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Intro & Notation

Abstract: an overview about the concept of this collection of articles is given as well as notational conventions. Also the most prominent vectors and matrices are introduced.

Contents:

1. Introduction

In this collection of articles I present some known identities about binomial-coefficients, Bernoulli-, Stirlingnumbers (and some other), as well as some heuristics about compositions of that numbers in context with geometric series (powerseries), harmonic- and more general: zeta-series.

I present the results in terms of a "*toolbox*" of matrices and vectors of infinite dimension.

Most of the matrices are of lower triangular shape, so that the common matrix-operations like addition, right-multiplication and inversion, even finding of Eigensystems etc are based on finite operations. Convergence/divergence is then an issue when leftmultiplication is applied and/or infinite square-matrices like the Vandermondematrix ZV are used - such situations must be considered specifically.

Thus a chapter about divergent summation (for instance Cesaro and Eulersummation) is appended where the matrix-representation can express some of the common techniques in a very concise and instructive manner. Some selected results of divergent sums are already included, which exhibit some much interesting relations, for instance the sums of bernoulli-numbers and some variants of that.

Expressing relations in terms of matrix-products is a rich source of identities, which are usually expressed as identities of sums of products of coefficients (like the "sum of products of binomial-coefficients and bernoulli-numbers"), since a matrix-product of infinite dimensioned matrices gives infinitely many such sums for each column (and each row) of the result-vector/matrix in one shot.

Over this collection of articles I'll often save the effort to express such relations in the conventional summation-notation; they may be simply reproduced; in some cases however, where the expressions are very common or special interesting, I'll write them out for convenience of the reader.

1.1. notational conventions

The *toolbox* contains the following vectors and matrices, with the following conventions:

1. vectors are primarily assumed as column-vector
2. the transpose-symbol *"~"* is used (as in the openly available number-theoretic program Pari/GP) for convenient translation of the formulae into the programming language, and to prevent confusion with the apostroph for the derivative, which shall also be used in some chapters.
3. the indices r,c for rows and columns are always assumed as beginning at zero
4. the superscript prefix ^{d} is added, if a vector is assumed as the coefficients of a diagonal matrix
5. the Hadamard-product of two matrices (elementwise multiplication) is denoted by "#": $A \# B = C$
6. matrices are generally assumed as lower triangular matrices (some exceptions)

1.2. Vectors

Basic vectors are

Powerseries	$V(x) = [1, x, x^2, x^3,...]$ ~
harmonic/Zeta-like series	$Z(s) = [1,1/2^s,1/3^s,1/4^s,...]$ ~
Summing vector	$E= V(1) = Z(0) = [1,1,1,1,...]$~
Factorials	$Fac(s) = [1, 1, 2!^s, 3!^s, 4!^s,...]$ ~
Bernoulli-numbers	$B = [\beta_0, \beta_1, \beta_2,...]$ where β_k are the k'th bernoulli-numbers
Bernoulli-numbers	$B_+ = [\beta_0, \beta_1, \beta_2,...]$ where $\beta_1 = +1/2$

I also use for convenience ***J*** and ***I*** for the vectors resp diagonalmatrices

Identity-matrix	$I = diag(1,1,1,...)$
alt.Identity	$J = {}^dV(-1) = diag([1,-1,1,...])$

1.3. Matrices

Basic lower-triangular matrices of number-theoretic coefficients are the following (more detailed description in the resp. chapter):

Pascalmatrix	$P := P_{r,c} = binomial(r,c)$ if $r>=c$
column-signed	$P_j := P * J$
row-signed	${}_jP:= J * P$

matrices representing the Bernoulli-polynomials

$BY:= BY_{r,c} = \beta_{r-c} * binomial(r,c)$ if $r>=c$
$BY_m :=$ BY, using the standard setting $\beta_1 = -1/2$
$BY_p :=$ similar to BY, only using $\beta_1 = +1/2$

G-matrices $G:= G_{r,c} = \beta_{r-c} * binomial(r,c) /(c+1)$ if $r>=c$
$G_m :=$ G, using the standard setting $\beta_1 = -1/2$
$G_p :=$ similar to G, only using $\beta_1 = +1/2$

Stirling-matrices

2'nd kind	$St2:= St2_{r,c} = stirling\_kind2_{r,c}$ if $r>=c$
1'st kind	$St1:= St1_{r,c} = stirling\_kind1_{r,c}$ if $r>=c$

the Vandermondematrix ZV as column-concatenation of Z-vectors Z(0),Z(-1),Z(-2),...

$ZV := ZV_{r,c} = (r+1)^c$

1.4. Examples for sums of numbers by matrix-products

Summing a powerseries:

$$E\sim * V(1/x) = V(1)\sim * V(1/x) = \Sigma_{r=0..inf} (1/x^r) = 1/(x-1)$$

$$V(1)\sim * V(1/x) = \Sigma_{r=0..inf} (1/x^r) = 1/(x-1)$$

$$x = \begin{bmatrix} 1 \\ 1/x \\ 1/x^2 \\ 1/x^3 \\ 1/x^4 \\ 1/x^5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/(1-x) \end{bmatrix}$$

Simple sign-inversion:

$$J * V(x) = V(-x)$$

$$2 = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \\ 32 \end{bmatrix}$$

$$dJ = \begin{bmatrix} 1 & . & . & . & . & . \\ . & -1 & . & . & . & . \\ . & . & 1 & . & . & . \\ . & . & . & -1 & . & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 4 \\ -8 \\ 16 \\ -32 \end{bmatrix}$$

Using ***P*** and **V***(x)* means to apply the binomial-rules:

$$P * V(x) = V(1+x)$$

$$\Sigma_{c=0..r} binomial(r,c)*x^c = (1+x)^r$$

$$2 = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \\ 32 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 9 \\ 27 \\ 81 \\ 243 \end{bmatrix}$$

Use of column-signed Pascalmatrix $\boldsymbol{P}_j$

$$P_j * V(x) = V(1-x)$$

$$\Sigma_{c=0..r} (-1)^c binomial(r,c)*x^c = (1-x)^r$$

$$2 = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \\ 32 \end{bmatrix}$$

$$Pj = \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & -1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ 1 & -3 & 3 & -1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Eigenvector-relations:

$P_j * V(1/2) = V(1/2)$
saying $V(1/2)$ is an eigenvector of P_j

$\Sigma_{c=0..r}\ binomial(r,c)*(-1/2)^c = 1/2^r$

/2

1
1/2
1/4
1/8
1/16
1/32

Pj

1	.	.	.	.	.
1	-1	.	.	.	.
1	-2	1	.	.	.
1	-3	3	-1	.	.
1	-4	6	-4	1	.
1	-5	10	-10	5	-1

/2

1
1/2
1/4
1/8
1/16
1/32

Powers of matrices:

Powers of P

$P^n * V(x) = V(n+x)$

1

1
1
1
1
1
1

P^3

1	.	.	.	.	.
3	1	.	.	.	.
9	6	1	.	.	.
27	27	9	1	.	.
81	108	54	12	1	.
243	405	270	90	15	1

4

1
4
16
64
256
1024

Summation of powerseries (includes to handle also their derivatives)

$lim_{r\text{->}oo}\ (1/2*V(1/2)\sim * P) = V(1)\sim$

P

	1				
	1	1			
	1	2	1		
	1	3	3	1	
	1	4	6	4	1
	1	5	10	10	5
lim r->oo=	1	1	1	1	1

1/2	1/4	1/8	1/16	1/32	1/64

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a slightly more general expression of power-series-summation:

$lim_{r\text{->}oo}\ (1/x*V(1/x)\sim * P^{x-1}) = V(1)\sim$

P^2

	1				
	2	1			
	4	4	1		
	8	12	6	1	
	16	32	24	8	1
	32	80	80	40	10
lim r->oo=	1	1	1	1	1

1/3	1/9	1/27	1/81	1/243	1/729

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2. References

[Project-Index] http://go.helms-net.de/math/binomial/index

[Intro] http://go.helms-net.de/math/binomial/intro.pdf

[binomialmatrix] http://go.helms-net.de/math/binomial/01_1_binomialmatrix.pdf

Gottfried Helms, 13.12.2006