	1	3/2	2	5/2
.	.	1	3/2	14/5	9/2
.	.	.	1	212/105	13/3
.	.	.	.	1	2695/1058
.	.	.	.	.	1

Inverses

(3.2.3.5.) $R^{-1} =$

1	-1/2	0	1/4	-1/210	-2039/6348
.	1	-1	0	4/5	-20/529
.	.	1	-3/2	8/35	750/529
.	.	.	1	-212/105	1285/1587
.	.	.	.	1	-2695/1058
.	.	.	.	.	1

(3.2.3.6.) $D^{-1} = diag()$

1/2	1/2	2/5	2/7	105/529	276/2021

(3.2.3.7.) $L^{-1} =$

1	.	.	.	.	.
0	1	.	.	.	.
0	-1/4	1	.	.	.
0	1/30	-7/15	1	.	.
0	2/105	1/15	-19/28	1	.
0	-1/92	37/690	33/230	-103/115	1

Float arithmetic:

L-D-U-components

(3.2.3.8.) L

1.00000	.	.	.	.	.
0	1.00000	.	.	.	.
0	0.250000	1.00000	.	.	.
0	0.0833333	0.466667	1.00000	.	.
0	0.0208333	0.250000	0.678571	1.00000	.
0	0.00416667	0.103333	0.464286	0.895652	1.00000

(3.2.3.9.) $D=$

2.00000	2.00000	2.50000	3.50000	5.03810	7.32246

(3.2.3.10.) R

1.00000	0.500000	0.500000	0.500000	0.500000	0.500000
.	1.00000	1.00000	1.50000	2.00000	2.50000
.	.	1.00000	1.50000	2.80000	4.50000
.	.	.	1.00000	2.01905	4.33333
.	.	.	.	1.00000	2.54726
.	.	.	.	.	1.00000

Inverses

(3.2.3.11.) R^{-1}

1.00000	-0.500000	0	0.250000	-0.00476190	-0.321204
.	1.00000	-1.00000	0	0.800000	-0.0378072
.	.	1.00000	-1.50000	0.228571	1.41777
.	.	.	1.00000	-2.01905	0.809704
.	.	.	.	1.00000	-2.54726
.	.	.	.	.	1.00000

(3.2.3.12.) D^{-1}

0.500000	0.500000	0.400000	0.285714	0.198488	0.136566

(3.2.3.13.) L^{-1}

1.00000	.	.	.	.	.
0	1.00000	.	.	.	.
0	-0.250000	1.00000	.	.	.
0	0.0333333	-0.466667	1.00000	.	.
0	0.0190476	0.0666667	-0.678571	1.00000	.
0	-0.0108696	0.0536232	0.143478	-0.895652	1.00000

3.2.4. Asymptotics for $M_{inf} = B1_{inf}^{-1} = R^{-1} * D^{-1} * L^{-1}$

Rational artihmetic

(3.2.4.1.) $_2M=$

1/2	-1/4
.	1/2

(3.2.4.2.) $_3M=$

1/2	-1/4	0
0	3/5	-2/5
0	-1/10	2/5

(3.2.4.3.) $_4M=$

1/2	-26/105	-1/30	1/14
0	3/5	-2/5	0
0	-4/35	3/5	-3/7
0	1/105	-2/15	2/7

(3.2.4.4.) $_5M=$

1/2	-131/529	-53/1587	305/4232	-1/1058
0	319/529	-206/529	-57/529	84/529
0	-60/529	319/529	-243/529	24/529
0	1/529	-254/1587	295/529	-212/529
0	2/529	7/529	-285/2116	105/529

Float arithmetic:

(3.2.4.5.) $_2M =$

0.500000	-0.250000
.	0.500000

(3.2.4.6.) $_3M =$

0.500000	-0.250000	0
0	0.600000	-0.400000
0	-0.100000	0.400000

(3.2.4.7.) $_4M =$

0.500000	-0.247619	-0.0333333	0.0714286
0	0.600000	-0.400000	0
0	-0.114286	0.600000	-0.428571
0	0.00952381	-0.133333	0.285714

(3.2.4.8.) $_5M =$

0.500000	-0.247637	-0.0333963	0.0720699	-0.000945180
0	0.603025	-0.389414	-0.107750	0.158790
0	-0.113422	0.603025	-0.459357	0.0453686
0	0.00189036	-0.160050	0.557656	-0.400756
0	0.00378072	0.0132325	-0.134688	0.198488

partial sums in M_e at dimension dim=64

(3.2.4.9.) $_{64}M =$

0.500000	-0.247037	-0.0361563	0.0642067	0.0423204	-0.0311399
0	0.603326	-0.386238	-0.118929	0.128640	0.125485
0	-0.115978	0.616023	-0.427685	-0.154026	0.174232
0	-0.00000372386	-0.158906	0.594296	-0.450245	-0.138258
0	0.00775416	-0.0116921	-0.177212	0.570928	-0.460983
0	-0.000218609	0.0134978	-0.0144036	-0.185198	0.531144

(3.2.4.10.) $_{64}M$ by Euler(2)-summation

0.500000	-0.246961	-0.0362767	0.0630316	0.0437109	-0.0240071
0	0.603298	-0.385134	-0.120426	0.119755	0.134773
0	-0.116309	0.616853	-0.423183	-0.162347	0.146958
0	-0.000133114	-0.160418	0.599577	-0.439816	-0.169389
0	0.00811902	-0.0135888	-0.180641	0.589765	-0.446365
0	0.0000731323	0.0144008	-0.0212528	-0.192534	0.584006

3.2.5. Remarkable simplifications for the computation

There are two significant simplifications for the computation of $S(s,1)$ possible.

1) We need only the second column of L_s^{-1}.

This is due the fact, that only in the second column of the result we get out interesting result.

(3.2.5.1.) $$S(s,1) = V(1)\sim * R_s^{-1} * D_s^{-1} * L_s^{-1} [,1]$$

2) We need only the first row of R_s^{-1}.

This is due the fact, that for infinite continuation

$$V(x)\sim * M_s[,1] = x - s^x + s^{s^x} - s^{s^{s^x}} + -$$

for $x=1$ and $x=0$ the results are the same except for their sign:

(3.2.5.2.)
$$\begin{aligned} S(s,1) &= V(1)\sim * M_s[,1] && = 1 - s^1 + s^{s^1} - s^{s^{s^1}} + \dots \\ S(s,0) &= V(0)\sim * M_s[,1] && = 0 - s^0 + s^{s^0} - s^{s^{s^0}} + \dots = -1 + s - s^s + s^{s^s} - \dots \\ &= - S(s,1) \end{aligned}$$

and conversely:

(3.2.5.3.) $$S(s,1) = - S(s,0)$$

In effect, we may write, using the letters $\boldsymbol{l_s}$, $\boldsymbol{d_s}$ and $\boldsymbol{r_s}$ for the inverses of $\boldsymbol{L_s}$,$\boldsymbol{D_s}$ and $\boldsymbol{R_s}$

(3.2.5.4.) $$S(s,1) = -\sum_{k=0}^{inf} rs_{0,k} * ds_{k,k} * ls_{k,1}$$

which may be replaced by Euler-summation, if the partial sums should appear as oscillating divergent.

Since previous coefficients in $\boldsymbol{r_s}$, $\boldsymbol{d_s}$, $\boldsymbol{l_s}$ do not change with increasing dimensions we are in a situation, that we can compute approximations of $S(s,1)$ by successively computed partial sums of scalar coefficients $a_{s,k}$:

(3.2.5.5.) $$_nS(s,1) = \sum_{k=0}^{n} a_{s,k}$$

which may then be analyzed in more detail for finding and removing singularities.

A table for coefficients $a_{s,k}$ for $s=1$ *to* 7 in small steps is in

http://go.helms-net.de/math/binomial_new/powertower/powertowertables.htm

These tables show the smoothness of computation of values of $S(s,1)$ for the powertowersums with bases $s=1$ *to* 7, which is beyond the bounds of convergence for a single infinite powertower. (Note that I used the letter x instead of s to make a clear difference to the capital letter S, which denotes the powertowersum, because of the low formatting-convenience in the html-generating program.)

An article, which deals specifically with a singularity at $s=1/e$, is in

http://go.helms-net.de/math/binomial_new/PowertowerproblemDocSummation.htm

The singularity occurs in intermediate computations and is due to the method itself, but it comes out, that this singularity can be removed by approximation from below and above the critical value in s.

3.3. Another check for a conventionally convergent parameter /Example

> What does your method give when the base of your iterated exponentiation
> is changed from e to sqrt(2)? When the latter base is used the terms
> are bounded and you will get an Euler-summable expression, whose Euler
> sum can be compared with what your method gives.
>

What I get is, even with (my) Eulersum of order 1 (direct summing)

```
dim = 64
%pri     B    =    F^-1*(P^-1*St2F) * P~         \\ the initial version of B, performing summing with
parameter exp(1)
%pri     Btest = dV(log(2)/2) * B       \\ test-variant which performs with parameter sqrt(2)
```

3.3.1. Check appropriateness of B_{test} : for a single step of transformation

V(1)~ ---> V(sqrt(2))~ = [1,1,1,1...] ---> [1, sqrt(2), sqrt(2)^2 , sqrt(2)^3, ...]

```
---------------------------------------------------
%pri ESum(1.0)*dV(1)*Btest              \\ example-summation , Eulersum of low order 1 is possible

  1.00000    1.41421    2.00000    2.82843    4.00000    5.65685    8.00000
```

V(1)~ ---> V(sqrt(2)$^{sqrt(2)}$)~ = [1,1,1,1...] ---> [1, sqrt(2)$^{sqrt(2)}$, (sqrt(2)$^{sqrt(2)}$)^2 , ...]

```
---------------------------------------------------
%pri ESum(1.0)*dV(1) * Btest^2          \\ example-summation , Eulersum of low order 1 is possible

  1.00000    1.63253    2.66514    4.35092    7.10299    11.5958    18.9305
```

writing *r2* as *sqrt(2)*:

V(1)~ ---> V($r2^{r2^{r2}}$)~ = [1,1,1,1...] ---> [1, $r2^{r2^{r2}}$, ($r2^{r2^{r2}}$)^2 , ($r2^{r2^{r2}}$)^3, ...]

```
---------------------------------------------------


%pri ESum(1.0) * dV(1) * Btest^3

  1.00000    1.76084    3.10056    5.45958    9.61345    16.9277    29.8070
```

3.3.2. Now check results of infinitely iterated use of B_{test} by use of M_{test}

V(1)~ --> V(1)~ - V(r2)~ + V(r2^r2)~ - V(r2^r2^r2)~ + ... -...
performed by
$V(1) \sim * M_{test} = V(1)\sim - V(r2)\sim + V(r2\hat{}r2)\sim - V(r2\hat{}r2\hat{}r2)\sim + ... -...$

```
B1test = matid(n) + Btest ;
tmp = CV_LR(B_test);                \\ compute L-D-U - components of tst for inversion
                                    \\ result in matrices CV_L  CV_D  CV_R

dim = 64   \\ compute the Inverse of (I + Btest) using different dimensions

Mtest = EMMul(VE(CV_RInv,dim)    ,  VE(CV_DInv*CV_LInv,  dim), 1.0)
             \\ compute the inverse matrix Mtest of B1test by inverses of L-D-U-components
             \\ with    dim as selectable dimension,
             \\         implicte euler-summation (here order 1.0)
             \\         in the matrix-multiplications



 \\ results
%pri ESum(1.0) * dV(1) * Mtest          \\ Eulersum(1.0)

  0.500000   0.362302    0.201330    0.0223437   -0.161780   -0.326291   -0.429142 ...
```

where in the second column is the interesting result.

Note, that for summation of the more right columns higher orders of the Euler-summation may be needed, so these numbers need be crosschecked.

3.3.3. The result of the above operations is now:

S(sqrt(2),1) ~ 0.3623015472668659
which represents
$S(sqrt(2),1) = \lim \quad 1 - sqrt(2)^1 + sqrt(2)^{sqrt(2)^1} - sqrt(2)^{sqrt(2)^{sqrt(2)^1}} + ... - ...$

which can be confirmed by ordinary Euler-summation of this series.

Note, that using ***V****(2)* as initial powerseries we get

$V(2)\sim * M_{test} = 2 * V(-1)\sim = [2,-2,2,-2,...]$

```
%pri ESum(1.0) * dV(2) * Mtest          \\ Eulersum(1.0)
     0.500000   1.00000   2.00000   4.00000   8.00000   16.0000   32.0000
```

and in the second column we have the implicte computation of:

$S(sqrt(2),2)= \lim \ 2 - sqrt(2)^2 + sqrt(2)^{sqrt(2)^2} - sqrt(2)^{sqrt(2)^{sqrt(2)^2}} + ... - ...$

S(sqrt(2),2) = 2 - 2 + 2 - 2 + ... - ....
= 1 \\ Euler-summed

which we would have expected by considering the formula before.

---

4. Citations and suggestions for further reading

I found the following resources worth reading further. To get an impression what they deal with and how I copied a bit of their contents here.

4.1. Powertower: encyclopedia...

4.1.1. Wikipedia

http://en.wikipedia.org/wiki/Tetration

Infinitely high power towers

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot}}}}}}}$$

converges to 2, and can therefore be said to be equal to 2.

The trend towards 2 can be seen by evaluating a small finite tower:

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{1.41}}}}} = \sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{1.63}}}} = \sqrt{2}^{\sqrt{2}^{\sqrt{2}^{1.76}}} = \sqrt{2}^{\sqrt{2}^{1.84}} = \sqrt{2}^{1.89} = 1.93$$

In general, the infinite power tower $x^{x^{\cdot^{\cdot}}}$ *converges for* $e^{-e} < x < e^{1/e}$.

For arbitrary real r *with* $0 < r < e$, *let* $x = r^{1/r}$, *then the limit is* r.

There is no convergence for $x > e^{1/e}$ *(max of* $r^{1/r}$ *is* $e^{1/e}$*).*

This may be extended to complex numbers z *with the definition:*

$$z^{z^{\cdot^{\cdot^{\cdot}}}} = -\frac{W(-\ln z)}{\ln z}$$

where W(z) represents Lambert's W function.

4.1.2. Mathworld

http://mathworld.wolfram.com/PowerTower.html

The value of the infinite power tower $h(z) = z \uparrow\uparrow \infty = z^{z^{\cdot^{\cdot^{\cdot}}}}$, *where* z^{z^z} *is an abbreviation for* $z^{(z^z)}$, *can be computed analytically by writing*

$$z^{z^{\cdot^{\cdot^{\cdot}}}} = h(z) \qquad (11)$$

taking the logarithm of both sides and plugging back in to obtain

$$z^{z^{\cdot^{\cdot^{\cdot}}}} \ln z = h(z) \ln z = \ln[h(z)]. \qquad (12)$$

Solving for $h(z)$ *gives*

$$h(z) = -\frac{W(-\ln z)}{\ln z}, \qquad (13)$$

where $W(z)$ *is the* Lambert W-function *(Corless* et al. *1996).*

$h(z)$ *converges* iff $e^{-e} \le x \le e^{1/e}$ *(*$0.0659 \le x \le 1.4446$*; Sloane's* A073230 *and* A073229*), as shown by Euler (1783) and Eisenstein (1844) (Le Lionnais 1983; Wells 1986, p. 35).*

Knoebel (1981) gave the following series for $h(z)$

$$h(z) = 1 + \ln z + \frac{3^2 (\ln z)^2}{3!} + \frac{4^3 (\ln z)^3}{4!} + \ldots \qquad (14)$$

(Vardi 1991),

The special value $h(i)$ *is given by*

$$i^{i^{\cdot^{\cdot^{\cdot}}}} = -\frac{W(-\ln i)}{\ln i} \qquad (16)$$

$$= \frac{2i}{\pi} W\left(-\tfrac{1}{2}\pi i\right) \qquad (17)$$

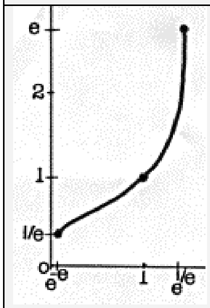
$$\approx 0.438283 + 0.3605924\, i$$

4.2. Some Critical Points of the....

http://www.faculty.fairfield.edu/jmac/ther/tower.htm

***Some Critical Points of the Hyperpower Function** $y = {}_x x^{x^{\cdot^{\cdot}}}$*

Introduction to the infinitely iterated exponential function $y = f(x) = {}_x x^{x^{\cdot^{\cdot}}}$



The interval of convergence for the HYPERPOWER function $y = {}_x x^{x^{\cdot^{\cdot}}}$

This has been established by Knoebel [I] and Mitchelmore [2] in their treatments of iterated exponentials.

Knoebel introduces the notation $^n x$ for these 'hyperpowers' where x, x^x, ${}_x x^{x^{\cdot}}$, . . . are written ^1x, ^2x, ^3x . . . and y in the example above would be the limit of the sequence $y = \lim {}^n x$. as n approaches infinity. He then proves that the sequence $\{^n x\}$ converges only in the interval

$e^{-e} < x < e^{1/e}$

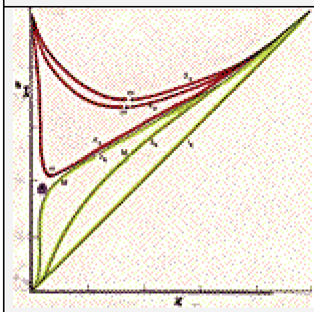
The members of this sequence of hyperpowers offer an interesting study in critical points and calculus teachers seeking computer-oriented problems which would challenge their brighter students may find the sequence useful. This is an attempt to organize the tools to help a teacher present this sequence $\{^n x\}$.

Comparison of the three sequences

Just as the computer has greatly facilitated the computation of the derivatives to study these three sequences, so also the computer can be used to check the values in the interval and outside. So if $x = 2$ this is in the interval repeated exponentiation and will give the answer $y = 2^{1/2}$. But for $x = 3$ we get $y = 2.4780$ and not $3^{1/3}$. This verifies an earlier claim. In fact for a value close to zero, for example, $x = .01$ we find that for after a large number of exponentiations

$^O(.01) = 0.0130925$ and $^E(.01) = .9414883$.
Also $(.01)^{.9414883} = .0130925$ and $(.01)^{.0130925} = .9414883$.

This illustrates that the sequence $\{^n.01\}$ is diverging (by oscillation) but the two subsequences $\{^O(.01)\}$ and $\{^E(.01)\}$ are converging to two distinct limits .0130925 and .9414883.



Points on the graphs of certain hyperpowers using values of n = 1, 2, 3, 4, . . . O, E, O, E, . . . in the interval (0,1). The sequence of functions $\{^n x\}$ converges as n-> only in the interval $[e^{-e}, e^{1/e}]$ where $e^{-1} < y < e$. At the bifurcation point $*(e^{-e}, e^{-1})$ the sequence $\{^n x\}$ divides into two (green) odd $\{^O x\}$ and (red) even $\{^E x\}$ sequences each of which converges in the interval $(0, e^{-e})$

4.3. Some tetration identities

http://tetration.itgo.com/ident.html

Integer Values of Tetration

- ${}^{-2}x = \pm\infty$
- ${}^{-1}x = 0$
- ${}^{0}x = 1$
- ${}^{1}x = x$
- ${}^{2}x = x^x$
- ${}^{\infty}x = -W(-log(z)) / log(z)$ — *in terms of the* **product-logarithm**, *the Lambert-W function.*
- ${}^{-\infty}x = {}^{\infty}x$ — *is a different branch of the infinite iterated exponential.*

Integer Values of Iterated Exponentiation

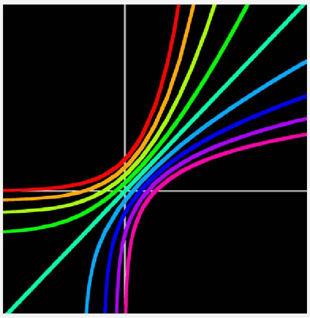
- ${}^{-n}x^a = log^n{}_x(a)$
- ${}^{-2}x^a = log_x(log_x(a))$
- ${}^{-1}x^a = log_x(a)$
- ${}^{0}x^a = a$
- ${}^{1}x^a = x^a$
- ${}^{2}x^a = x\text{^}x\text{^}a$ — *the graph of which made this website's logo*
- ${}^{n}x^a = exp^n{}_x(a)$
- ${}^{\infty}x^a = {}^{\infty}x$ — *in terms of the* **infinite iterated exponential**.
- ${}^{-\infty}x^a = {}^{\infty}x$ — *is a different branch of the infinite iterated exponential.*

The Infinite Iterated Exponential and Company

This table shows an interesting relationship: that these functions are so similar, they can all be expressed in terms of each other.

• ${}^{\infty}x$	$= -W(-log(z)) / log(z)$	$= 1 / srt_2(1 / x)$	*the infinite iterated exponential*
• ${}^{\infty}(\mathbf{e}^{-x})x$	$= W(x)$	$= x / srt_2(\mathbf{e}^x)$	*the product-logarithm*
• $1 / {}^{\infty}(1 / x)$	$= log(z) / W(log(z))$	$= srt_2(x)$	*the square super-root*

About the Graphs



${}^{y}e^{a}$ for y in { 1, ¾, ½, ¼, 0, -¼, -½, -¾, -1 }

5. References

From my project for this article:

SumLikePow Summing of like powers
http://go.helms-net.de/math/binomial_new/04_3_SummingOfLikePowers.pdf

PowTowCrit Critical point for this method of summation at *s=exp(-1)*
http://go.helms-net.de/math/binomial_new/PowertowerproblemDocSummation.htm

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[Project-Index] http://go.helms-net.de/math/binomial_new/index

[Intro] http://go.helms-net.de/math/binomial_new/00_0_Intro.pdf

[List] http://go.helms-net.de/math/binomial_new/00_1_ListOfMatrices.pdf

[binomialmatrix] http://go.helms-net.de/math/binomial_new/01_1_binomialmatrix.pdf

[signed binomial] http://go.helms-net.de/math/binomial_new/01_2_signedbinomialmatrix.pdf

[Stirlingmatrix] http://go.helms-net.de/math/binomial_new/01_3_stirling.pdf

[Gaussmatrix] http://go.helms-net.de/math/binomial_new/01_5_gaussmatrix.pdf

[GenBernRec] (Generalized Bernoulli-recursion)
http://go.helms-net.de/math/binomial_new/02_2_GeneralizedBernoulliRecursion.pdf

[SumLikePow] (Sums of like powers)
http://go.helms-net.de/math/binomial_new/04_3_SummingOfLikePowers.pdf

[Hasse] http://go.helms-net.de/math/binomial_new/10_2_recihasse.pdf

[Vandermonde] http://go.helms-net.de/math/binomial_new/10_3_InverseVandermonde1.pdf

Projekt ***Bernoulli-numbers***, first versions of the above, contain a ***first rough exploratory*** course but already cover most central topics and contain also the basic material about **Gp** and **Gm** which is still missing in the above list:

[Bernoulli] http://go.helms-net.de/math/binomial_new/bernoulli_en.pdf

[Summation] http://go.helms-net.de/math/binomial_new/pmatrix.pdf

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Gottfried Helms, Kassel.

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