Identities involving binomial-coefficients,
Bernoulli- and Stirlingnumbers

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Some Zeta-identities, which occur from the summations of the binomial matrix P and bernoullian matrices G_p and G_m

Abstract: Some simple identities of zeta-sums are given. They occur as results from a Borel-Summation over the bernoulli-numbers connected to some known matrix-identities.

Gottfried Helms Version: 17.Feb.07

1. Intro:

In a discussion-thread in new:sci.math Robert Israel answered to my question concerning the possibility of (divergent) summation of Bernoulli-numbers and gave a solution which involved the values of the zeta-function at positive integer arguments using Borel-summation. From this evolve then further identities for zeta-sums or sums of the fractional parts of zeta-values, some of them not shown in my books or in internet ressources.

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2. Preliminaries

2.1. Notation, Definitions:

I present the discussion with my current notation in my matrix-identities-project.

(2.1.1.) All matrices are understood as of infinite dimension

(2.1.2.) vectors are assumed as column-vectors (again with infinite dimension)

(2.1.3.) row/column-indices of matrices/ vectors are *r (row)* and *c(col)* and are zero-based

(2.1.4.) from the convention of the Pari/GP-softwareprogram I take the notation "~" for matrix-transposition.

Vectors

A vector used for powerseries-representation:

(2.1.5.) $V(x) = [1, x, x^2, x^3,....]\sim$

a shortcut for a simple summation:

(2.1.6.) $E = V(x) = [1, 1, 1, 1,....]\sim$

so that the sum of the series

$$1*1 + 1*1/2 + 1*1/4 + 1*1/8 + ... = \Sigma_{r=0..oo} 1/2^r$$

can be written as

$$E\sim * V(1/2) = \Sigma_{r=0..oo} 1/2^r = 2$$

The alternating sum can be written using $V(-1)$, I introduce the diagonal-matrix ***J***:

(2.1.7.) $J = V(-1) = [1,-1,1,-1,...]$

A vector, which represents a zeta-series is noted as :

(2.1.8.) $Z(s) = [1^{-s}, 2^{-s}, 3^{-s}, ...]\sim$
$Z := Z(1)$ *for brevity*

The factorial vector is noted as

(2.1.9.) $F(s) = [0!^s, 1!^s, 2!^s, ...]\sim$
$F = F(1)$ *for brevity*

and, for instance

$$F(-1)\sim * V(2) = 1/0! + 2/1! + 4/2! + ... = exp(2)$$

Generally, if these vectors are used as diagonalmatrix in a formula, then I prefix them with a superscript ^{d} :

(2.1.10.) $\quad {}^dJ = diag(J)$

(2.1.11.) $\quad {}^dI = diag(E) \qquad$ *(Identity-matrix)*

However, for the most common matrices and in longer formulae I shall omit the prefix for brevity, if the diagonal use is clear from the context; so generally I write simply ***I*** or ***J***.

Matrices

Basic matrices are the pascalmatrix P:

$$P = \begin{bmatrix} 1 & . & . & . \\ 1 & 1 & . & . \\ 1 & 2 & 1 & . \\ 1 & 3 & 3 & 1 \end{bmatrix}$$

the signed pascalmatrix $P_j = P * {}^dJ$

$$P_j = \begin{bmatrix} 1 & . & . & . \\ 1 & -1 & . & . \\ 1 & -2 & 1 & . \\ 1 & -3 & 3 & -1 \end{bmatrix}$$

It is worth to mention that

$$P^{-1} = J * P * J = J * P_j$$

and thus $\boldsymbol{P_j}$ is its own inverse.

The eigenmatrices of P_j , G_p and G_m with the properties:

(2.1.12.) $\quad G_p = J * G_m * J$

(2.1.13.) $\quad P_j = G_p * J * G_p^{-1}$

(2.1.14.) $\quad J{*}P = G_m * J * G_m^{-1}$

or

(2.1.15.) $\quad P_j \ G_p = G_p * J$

(2.1.16.) $\quad J{*}P \ \ G_m = G_m * J$

(2.1.17.) $\quad P \quad = G_p \ G_m^{-1}$

$$* \begin{bmatrix} 1 & . & . & . \\ 1/2 & 1/2 & . & . \\ 1/6 & 1/2 & 1/3 & . \\ 0 & 1/4 & 1/2 & 1/4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & . & . & . \\ 1 & -1 & . & . \\ 1 & -2 & 1 & . \\ 1 & -3 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & . & . & . \\ 1/2 & -1/2 & . & . \\ 1/6 & -1/2 & 1/3 & . \\ 0 & -1/4 & 1/2 & -1/4 \end{bmatrix}$$

G_p and G_m contain the Bernoulli-numbers $ß_r$ similar to the bernoulli-polynoms, for instance the top-left submatrix of G_p is:

(2.1.18.) $\quad G_p := g_{p\,r,c} = binomial(r,c) * ß_{r-c} / (c+1)$
using $b_1 = +1/2$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1/2 & 1/2 & . & . & . & . \\ 1/6 & 1/2 & 1/3 & . & . & . \\ 0 & 1/4 & 1/2 & 1/4 & . & . \\ -1/30 & 0 & 1/3 & 1/2 & 1/5 & . \\ 0 & -1/12 & 0 & 5/12 & 1/2 & 1/6 \end{bmatrix}$$

The entries of a row represent the coefficients of the integral of the r'th Bernoulli-polynomial; $\boldsymbol{G_m}$ with the original setting $ß_1=-½$, $\boldsymbol{G_p}$ with the alternate setting $ß_1=+½$ and also simply $\boldsymbol{G_m = J\, G_p\, J}$.

If **B** is the matrix of bernoulli-polynomials, then

(2.1.19.) $G_m = B * {}^dZ(1)$

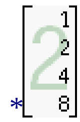
The definition of the entries is exactly:

(2.1.20.) $G_{m\ r,c} := ß_{r-c} * binomial(r,c) / (c+1)$ // $ß_1 = -½$
(2.1.21.) $G_{p\ r,c} := ß_{r-c} * binomial(r,c) / (c+1)$ // $ß_1 = +½$

2.2. a couple of important properties of P and P_j :

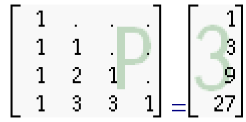
The most simple propety of **P** is, to transform a powerseries according to the binomial-rule

(2.2.1.) $P * V(s) = V(s+1)$ for all complex s



for instance

$P * V(2) = V(3)$



The column-signed matrix P_j has the more interesting property to have an invariant vector, since from

(2.2.2.) $P * V(c) = V(c + 1)$

follows first:

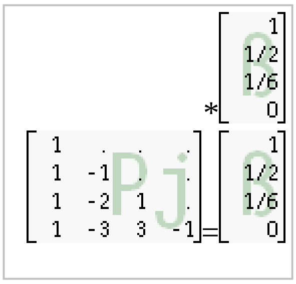
(2.2.3.) $P \ * V(-c) = V(-c+1) = V(1 - c)$
$P\ J * V(c) = V(1 - c)$

and then the symmetry about the center:

$P_j * V(½) = P * V(-½) = V(-½+1) = V(½)$
(2.2.4.) $P_j * V(½) = V(½)$

and $V(½)$ is an invariant with respect to the signed binomial-transformation, or shorter: w.r. P_j-transformation. $V(½)$ is thus an eigenvector of Pj associated with the eigenvalue 1.

Even more interesting is P_j , in that among the infinite number of eigenvectors, which occur from the infinite multiplicity of this eigenvalue, is also the vector B^+ containing the Bernoulli-numbers (B^+ meaning $ß_1$ is $+1/2$ assumed), so that



(2.2.5.) $Pj * B^+ = B^+ * 1$
(2.2.6.) ${}_dJ * P * B^- = B^- * 1$

In [Binomialmatrix] I have considered some more properties; the reader may find the explicite derivation of this property, and moreover, how a full set of eigenvectors can be found, namely the matrices G_p and G_m .

3. Sums of Bernoulli-numbers, ζ()- and fractional parts of ζ()-values

3.1. Sums of Bernoulli-numbers give ζ()-values

Here I present the relation between the matrices G_p and G_m and the values of the zeta-function, as Robert Israel has shown it in the *sci.math* discussion *(see Appendix)*.

The basic information is here:

(3.1.1.) $E\sim * G_m = [\zeta(2)-1, \zeta(3)-1, \zeta(4)-1, \dots]$
(3.1.2.) $E\sim * G_p = [\zeta(2) \quad, \zeta(3) , \quad \zeta(4) \quad, \dots]$

Example:

$$E\sim * G_m = Zet_m\sim \qquad * \begin{bmatrix} 1 & . & . & . & . & . \\ -1/2 & 1/2 & . & . & . & . \\ 1/6 & -1/2 & 1/3 & . & . & . \\ 0 & 1/4 & -1/2 & 1/4 & . & . \\ -1/30 & 0 & 1/3 & -1/2 & 1/5 & . \\ 0 & -1/12 & 0 & 5/12 & -1/2 & 1/6 \end{bmatrix} \; Gm$$

$$[\ \dots \quad \dots \ ]$$

$$\lim_{r\to oo} [1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1] = [z2-1 \;\; z3-1 \;\; z4-1 \;\; z5-1 \;\; z6-1 \;\; z7-1]$$

z2-1 etc is here ζ(2)-1 etc

which can be found by Borel-summation:

(3.1.3.) $$\zeta(2)=\sum_{r=0}^{oo} ß_r^{+} \qquad \qquad \zeta(2)-1=\sum_{r=0}^{oo} ß_r^{-} \quad (Borel)$$

and appropriate modifications. (*see 4.1 what is*).

My question concerned the column-sums of $\mathbf{G_m}$; which led to the result in *(3.1.1)*; from there more modifications can be derived[1]: The replacement of $\mathbf{G_m}$ by $\mathbf{G_p}$ requires for each column only the double addition of the reverse-signed value of $ß_1$, thus simply an addition of the unit in each column-sum:

(3.1.4.) $ß^{+}_1 - ß^{-}_1 = 1/2 - (-1/2) = 1$

For brevity, I note the vectors

(3.1.5.) $[\zeta(2) \quad , \ \zeta(3) \quad , \quad \zeta(4) \quad ,\dots] \ = Zet$
(3.1.6.) $[\zeta(2)-1, \ \zeta(3)-1, \ \ \zeta(4)-1,\dots] \ = Zet_m$

where the second row contains just the fractional part of the zeta-values:

(3.1.7.) $[\{\zeta(2)\} , \{ \zeta(3)\} , \ \{ \zeta(4)\} ,\dots] \ = Zet_m$

whose sum is convergent and is thus easier to handle in analytical formulae, where convergence is assumed.

Here some variants of these zeta-sums are of interest.

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[1] the Borel-summation is regular in the sense, that finitely many arithmetical modifications do not affect the consistency of the result, for instance the adding of the unit as done in (3.1.3)

3.2. Sum of {ζ(2)} + {ζ(3)} + {ζ(4)} +....

The first variant can be constructed by the expansion of the trivial matrix-product:

$$E\sim \; = E\sim \; * I$$

into

(3.2.1.) $$E\sim \; = E\sim \; * (G_m * G_m^{-1})$$

I prefer G_m and G_m^{-1} here over G_p und G_p^{-1}, since the sum of the sum-vectors is also convergent. This sum occurs in the postmultiplication by G_m^{-1} - if we would use the G_p matrices we had already divergent sums at this point.

Interchanging the order of summation and applying the already known sum-formula for the columns of G_m giving Zet_m:we have:

(3.2.2.) $$E\sim \; = (E\sim \; * G_m) * G_m^{-1}$$

(3.2.3.) $$E\sim \; = \; Zet_m \; * G_m^{-1}$$

$$Zet_m\sim \; * G_m^{-1} = E\sim$$

$$* \begin{bmatrix} 1 & . & . & . \\ 1 & 2 & . & . \\ 1 & 3 & 3 & . \\ 1 & 4 & 6 & 4 \end{bmatrix}$$

$$\begin{bmatrix} z2-1 & z3-1 & z4-1 & z5-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

G_m^{-1} has the form:

(3.2.4.) $$G_m^{-1} = \begin{bmatrix} 1 & . & . & . \\ 1 & 2 & . & . \\ 1 & 3 & 3 & . \\ 1 & 4 & 6 & 4 \end{bmatrix}$$

and using the first column, for instance, is identical to using the simple summing-vector E.

Multiplication of Zet_m with the first column in G_m^{-1} must give 1, because of the first entry in the result (which equals E according to 3.2.2):

(3.2.5.)
$$Zet_m \sim \; * E = E\sim_0 \quad = 1$$
$$\sum_{r=0}^{oo} (\zeta(2+r)-1)*1 \quad = 1$$

and, since $\zeta(2+r)-1$ means also the fractional part $\{\zeta(2+r)\}$ of $\zeta(2+r)$ we can say:

(3.2.6.) *the sum of the fractional parts of ζ(2) up to ζ(inf) is = 1*

or, using the $\{x\}$-notation for the fractional part of a number x:

(3.2.7.) $$\{\zeta(2)\} + \{\zeta(3)\} + \{\zeta(4)\} + \{\zeta(5)\} + \ldots. = 1$$

More over, because we deal with matrices, the same result is valid for each of the columns in G_m and we get the infinite list of identities:

(3.2.8.)
$$\begin{aligned} 1 &= \sum_{r=0}^{oo} \{\zeta(2+r)\} &&= 1*\{\zeta(2)\}+1*\{\zeta(3)\}+1*\{\zeta(4)\}+\dots \\ 1 &= \sum_{r=1}^{oo} \{\zeta(2+r)\}*\binom{r+1}{1} &&= 2*\{\zeta(3)\}+3*\{\zeta(4)\}+4*\{\zeta(5)\}+\dots \\ 1 &= \sum_{r=2}^{oo} \{\zeta(2+r)\}*\binom{r+1}{2} &&= 3*\{\zeta(4)\}+6*\{\zeta(5)\}+10*\{\zeta(6)\}+\dots \\ &\dots \end{aligned}$$

If we reorder the formulae and rearrange the results, the beautiful result occurs:

(3.2.9.)
$$\begin{aligned} 1\{\zeta(2)\} +1\{\zeta(3)\} + 1\{\zeta(4)\} + 1\{\zeta(5)\} + \dots &= 1 \\ 1\{\zeta(2)\} +2\{\zeta(3)\} + 3\{\zeta(4)\} + 4\{\zeta(5)\} + \dots &= \zeta(2) \\ 1\{\zeta(3)\} + 3\{\zeta(4)\} + 6\{\zeta(5)\} + \dots &= \zeta(3) \\ 1\{\zeta(4)\} + 4\{\zeta(5)\} + \dots &= \zeta(4) \end{aligned}$$

The sum of the original zeta-values cannot simply be given, since each value is greater than 1 and the partial sums would diverge.

3.3. Alternating sum {ζ(2)} - {ζ(3)} + {ζ(4)} -....

The alternating sum is "technically" interesting, because in the formula of the previous chapter

(3.3.1.) $$E\sim \;=\; Zet_m\sim * G_m^{-1}$$

the diagonalmatrix J is inserted, and the result X is not obvious:

(3.3.2.) $$X\sim \;=\; (Zet_m\sim *J) * G_m^{-1}$$

But recalling, that $Zet_m\sim \;= E\sim * G_m$ we have:

(3.3.3.) $$X\sim \;=\; (E\sim * G_m) * (J * G_m^{-1})$$

and the rhs is the known eigensystem from ${}_JP \;=\; G_m * \; J * G_m^{-1}$ so (3.3.3) is simply:

$$\begin{aligned} &(3.3.4.) & X\sim &= E\sim * G_m * J * G_m^{-1} \\ & & &= E\sim * J * P \\ &(3.3.5.) & &= V(-1)\sim * P \end{aligned}$$

which means, that the sought result $X\sim$ is nothing else than the alternating summed columns of the pascalmatrix P.

These sums are not convergent, but values can be assigned by the techniques of divergent summation. The domain for the sums in

(3.3.6.) $Y\sim \;=\; V(x)\sim * P$

can meaningfully be extended to all $x<>1$ according to

(3.3.7.) $$Y\sim \;=\frac{1}{(1-x)}[1, \quad \frac{x}{(1-x)}, \qquad (\frac{x}{1-x})^2, ...]$$

For the given case $x=-1$ this is then:

(3.3.8.) $$Y\sim \;= ½ \; * \; [1, -1/2, \; 1/4, \; -1/8, ...].$$
$$Y\sim \;=\; ½ * V(-½)\sim$$

The complete derivation:

$$\begin{aligned} X\sim &= Zet_m\sim * J * G_m^{-1} \\ &= E\sim * G_m * J * G_m^{-1} \\ &= E\sim * J * P \\ &= V(-1)\sim * P \qquad \textbf{(3.3.9.)} \\ &= ½ \; V(-½)\sim \end{aligned}$$

thus

(3.3.10.)	$Zet_m\sim * J * \; G_m^{-1}$	$= ½ \; V(-½)$
and		
(3.3.11.)	$Zet_m\sim * (J * \; G_m^{-1} \; J) = Zet_m\sim * \; G_p^{-1}$	$= ½ \; V(½)$

The alternating sums of the fractional parts of zeta-values of $\zeta(2)$ to $\zeta(inf)$ are more explicite as follows (according to the different columns in $\boldsymbol{G_m^{-1}}$):

(3.3.12.)
$$\begin{array}{ll|ll} \sum_{r=0}^{oo}(-1)^r*\{\zeta(2+r)\} & =\frac{1}{2} & 1\{\zeta(2)\}-1\{\zeta(3)\}+1\{\zeta(4)\}-... & =\frac{1}{2} \\ \sum_{r=1}^{oo}(-1)^r*\{\zeta(2+r)\}*\binom{r+1}{1} & =-\frac{1}{4} & 2\{\zeta(3)\}-3\{\zeta(4)\}+4\{\zeta(4)\}-... & =\frac{1}{4} \\ \sum_{r=2}^{oo}(-1)^r*\{\zeta(2+r)\}*\binom{r+1}{2} & =\frac{1}{8} & 3\{\zeta(4)\}-6*\{\zeta(5)\}+10\{\zeta(6)\}-... & =\frac{1}{8} \\ ... & & ... & \end{array}$$

another display: Example:

$$\begin{array}{ll} & \begin{bmatrix} 1 & . & . & . \\ -1 & 2 & . & . \\ 1 & -3 & 3 & . \\ -1 & 4 & -6 & 4 \end{bmatrix} \\ Zet_m\sim * (J \; G_m^{-1} \; J) = ½ \; V(½) \sim & * \\ & [\; ... \quad ... \quad ... \quad ...] \\ lim \quad [\; z2-1 \quad z3-1 \quad z4-1 \quad z5-1 \;] & = [\; 1/2 \quad 1/4 \quad 1/8 \quad 1/16 \;] \end{array}$$

3.4. A differently weighted sum 1*{ζ(2)} +2* {ζ(3)} + 3 * {ζ(4)} + ...

Precisely said, the following summation assigns an additional cofactor to the previous example. In matrix-notation expressed with an additional diagonal zeta-vector ${}^{d}\mathbf{Z}(-1)$:

(3.4.1.) $$X\sim = Zet_m\sim * {}^{d}Z(-1) * G_m^{-1}$$

I don't have the analysis for these sums, but with Euler-summation I get the interesting and much likely proposition:

(3.4.2.)
$$\begin{aligned}
1\{\zeta(2)\} + 2\{\zeta(3)\} + 3\{\zeta(4)\} + 4\{\zeta(5)\} + \dots &= 1*(\zeta(2)+1) - 1 \\
2*2\{\zeta(3)\} + 3*3\{\zeta(4)\} + 4*4\{\zeta(5)\} + \dots &= 2*(\zeta(3)+1) - 1 \\
3*3\{\zeta(4)\} + 4*6\{\zeta(5)\} + \dots &= 3*(\zeta(4)+1) - 1 \\
4*4\{\zeta(5)\} + \dots &= 4*(\zeta(5)+1) - 1
\end{aligned}$$

thus

(3.4.3.) $$Y\sim = [\ 1*(\zeta(2)+1)\ ,\ 2*(\zeta(3)+1)\ ,\ 3*(\zeta(4)+1)\ ,\dots\ ] - 1$$
(3.4.4.) $$= [\ \{\zeta(2)\}+2\ ,\ 2\{\zeta(3)\}+4\ ,\ 3*(\zeta(4)+1)\ ,\dots\ ] - 1$$
(3.4.5.) $$= (Zet\sim + (Z(0)\sim - Z(1)\sim)) * {}_{d}Z(-1)$$

or rewritten:

(3.4.6.)
$$\begin{aligned}
1\{\zeta(2)\} + 2\{\zeta(3)\} + 3\{\zeta(4)\} + 4\{\zeta(5)\} + \dots &= 1\,\zeta(2) \\
1\{\zeta(2)\} + 2*2\{\zeta(3)\} + 3*3\{\zeta(4)\} + 4*4\{\zeta(5)\} + \dots &= 2\,\zeta(3) + 1\,\zeta(2) \\
2\{\zeta(3)\} + 3*3\{\zeta(4)\} + 4*6\{\zeta(5)\} + \dots &= 3\,\zeta(4) + 2\,\zeta(3) \\
3\{\zeta(4)\} + 4*4\{\zeta(5)\} + \dots &= 4\,\zeta(5) + 3\,\zeta(4)
\end{aligned}$$

3.5. Sums using the original zeta-values [ζ(2), ζ(3), ζ(4)...]

To get simple results for the original zeta-values a set of cofactors must be chosen, which forces convergence.

Idea:

The vector ***Zet*** occurs from summation using the $\mathbf{G_p}$ matrix instead of the $\mathbf{G_m}$-matrix. The $\mathbf{G_p}$-matrix has the property to transform a vector, containing a powerseries in x, into a sum of such vectors, for instance:

(3.5.1.) $$G_p * 2*V(2) = V(1) + V(2)$$

generally

(3.5.2.) $$G_p * n*V(n) = V(1) + V(2) + \dots + V(n)$$

Conversely, to get a vector $\mathbf{V(n)}$ as result of a $\mathbf{G_p}$-transformation, one can subtract:

(3.5.3.) $$G_p * (\ n*V(n)\ -\ (n-1)*V(n-1)\ ) = V(n)$$

Example 1

To have a simple example for the sum of zeta-values with an apropriate set of cofactors in a vector ***W***

$$Zet\sim * W = x_0$$

we choose a vector ***W***, which assures convergence, for instance a vector ***V(½)*** or a variation. Using G_p we already know, that in a formula

$$G_p * W = X$$

if ***X*** is a powerseries ***V(n)***, ***W*** is the following variant:

$$G_p * W = V(n)$$
$$W = n\, V(n) - (n-1)\, V(n-1)$$

and since the summation of ***X*** = ***V(n)*** in

$$E\sim * G_p * W = E\sim * V(n)$$

eventually shall be convergent, we choose first ***X*** = ***V(½)***.

The sums of lhs and rhs

lhs: $Zet\sim * W = Zet\sim * (½\, V(½) - (½ -1)\, V(½-1))$
rhs: $E\sim * V(½)$

are both convergent and we get for ***W***:

(3.5.4.) $G_p * W = V(½)$
(3.5.5.) $W = ½*(V(½)+V(-½))$
$= [1\ 0\ 1/4\ 0\ 1/16\ 0\ 1/64\ 0 \ldots]\sim$

From the known sum ***E~ * V(½) = 2*** we get thus:

$$E\sim * (G_p * W) = E\sim * X$$
$$(E\sim * G_p) * W = E\sim * V(½)$$
$$Zet\sim * W = 2$$
(3.5.6.) $Zet\sim * [1\ 0\ 1/4\ 0\ 1/16\ 0\ 1/64\ 0 \ldots]\sim = 2$

or.

(3.5.7.) $$\sum_{c=0}^{oo} \frac{\zeta(2+2c)}{2^{2c}} = 2$$

which can be verified easily due to good convergence.

Example 2

Analoguously ony may choose

$$X = V(1/3)$$

For ***W*** we get

$W = 1/3*V(1/3) - (-2/3)*V(-2/3)$
(3.5.8.) $= [1, -1/3, 3/9, -5/27, 11/81, -21/243, \ldots]\sim$

and

$$E\sim * (Gp * W) = E\sim * V(1/3)$$
$$Zet\sim * W = 3/2$$
(3.5.9.) $Zet\sim * [1, -1/3, 3/9, -5/27, 11/81, -21/243, \ldots]\sim = 3/2$

or

(3.5.10.) $$\sum_{c=0}^{oo} \zeta(2+c)\left(\left(\frac{1}{3}\right)^{c+1} - \left(-\frac{2}{3}\right)^{c+1}\right) = \frac{3}{2}$$

Proposition for a general result

Generally this should give:

(3.5.11.) $$\sum_{c=0}^{oo} \zeta(2+c)\left(\left(\frac{1}{n}\right)^{c+1} - \left(\frac{1-n}{n}\right)^{c+1}\right) = \frac{n}{n-1} \quad \text{for } n>=1$$

(see the similar expression of P. Abbott at 4.2)

3.6. Summation of powerseries-weighted Bernoulli-numbers to ζ-values

The successful summation of bernoulli-numbers in the columns of $\mathbf{G_p}$ suggests to consider, which else sums may meaningfully be constructed. In the context of $\mathbf{P_J}$, whose eigenmatrix is $\mathbf{G_p}$, a further interesting possibility may be considered.

First recall, that from the eigen-decomposition of $\mathbf{P_J}$ it is:

(3.6.1.) $P_j * G_p * {}_dJ = G_p$

and the summing along columns on the lhs should equal the same sum in the rhs.

Now, we have:

(3.6.2.) $1/s\ V(1/s) * P = 1/(s-1)\ V(1/(s-1))$

for instance

$[1/2, 1/4, 1/8, ...] * P = [1, 1, 1, 1, ...]$

and on the other hand

(3.6.3.) $P_j * G_p * {}_dJ = P * ({}_dJ\ G_p\ J) = P * G_m$

The summation

$½\ V(½)\sim \ * \ (P * G_m)$

can be rewritten as follows:

(3.6.4.) $(½\ V(½)\sim * P\ ) \ * G_m = V(1)\sim * G_m$

and the rhs was already analytically determined as

(3.6.5.) $V(1)\sim \ * \ G_m = Zet_m\sim \qquad = [\zeta(2)\text{-}1, \zeta(3)\text{-}1, ...]$

so that also the lhs must be

(3.6.6.) $½\ V(½)\sim * P * G_m = Zet_m\sim \qquad = [\zeta(2)\text{-}1, \zeta(3)\text{-}1, ...]$

and since $P * G_m = G_p$ it is now also

(3.6.7.) $½\ V(½)\sim \ * G_p \ = Zet_m\sim \qquad = [\zeta(2)\text{-}1, \zeta(3)\text{-}1, ...]$

Thus we arrived at the following two results

(3.6.8.)	$V(1) \sim\ *\ G_p = [\zeta(2)\ ,\ \zeta(3)\ ,\ ...]$
(3.6.9.)	$½\ V(½) \sim\ *\ G_p = [\zeta(2)-1,\ \zeta(3)-1,\ ...]$

and this suggest, to construct this way a systematic gemeralization, like

$$1/n\ V(1/n)\sim\ *\ G_p = [\ x_0,\quad x_1\quad, x_2, ...\ ]$$

Using Euler-summation one finds the very likely general limit for a column *c* :

(3.6.10.)

$$1/n\ V(1/n)\sim\ *\ G_{p[c]} = \zeta(2+c) - \Sigma_{k=1..n-1} (1/k^{c+2})$$

3.7. Summation using Bernoulli-numbers to get harmonic numbers (any order)

Eq. (3.6.1) implies a surprising and seemingly little known summation-property of the Bernoulli-numbers: the summing of negative powers of natural numbers.

If subtracting

$$(V(1)\sim \ * \ G_p) \quad - \quad (1/n\ V(1/n)\sim \ * \ G_p)$$

one gets -after collecting cofactors of G_p for a column c :

(3.7.1.) $$(V(1) - 1/n\ V(1/n))\sim \ * \ G_{p[c]} = \Sigma_{k=1..\,n-1} (1 / k^{c+2})$$

For column $c=0$ (the unweighted sequence of Bernoulli-numbers, $ß_1=+1/2$), this gives the general expression for $n>=1$:

(3.7.2.)
$$\Sigma_{k=0..oo} (1-1/n^{k+1}) \ ß_k = \Sigma_{k=1..\,n-1} (1/k^2) = 1 + 1/2^2 + 1/3^2 + ... + 1/(n-1)^2$$

for columns $c=1,2,3$:

(3.7.3.)
$$\Sigma_{k=1..oo} (1-1/n^{k+1}) \quad k \quad ß_{k-1} /2 = \Sigma_{k=1..\,n-1} (1/k^3) = 1 + 1/2^3 + ... + 1/(n-1)^3$$
$$\Sigma_{k=2..oo} (1-1/n^{k+1})\ bi(k,2))\ ß_{k-2} /3 = \Sigma_{k=1..\,n-1} (1/k^4) = 1 + 1/2^4 + ... + 1/(n-1)^4$$
$$\Sigma_{k=3..oo} (1-1/n^{k+1})\ bi(k,3))\ ß_{k-3} /4 = \Sigma_{k=1..\,n-1} (1/k^5) = 1 + 1/2^5 + ... + 1/(n-1)^5$$

This suggests that generally, for a column c, we get the harmonic numbers of varying orders:

(3.7.4.) $$\sum_{k=c}^{oo}\left(1-\frac{1}{n^{k+1}}\right)\binom{k}{c}\frac{ß_{k-c}}{c+1}=\sum_{k=1}^{n-1}\frac{1}{k^{2+c}}=1+\frac{1}{2^{2+c}}+\frac{1}{3^{2+c}}+...+\frac{1}{(n-1)^{2+c}}$$

which is in the limit for for $n\to oo$ the already known result:

(3.7.5.) $$\sum_{k=c}^{oo}\binom{k}{c}\frac{ß_{k-c}}{c+1}=\sum_{k=1}^{oo}\frac{1}{k^{2+c}}=\zeta(2+c)$$

3.8. provisorial concluding remark

All the above sums-of-*ζ()* approaches may be seen as consequences of the simple matrix-product of the horizontally flipped ZV-matrix (with increasing positive *ζ()* exponents to the right) with the binomial matrix, which immediately holds, as far as convergence in one or another way of using associativity is given. Remember that in the following scheme:

Example:

*lim 1/x*V(1/x)~ * P = 1/(x-1)*V(1/(x-1)) ~*

P

1	.	.	.	.	.
1	1	.	.	.	.
1	2	1	.	.	.
1	3	3	1	.	.
1	4	6	4	1	.
1	5	10	10	5	1

1/2	1/4	1/8	1/16	1/32	1/64
1/3	1/9	1/27	1/81	1/243	1/729
1/4	1/16	1/64	1/256	1/1024	1/4096
1/5	1/25	1/125	1/625	1/3125	1/15625
1/6	1/36	1/216	1/1296	1/7776	1/46656
1/7	1/49	1/343	1/2401	1/16807	1/117649

1	1	1	1	1	1
1/2	1/4	1/8	1/16	1/32	1/64
1/3	1/9	1/27	1/81	1/243	1/729
1/4	1/16	1/64	1/256	1/1024	1/4096
1/5	1/25	1/125	1/625	1/3125	1/15625
1/6	1/36	1/216	1/1296	1/7776	1/46656

the postmultiplication using the pascalmatrix ***P*** provides simply a row-shift of the lhs-matrix, whose column-sums represent the *ζ(n)-values-minus-1*, thus the fractional parts of the *ζ(n)*-values.

For all product-sums (rows of lhs-matrix by columns of ***P***-matrix) we have convergent summations. The lhs-column-sums and those of the result are convergent except for the first column - and thus convergent column-sums in the resulting matrix - in the result just the lhs-sums *+1* , thus the original *ζ()*-values.

Using ***P*** or powers of ***P*** (including the reciprocal P^{-1}) such *ζ()*-sums can be constructed easily. Powers of ***P*** simply include powerseries vectors as cofactors in the columns of P^n (for details see article: *[binomialmatrix]*), which then only iterates the shifting process: so also sums involving powerseries-weighted *ζ()*-values are easily accessible this way.

4. a couple of formulae and snippets

4.1. Sci.math: what is the sum of

$$\sum_{k=0}^{oo} c(k,n) * ß_{k-n} = ?$$

Robert Israel in sci.math:

first example; n=1. Use

(4.1.1.) $$F(z) = \sum_{k=0}^{oo} ß_k * z^k = \Psi\left(1,1+\frac{1}{z}\right) * \frac{1}{z}$$

then

(4.1.2.) $$(z F(z))' = \sum_{k=0}^{oo} ß_k * (k+1) * z^k = -\Psi\left(2,1+\frac{1}{z}\right) * \frac{1}{z^2}$$
$$S(1) = 2(\zeta(3)-1)$$

(4.1.3.) *(In the current notation this means :)*
$E' * G_m[*,1] = S(1)/2 = \zeta(3)-1$

Generally for any n

(4.1.4.) $$\frac{d^n}{dz}(z^n F(z)) = \sum_{k=0}^{oo} ß_k * (k+n)...(k+1) * z^k = (-1)^n \Psi\left(n+1,1+\frac{1}{z}\right) * \frac{1}{z^{n+1}}$$
$$S(n) = (-1)^n \frac{\Psi(n+1,2)}{n!} = (n+1)(\zeta(n+2)-1)$$

(4.1.5.) *(In the current notation this means :)*

$E' * G_p[*,n] \quad = S(n)/(n+1) - ß_1^- + ß_1^+$
$\quad = \zeta(n+2)$

where $ß_1^-$ -1/2, $ß_1^+ = +1/2$

4.2. P. Abbot

Am 18.09.2006 12:14 schrieb Paul Abbott:

(4.2.1.) $$\frac{1}{1-p} = \sum_{k=0}^{oo} \zeta(2+k)\left(p^{k+1} - (p-1)^{k+1}\right) \qquad 0 \le p \le 1$$

(4.2.2.) $$\frac{1}{p} = \sum_{k=0}^{oo} (-1)^k \zeta(2+k)\left(p^{k+1} - (p-1)^{k+1}\right) \qquad 0 \le p \le 1$$

4.3. a representation of the alternating sum of ζ(2k) k=1..inf

(4.3.1.) $$\sum_{k=1}^{oo} i^{2k}\zeta(2k) = -\pi(1+ß_1+\frac{1}{e^{2\pi}-1})+\frac{1}{2} = \frac{1}{2}-\pi(\frac{1}{2}+\frac{1}{e^{2\pi}-1})$$

(Pari/GP) = 0.57667404746858117413405079 47

Plouffe: Real and Imaginary parts of the Digamma or Psi function at a+b*i
0.5766740474685811 = Im(Psi(2+I*1)

4.4. Reflectionformula Zeta

(4.4.1.) $$\zeta(1-s) = \frac{2}{(2\pi)^s}\Gamma(s)\cos(\frac{s\pi}{2})\zeta(s) = -\frac{ß_s}{s}$$ for even integer s

(4.4.2.) $$\boxed{\zeta(k) = \frac{i^2}{2!}\frac{(2\pi i)^k}{k!}ß_k}$$ for even k

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4.5. Gene Ward Smith in sci.math

http://groups.google.com/group/sci.math/browse_frm/thread/cbcd1d11e797e810/15e805d436f3c8ac?lnk=st&q=&rnum=4#15e805d436f3c8ac

Here's another interesting Borel sum:

$$\zeta(0) + \zeta(-1) + \zeta(-2)+... = -1/2 - \Sigma ß_n / n$$

where ζ(s) is the Riemann zeta function. Then we get

$$\Sigma_{n=0..infinity}\ \zeta(-n)\ x^n/n! = ((x-1)exp(x)+1)/(x\ (exp(x)-1))$$

The Borel sum is

$$\int_0^{inf} exp(-x)((x-1)exp(x)+1)/(x\ (exp(x)-1))\ dx = -\gamma$$

where γ is the Euler constant.

4.6. Another posting:

(4.6.1.) $$PG(0,1-p)+\gamma \quad = -\sum_{k=0}^{oo}\zeta(2+k)p^{k+1} \qquad 0\le p\le 1$$

wobei PG(a,b)=PolyGamma(a,b)

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5. References

Projekt Binomial-Matrix

[Project-Index] http://go.helms-net.de/math/binomial/index

[Intro] http://go.helms-net.de/math/binomial/intro.pdf

[binomialmatrix] http://go.helms-net.de/math/binomial/01_1_binomialmatrix.pdf

[signed binomial] http://go.helms-net.de/math/binomial/01_2_signedbinomialmatrix.pdf

[Gaussmatrix] http://go.helms-net.de/math/binomial/04_1_gaussmatrix.pdf

[Stirlingmatrix] http://go.helms-net.de/math/binomial/05_1_stirling.pdf

[Hasse] http://go.helms-net.de/math/binomial/01_x_recihasse.pdf

[Vandermonde] http://go.helms-net.de/math/binomial/10_3_InverseVandermonde1.pdf

[GenBernRec] (Generalized Bernoulli-recursion)
http://go.helms-net.de/math/binomial/GeneralizedBernoulliRecursion.pdf

Projekt Bernoulli-numbers, older versions of the above, contains the basic material about G_p and G_m

[Bernoulli] http://go.helms-net.de/math/binomial/bernoulli_en.pdf

[Summation] http://go.helms-net.de/math/binomial/pmatrix.pdf

Gottfried Helms