Mathematical Miniatures

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1-12 Eulermatrix

Abstract: The lower triangular matrix of Eulerian numbers is considered. Basic properties are documented. Also the possibility of interpolation to fractional row-indices is discussed. The property, that the rows of the Eulerian triangle can also sum to values of the eta-function of integer or fractional arguments seems to be much less widely known.

In a third step the triangle is tried as tool for summing of the strongly divergent alternating factorial series $su = 0! - 1! + 2! - 3! + \ldots$ *whose sum under the concept of divergent summation was already considered by L. Euler (while not in that context). The approach shown here agrees numerically well with the known value* $su = 0.5963473623231\ldots$ *respectively to the finite tunctation of powerseries (30 correct digits using matrix-dimension 128x128).*

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1. Basic definitions and identities

1.1. Intro: sum of like powers

The coefficients of the Eulerian triangle were found by L. Euler; in [wikipedia] we find a short reproduction of one table in his book " Institutiones calculi differentialis"[1]. Here, Euler considered the evaluation of series of like powers and arrived at these coefficients. Because it is truly a nice pattern which is involved here, I'll describe the Euler-matrix via the sum-of-like-powers problem which was my own approach and brought me –by chance– in contact with that numbers.

We may observe, that when looking at the binomial numbers we find the perfect powers by simple summing of neighboured binomials:

```
1 3 6 10 15 …..
  1 3  6 10 …
------------------------
1 4 9 16 25 …
```

This is a nice and striking pattern, of course being explainable by the binomial theorem

$$\begin{aligned}(n-1)n/2 + n(n+1)/2 &= 1/2\,(n^2 - n + n^2 + n)\\ &= 1/2\,(2n^2)\\ &= n^2\end{aligned}$$

We may try whether this is somehow generalizable, we could look at the third powers and see, whether we can combine the binomial-numbers of next order:

```
    1   4  10  20  35  56.
        1   4  10  20  35 ...
            1   4  10  20  35...
    ------------------------
    1   5  15  34  65 111 ….
```

This is not working well, but if we take a multiple of the middle row, we find

```
1*  1   4  10  20  35  56 ….
4*      1   4  10  20  35 …
1*          1   4  10  20 …
------------------------
    1   8  27  64 125 216 ….
```

Heuristically we will find that this holds when analoguously continued with higher orders of sums; and to complete the scheme at the beginning we may look back to the low orders.

[1] in part 2, chap 7. See the reproduction in [wikipedia]

We can write

```
1 *   1 1 1 1 1 …
- -----------------------
k^0=  1 1 1 1 1 ….

1 *   1 2 3 4 5 …
- -----------------------
k^1=  1 2 3 4 5 …

1 *   1 3 6 10 15 … ..
1 *     1 3  6 10 …
- -----------------------
k^2=  1 4 9 16 25 …

1 *   1    4   10   20   35   56 ….
4 *        1    4   10   20   35 …
1 *             1    4   10   20 …
- -----------------------
k^3=  1    8   27   64  125  216 …

…
```

and extract a scheme for the multiplicators. This gives a triangle of coefficients

```
1
1
1   1
1   4   1
1  11  11   1
1  26  66  26   1
…
```

which is known as the "triangle of Eulerian numbers" and what we have done so far was to relate them to the perfect powers of natural numbers.

But besides this nice pattern we can extend this one more step to arrive at similar formulae for the *sums* of like powers.

The given binomial numbers in the sequences of some order are also the the *sums* of that of one less order. So we have in the square arrangement

```
1   1   1   1   1 …
1   2   3   4   5 …
1   3   6  10  15 …
1   4  10  20  35 …
…
```

that always the sums of the coefficients of one row up to a certain column-position are the values at the column position in one row below.

So we can simply insert the plus-operators in our scheme above and evaluate to some partial sum:

```
1 *   1+3+6+10+15 ….
1 *   0+1+3+ 6+10 …
------------------------
k^2=  1+4+9+16+25 ….
```

If we actually write out the partial sums as values we get simply

```
1   *    1    4   10   20   35   56 ….
1   *         1    4   10   20   35 …
-----------------------------
Σk²=     1    5   14   30   55   81 …
```

and

```
 1  =  1²
 5  =  1² + 2²
14  =  1² + 2² + 3²
30  =  1² + 2² + 3² + 4²
…
```

which is valid the same way for the other orders.

It is obvious that this general scheme can be extended to partial-sums of partial sums ad libitum, but also to their differences.

Having this general pattern, we can describe a unique formula for the generation of the coefficients of the triangle. For example, if we would not know the coefficients for the second row of the triangle. We set the first coefficient *= 1*, and treat the second as unknown.

```
1 *   1 +3  +6  +10  +15  ….
x *   0 +1  +3  + 6  +10  …
------------------------
k²=   1 +4  +9  +16  +25  ….
```

This gives an immediate solution for *x*: just subtract the first row from the sum and divide to get *x*:

```
k²       1 +4  +9  +16  +25  ….
-1*      1 +3  +6  +10  +15  ….
------------------------
=x*      0 +1  +3  + 6  +10  …

1 - 1 = 0*x
4 - 3 = 1*x
9 - 6 = 3*x
```

and we find, that all these equations can be simultanously satisfied assuming *x=1*.

Next example:

```
k³       1    8   27   64  125  216 …
-1*      1    4   10   20   35   56 …
-------------------------------
 x*      0    1    4   10   20   35 …
 y*      0    0    1    4   10   20 …

 8 -  4 = 1*x                 => x = 4
27 - 10 = 4*4 + 1*y           => y = 1
```

and again these constant coefficients satisfy all equations which result from the evaluation of all other columns.

However, this all was only heuristic; using the binomial identities it is not difficult to be proven for a specific order [2].

Moreover, we can find two different ways to compute the coefficients of the Euler-triangle:

- one using the combinations of the binomial-coefficients, and even
- one which allows to compute a row of the Eulerian triangle recursively only using the previous row and its row-index.

See for that rules the sections below.

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[2] which I'm not going to do here, see *References* for this

1.2. Euler-matrix

1.2.1. Appearance

The Eulerian triangle is seen in two different variations; so I'll denote the Euler-matrix giving two names:

E:

$$\begin{bmatrix} 1 & . & . & . & . \\ 1 & 0 & . & . & . \\ 1 & 1 & 0 & . & . \\ 1 & 4 & 1 & 0 & . \\ 1 & 11 & 11 & 1 & 0 \end{bmatrix}$$

E1:

$$\begin{bmatrix} 1 & . & . & . & . \\ 1 & 1 & . & . & . \\ 1 & 4 & 1 & . & . \\ 1 & 11 & 11 & 1 & . \\ 1 & 26 & 66 & 26 & 1 \end{bmatrix}$$

1.2.2. recursive definition

The coefficients can be computed recursively. Assume the row/col-indexes *(r,c)* beginning at zero, and elements outside the matrix as zero. Then

$e_{r,0} = 1$
$e_{r,c} = (r-c)*e_{r-1,c-1} + (1+c)*e_{r-1,c}$

1.2.3. direct definition

The coefficients can also be described by the following direct formula:

$$E \overset{def}{:=} \quad e_{r,c} = \sum_{k=0}^{c} (-1)^k \binom{r+1}{k} (c-k+1)^r$$

Note that this direct definition formally allows generalization to fractional row-indexes. See more about this in chap. 2.

1.2.4. Generation function

In the OEIS we find an exponential generating-function for the Eulerian-triangle [OEISA000142]:

$$f(x,t) =_{s=t-1} \frac{s}{s-(\exp(s*x)-1)}$$

translated to ***Pari/GP***:

```
\\ ==== also using functions from helms's basic matrixlib =====
f(x,t) = (t-1)/(t-exp((t-1)*x  ))
pc = polcoeffs( f(x,t) )  \\ extract coefficients at powers of x
dFac(1,13)*(Mat(pc)~)     \\ show coefficients in a column, rescale by factorials


[1]
[1]
[t + 1]
[t^2 + 4*t + 1]
[t^3 + 11*t^2 + 11*t + 1]
[t^4 + 26*t^3 + 66*t^2 + 26*t + 1]
[t^5 + 57*t^4 + 302*t^3 + 302*t^2 + 57*t + 1]
[t^6 + 120*t^5 + 1191*t^4 + 2416*t^3 + 1191*t^2 + 120*t + 1]
[t^7 + 247*t^6 + 4293*t^5 + 15619*t^4 + 15619*t^3 + 4293*t^2 + 247*t + 1]
…
```

2. Basic observations

2.1. Rowsums and alternating rowsums

The rowsums give the factorials,

$$\Sigma_{c=0..r}(e_{r,c}) = r!$$

The alternating rowsums give – not so obvious –

$$\Sigma_{c=0..r}((-1)^c * e_{r,c}) = 2^{r+1}\ \eta(-r)$$

where $\eta(-r)$ (="*eta*") is the alternating *zeta*-function.

(see [wikipedia] and [mathworld] for more properties)

(This was also proven in [Stopple], 2003) Using the coefficients with appropriate binomials we get powers or sums of powers as indicated in the first paragraph.

2.2. Rowsums in terms of vectors

$$E * V(1) \quad = F(1) \quad = [0!, 1!, 2!, 3!,...]$$

$$E * V(-1) \quad = 2\ {}^{d}V(2)\ H \quad = 2*[\ \eta(0), 2\ \eta(-1), 2^2\ \eta(-2), 2^3\ \eta(-3), ...]$$

2.3. The inverse (of E1)

E1^-1

$$\begin{bmatrix} 1 & . & . & . & . \\ -1 & 1 & . & . & . \\ 3 & -4 & 1 & . & . \\ -23 & 33 & -11 & 1 & . \\ 425 & -620 & 220 & -26 & 1 \end{bmatrix}$$

Obviously the inverse of the Eulerian-triangle allows to compute the binomials from powers or sum-of-powers – just consider the inverse relations from that above.

2.4. The matrixlog (of E1)

E1L

$$\begin{bmatrix} 0 & . & . & . & . & . \\ 1 & 0 & . & . & . & . \\ -1 & 4 & 0 & . & . & . \\ 14/3 & -11 & 11 & 0 & . & . \\ -196/3 & 397/3 & -77 & 26 & 0 & . \\ 34619/15 & -13115/3 & 2194 & -439 & 57 & 0 \end{bmatrix}$$

2.5. The matrixexponential (of E1)

E1E

$$exp(1)* \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 3 & 4 & 1 & . & . & . \\ 58/3 & 33 & 11 & 1 & . & . \\ 247 & 1475/3 & 209 & 26 & 1 & . \\ 176167/30 & 39659/3 & 6561 & 1043 & 57 & 1 \end{bmatrix}$$

3. Advanced operations

3.1. Generalization to negative and interpolated rowindexes

We can continue the triangle to negative rowindexes, keeping the same properties valid. Here is a segment of the extension of the Euler-matrix to negative row-indexes also indicating some example compositions, which can be found heuristically, but can also be defined by extension of the range of the binomial-formula for the Eulerian-numbers:

Row index	column-entries					row sum	alternating rowsum	
$-3:$	1	$\frac{17}{8}$	$\frac{355}{108}$	$\frac{7715}{1728}$	...	$-3!$	$2^{-2}\eta(3)$	$=$
$-2:$	1	$\frac{5}{4}$	$\frac{49}{36}$	$\frac{205}{144}$	...	$-2!$	$2^{-1}\eta(2)$	$=$
$-1:$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	...	$-1!$	$2^{0}\eta(1)$	$=\ln(2)$
$0:$	1	.	.	.	... =	$0!$	$2^{1}\eta(0)$	$=1$
$1:$	1	.	.	.	...	$1!$	$2^{2}\eta(-1)$	$=1$
$2:$	1	1	.	.	...	$2!$	$2^{3}\eta(-2)$	$=0$
$3:$	1	4	1	.	...	$3!$	$2^{4}\eta(-3)$	$=-2$
$4:$	1	11	11	1	...	$4!$	$2^{5}\eta(-4)$	$=0$

and the composition by binomials and powers:

Row index	column-entries					row sum	alternating rowsum	
$-3:$	1	$2+\frac{1}{8}$	$3+\frac{2}{8}+\frac{1}{27}$	$4+\frac{3}{8}+\frac{2}{27}+\frac{1}{64}$	...	$-3!$	$2^{-2}\eta(3)$	$=$
$-2:$	1	$1+\frac{1}{4}$	$1+\frac{1}{4}+\frac{1}{9}$	$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}$	...	$-2!$	$2^{-1}\eta(2)$	$=$
$-1:$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	...	$-1!$	$2^{0}\eta(1)$	$=\ln(2)$
$0:$	1	$-1+2^0$	$0-1*2^0+3^0$	$0+0*2^0-1*3^0+4^0$	... =	$0!$	$2^{1}\eta(0)$	$=1$
$1:$	1	$-2+2^1$	$1-2*2^1+3^1$	$-0+1*2^1-2*3^1+4^1$	...	$1!$	$2^{2}\eta(-1)$	$=1$
$2:$	1	$-3+2^2$	$3-3*2^2+3^2$	$-1+3*2^2-3*3^2+4^2$	...	$2!$	$2^{3}\eta(-2)$	$=0$
$3:$	1	$-4+2^3$	$6-4*2^3+3^3$	$-4+6*2^3-4*3^3+4^3$	...	$3!$	$2^{4}\eta(-3)$	$=-2$
$4:$	1	$-5+2^4$	$10-5*2^4+3^4$	$-10+10*2^4-5*3^4+4^4$	...	$4!$	$2^{5}\eta(-4)$	$=0$

Note, that the infinite harmonic series at row-index *r=–1* is identified with *(-1)!* or *gamma(0)* and analoguously the other infinities of gamma at the other negative rowindexes.

Note also, that for non-natural row-indices the rows have infinitely many nonzero entries.

The interpolation to fractional row-indexes is an interesting feature. If we assume, that the formula for the direct computation of entries can assume fractional binominal coefficients, then we get still meaningful results:

- the rowsums generalize to *gamma( )*-values at fractional arguments, and
- the alternating rowsums to *η( )*–values at fractional negative arguments.

Table: Euler-triangle with entries at fractional row-indexes summing to Γ() and η()

E_{05} * [V(1) V(-1)] = *

1	1
1	-1
1	1
1	-1
1	1
1	-1
1	1
1	-1
1	1

E_{05}

1	.	.	.	.
1.00000	-0.0857864	-0.0142695	-0.00524613	-0.00258712
1	.	.	.	.
1.00000	0.328427	0.0000846108	0.000419802	0.000179704
1	1	.	.	.
1.00000	2.15685	0.164467	0.00163690	0.000268815
1	4	1	.	.
1.00000	6.81371	3.72868	0.0887813	0.000525622
1	11	11	1	.

=

Result E_{05}*V(1)	Result E_{05}*V(-1)	Check Γ(1+r/2)	Check $2^{r+1}\eta(-r/2)$
1	1	1.00000	1
0.886268	1.07510	0.886227	1.07510
1	1	1.00000	1.00000
1.32934	0.671360	1.32934	0.671360
2	0	2.00000	0
3.32335	-0.993809	3.32335	-0.993809
6	-2	6.00000	-2.00000
11.6317	-2.17331	11.6317	-2.17331
24	0	24.0000	0

(Matrixsize for approximation of results: 128 columns; the rowindices are integers, so the gamma- and eta-formula must be adapted to that indexes)

3.1.1. Eulerian triangle and *ζ()-/ η( )*-zeroes at "complex row-indexes"

Since *ζ()* and *η( )*-values coincide at the functional zeros, we can find a powerseries expression for the *η( )*-value using the known roots of *ζ()*to let the alternating rowsum at the - now complex(!) - index vanish.

We have then, using ρ_n (greek: "rho") for the *n*'th root of *zeta*

$$e(r,c)=\sum_{k=0}^{c}(-1)^k\binom{1+r}{k}(1+(c-k))^r \qquad\qquad 0=\sum_{c=0}^{oo}(-1)^c e(-\rho_n,c)$$

at the complex ρ_n'th row-index .

3.2. Coefficients for summation of a strongly divergent series[3]

3.2.1. The approach

Looking at the question how to sum $su = 0! - 1! + 2! - 3! \ldots$ we can try to use the Euler-matrix as coefficients for a double/multiple sum scheme, where we can then resolve the divergent computation into some handsome elements by change of order of summation.

Let's restate the problem in terms of limit of a powerseries:

$$su(x) = 0! + 1!\,x + 2!\,x^2 + 3!\,x^3 + 4!\,x^4 + \ldots$$
$$su = \lim_{x\to 1} su(x)$$

This series is divergent for all x, or we can say, it has zero radius of convergence. It is a classical problem to find a meaningful answer for this expression and was first solved by L. Euler.

Now since the rowsums of the Eulerian triangle just provide the factorials we attempt to base a matrix-summation-method on it.

The above sum in question can be seen as alternating sum of the row-sums of **E**. Formally expressed as powerseries in x, where we use later $x=-1$, using the matrix-notation:

$$\begin{aligned} su(x) &= V(x)\sim * [0!,1!,2!,\ldots] \\ &= V(x)\sim * (E * V(1)) \end{aligned}$$

Now change order of summation (we'll have to make sure whether this is meaningful with the occuring divergent series):

$$su(x) = (V(x)\sim * E) * V(1)$$

then using $x=-1$ we come from

$$\begin{aligned} su &= V(-1)\sim * [0!,1!,2!,\ldots] \\ &= V(-1)\sim * (E * V(1)) \end{aligned}$$

to the notation

$$\begin{aligned} &= (V(-1)\sim E) * V(1)) \\ &= (AS \sim) * V(1)) \end{aligned}$$

Thus we compute the alternating columnsums **AS** of the Eulertriangle first.

However, also these are divergent sums.

But when we look at the composition of the entries of one column, we see, that we can dissolve each column-sum into a finite sum of binomially weighted ***geometric series*** – precisely: finite sums of geometric series and their derivatives. For these we have closed forms which are valid even if the underlying series is divergent.

[3] For another very nice introduction to this problem see Ed Sandifer's discussion of the article for L.Euler [Sandifer]

3.2.2. The formulae for columnsums

Given a fixed column *c*, then the entries of *E[,c]* are

$$E_{r,c} = \sum_{k=0}^{c} (-1)^k \binom{r+1}{k} (c-k+1)^r$$

and if we want to formulate a function $f_c(x)$ using the entries along that column we have

$$f_c(x) = \sum_{r=0}^{oo} \left(\sum_{k=0}^{c} (-1)^k \binom{r+1}{k} (c-k+1)^r x^r \right)$$

We change order of summation and get

$$f_c(x) = \sum_{k=0}^{c} (-1)^k \left(\sum_{r=0}^{oo} \binom{r+1}{k} (c-k+1)^r x^r \right)$$

For notational convenience we set $x_k = x*(1+(c-k))$

$$f_c(x) = \sum_{k=0}^{c} (-1)^k \left(\sum_{r=0}^{oo} \binom{r+1}{k} x_k^{\ r} \right)$$

Then we rewrite this to adapt the index of the inner sum:

$$\begin{aligned} f_c(x) &= \sum_{k=0}^{c} (-1)^k \left(\sum_{r=1}^{oo} \binom{r}{k} x_k^{\ r-1} \right) \\ &= \sum_{k=0}^{c} (-1)^k \left(\sum_{r=k}^{oo} \binom{r}{k} x_k^{\ r-1} \right) \\ &= \sum_{k=0}^{c} (-1)^k x_k^{\ k-1} \left(\sum_{r=k}^{oo} \binom{r}{k} x_k^{\ r-k} \right) \end{aligned}$$

Here the inner sums are just derivatives of the geometric series of x_k, and for those we have closed forms. We get

$$\begin{aligned} f_c(x) &= -\frac{1}{x(1+c)} + \sum_{k=0}^{c} (-1)^k \frac{x_k^{\ k-1}}{(1-x_k)^{k+1}} \\ &= -\frac{1}{x(1+c)} + \sum_{k=0}^{c} (-1)^k \frac{(x(c+1-k))^{k-1}}{(1-x(c+1-k))^{k+1}} \\ &= \frac{1}{1-x(1+c)} + \sum_{k=1}^{c} (-1)^k \frac{(x(c+1-k))^{k-1}}{(1-x(c+1-k))^{k+1}} \\ &= \frac{1}{1-x(1+c)} - \sum_{k=0}^{c-1} (-1)^k \frac{1}{(1-x(c-k))^2} \left(\frac{x(c-k)}{1-x(c-k)} \right)^k \end{aligned}$$

For instance. at column *c=4* this is

$$f_4(x) = \frac{1}{1-5x} - \frac{1}{(4x)^2} \left(\frac{4x}{1-4x} \right)^2 + \frac{1}{(3x)^2} \left(\frac{3x}{1-3x} \right)^3 - \frac{1}{(2x)^2} \left(\frac{2x}{2x-1} \right)^4 + \frac{1}{(1x)^2} \left(\frac{1x}{1x-1} \right)^5$$

where the parentheses were a bit rearranged to focus a more compact generation rule.

3.2.3. The alternating sum of factorials (*x=-1*)

We use

$su(x) = 0! + 1!\, x + 2!\, x^2 + 3!\, x^3 \ldots.$

setting x=-1

$su \quad = 0! - 1! + 2! - 3! + \ldots.$

Example for *x=-1*:

Using *x=–1* means simply to compute the *alternating columnsum* at a column *c* and we have

$$f_c(-1) = \frac{1}{2+c} - \sum_{k=0}^{c-1} \left(\frac{1}{(1+(c-k))^2} \left(\frac{c-k}{1+(c-k)} \right)^k \right)$$

Example for column *c=4*:

$$f_4(-1) = \frac{1}{6} - \left(\frac{4^0}{5^2} + \frac{3^1}{4^3} + \frac{2^2}{3^4} + \frac{1^3}{2^5} \right) = -\frac{109}{129600}$$

and in the previous matrix–formula we can insert formally:

$V(-1)\sim E \quad = AS\sim$

$\quad = [\, f_0(-1),\ f_1(-1),\ f_2(-1),\ \ldots\ ]\sim$

For the first five columnsums *c=0..4* this is

c=0	$f_0(-1) = 1/2$	$= 1/2 - (\ )$
c=1	$f_1(-1) = 1/12$	$= 1/3 - (1/2^2)$
c=2	$f_2(-1) = 1/72$	$= 1/4 - (1/3^2 + 1/2^2*(1/2)^1)$
c=3	$f_3(-1) = 1/1080$	$= 1/5 - (1/4^2 + 1/3^2*(2/3)^1 + 1/2^2*(1/2)^2)$
c=4	$f_4(-1) = -109/129600$	$= 1/6 - (1/5^2 + 1/4^2*(3/4)^1 + 1/3^2*(2/3)^2 + 1/2^2*(1/2)^3)$

The first few columnsums as real numbers

[1/2,	1/12,	1/72,
0.0009259259259,	-0.000841049383,	-0.000623695620,
-0.0002838727132,	-0.000090280207,	-0.000009688042,
0.0000140629265,	0.000015479872,	0.000010719082,
0.0000058791663,	0.000002559936,	...]

and looking at *128* entries it seems as if

a) the sequence of absolute values of columnsums converges monotonuously to zero and

b) it appears also to be conventionally summable.

The approximated value is –no surprise– very well near the expected value

$su(-1) \quad = AS\sim * V(1)$

$\quad = \Sigma_{c=0..inf} f_c(-1)$

$\quad \sim 0.596347362323194074341078499369\ldots$ *(using 512 terms)*

giving *34* correct digits compared to the value as documented in the OEIS [OEIS_A073003]. According to OEIS this value is also known as "Gompertz's constant" and is also *su(-1) = -exp(1)*EI(-1)*

3.2.4. Further examples with other values of *x*

Extending this to other arguments *x* (for this formal powerseries with the factorials) I get using the *sumalt*-procedure in ***Pari/GP*** the values in the following table. For crosscheck I inserted also the *x*-values into the integral formula

$$\text{intnum}(x) = \frac{e^{\frac{1}{-x}}}{-x}\int_0^{-x}\frac{e^{\frac{-1}{t}}}{t}dx \qquad \text{(adaption of a version in [Sandifer])}$$

Table for $su(x) = 0! + 1!\,x + 2!\,x^2 + 3!\,x^3 + \ldots$

	Euler-matrix/sumalt	*intnum()*
su(-1) =	0.596347362323	0.596347362323
su(-2) =	0.461455316242	0.461455316242
su(-3) =	0.385602012137	0.385602012137
su(-4) =	0.335221361210	0.335221361208
su(-5) =	0.298669749329	0.298669749386
su(-6) =	0.270633013639	0.270633013429
su(-7) =	0.248281352547	0.248281351295
su(-8) =	0.229947781627	0.229947781662
su(-9) =	0.214577094581	0.214577102695
su(-10) =	0.201464233646	0.201464254471
su(-11) =	0.190117766778	0.190117792894
su(-12) =	0.180183310425	0.180183317779
…	…	

```
\\ Euler-matrix: compute one column when taken as powerseries with parameter x
\\ in this implementation x cannot assume 1/2, 1/3,…,1/(c-1)
\p 200   \\ internal float precision used 200 dec digits
{ Ecolsum(x,c)= local(xk,res);
   res = 1/(1-x*(1+c)) ;
   for(k=2,c+1, \\ each column is a finite sum of geometric series
         xk = (c+2-k)*x;
         res = res - (-1)^k / (xk)^2 *(xk/(1-xk))^k;
       );
   return(res); }

 su(x)  = sumalt(c=0,Ecolsum(x,c))
```

3.3. An alternating sum of η(-k) ,connection to the tanh()-function

It is a nice feature, that the alternating rowsums in ***E*** give the according *η()* –value, such that

$$E * V(-1) \qquad = 2*[\ \eta(0),\ 2\ \eta(-1),\ 2^2\ \eta(-2),\ 2^3\ \eta(-3),\ \ldots]$$

The above list of coefficients (in 3.2.3)

[1/2, 1/12, 1/72,
0.0009259259259, -0.000841049383, -0.000623695620,
-0.0002838727132, -0.000090280207, -0.000009688042,
0.0000140629265, 0.000015479872, 0.000010719082,
0.0000058791663, 0.000002559936, ...]

are the alternating column-sums of the ***E***-matrix, whose total sum we already found to equal the Gompertz's constant, approximately *0.596347362323194074341...* .

Now we expect, that the ***alternating*** sum of that columnsums should accordingly give the alternating sum of *η()* at nonpositive integer values like

$$AS = V(-1) * E$$
$$AS * V(-1) = 2 * \Sigma_{k=0..inf} (-1)^k\ \eta(-k) * 2^k = as_\eta$$

and indeed we get the value

$$as_\eta = 0.429203673205\ldots$$

which also agrees with a *Nörlund/Voronoi*-sum for that divergent alternating sequence of *η( )*-values, each scaled by the according power of *2*.

The coefficients of the formal (infinite) sum of *η( )* – function of consecutive negative parameters *k = 0, -1, -2,...* are also the coefficients of the *tanh()*-function (see also: [Hirzebruch]):

```
    x = 1.125              \\  check identity using an actual value
     tanh( x )
     sumalt(k=1, x^k/k!*aeta(-k)*2^k)*2
     tanh( x )   - %       \\  should be asymptotic to zero
%1247 = 0.809301070202
%1248 = 0.809301070202
%1249 = 4.28780487991 E-57

                           \\ check identity of coefficients
bestappr(sum(k=1,11,x^k/k!*aeta(-k)*2^k)*2),1e6) + O(x^11)
tanh(x) + O(x^11)
 %1257 = x - 1/3*x^3 + 2/15*x^5 - 17/315*x^7 + 62/2835*x^9 + O(x^11)
 %1258 = x - 1/3*x^3 + 2/15*x^5 - 17/315*x^7 + 62/2835*x^9 + O(x^11)
```

3.4. Factorially scaling and a variant of summing the geometric series

If we scale the rows of the Eulerian-matrix ***E*** by the reciprocal factorials,

$fE = {}^{d}f * E = diag(1/0!,1/1!,1/2!,\dots)*E$

then by the double-sum-principle

$s(x) = V(x)\sim * fE * V(1)$

we have first

${}^{d}f * E * V(1) = {}^{d}f * F = V(1)$

and then

$$\begin{aligned} V(x)\sim * V(1) &= V(x) * ({}^{d}f * E * V(1)) \\ &= (V(x) * {}^{d}f * E) * V(1) \\ &= [fe(x,0), fe(x,1), fe(x,2),\dots] * V(1) \\ &= \Sigma_{c=0..inf}\, fe(x,c) \end{aligned}$$

Like with the non-scaled version ***E***, where we get columnsums which are finitely composed by geometric series and their derivatives, we get here the columnsums as finitely composed by exponential-series and their derivatives. Using the columns of ***fE*** as coefficients for power-sieres in *x* we get the following closed-form formulas for that powerseries:

$$fe(x,c) = e^{x(1+c)} + \sum_{k=1}^{c}{}_{z=(c+1-k)*x} (-1)^k \frac{z^k + kz^{k-1}}{k!} e^z$$

for programmed version see footnote[4]

So we get

$V(x)\sim * fE = Y(x)\sim = [\, fe(x,0),\ fe(x,1), fe(x,2), \dots]$

and the second step, which should sum to the value of the geometric-series in its closed form:

$Y(x)\sim * V(1) = 1/(1-x)$

Indeed, we get a sequence of values in *Y(x)* whose sum agree with the geometric series of *x*.

However, there is an interesting aspect: we can even insert *x=1*, for which the geometric series has a singularity. Then the sequence of *fe(x,c)=fe(1,c)* converges to the constant value *2*, with diminuishing oscillation. The sequence of the deviations from *2* is convergent, its sum converges very well to give a "residue" of *2/3* (supposedly). I don't know currently what this might say to us.

The sequence *fe(x,0), fe(x,1),…* seems itself to form asymptotically a geometric series with a quotient *q* (whose relation to *x* is not simple). The consequence is, that we have by all this just another representation of the geometric series

$1/(1-x) = a * 1/(1-q) + res \qquad // q=/= 1$

where *a*, *q* and *res* depend on *x*. If *x=1*, we have *a=2*,*q=1* and *res = 2/3*.

[4]
```
fe(x,c) = local(m);
  sum(k=0,c,if(1,            \\ if-clause to allow sequel in sum
                m = (c+1-k)*x;
                if(k==0, 1,
                       (-1)^k * m^(k-1)*(m + k) /k!
                   ) * exp(m)
                )
      )
```

Here is a table for different values of *x=1 ± eps* , where *eps* aproximates zero

$x=1\pm 2^k$	a	q	res	a*1/(1-q)+res	1/(1-x)
$1-2^{-1}=$	0.945666776835	0.284668137041	0.678002720411	2	2
$1-2^{-2}=$	1.40961009463	0.576834004142	0.668895638054	4	4
$1-2^{-3}=$	1.68659712899	0.769993632100	0.667172024896	8	8
$1-2^{-4}=$	1.83842740033	0.880101616339	0.666787456187	16	16
$1-2^{-5}=$	1.91795437558	0.938788632410	0.666696218458	32	32
1=	**2.**	**1.**	**0.666666666667**	**->inf**	**->inf**
$1+2^{-5}=$	2.08464997945	1.06381576032	0.666695012174	-32	-32
$1+2^{-4}=$	2.17199227979	1.13031866790	0.666777790260	-16	-16
$1+2^{-3}=$	2.35511772204	1.27173094812	0.667094191416	-8	-8
$1+2^{-2}=$	2.75782540401	1.59076137589	0.668256112480	-4	-4
$1+2^{-1}=$	3.73312032168	2.39699882630	0.672242990758	-2	-2
$1+2^{0}=$	6.60612090601	4.92155363457	0.684567271446	-1	-1

Note: for this table a and q were computed from values of fe(x,n) and fe(x,n+1) at n=64. Increasing n gives only better approximations to the exact values but the change is not seen in that first few digits.

Here are the columnsums in a more explicite form:

$$fe(x,0) = e^{x}$$

$$fe(x,1) = e^{2x} - \frac{1x+1}{1!} e^{1x}$$

$$fe(x,2) = e^{3x} - \frac{2x+1}{1!} e^{2x} + \frac{1x+2}{2!} (1x)^1 e^{1x}$$

$$fe(x,3) = e^{4x} - \frac{3x+1}{1!} e^{3x} + \frac{2x+2}{2!} (2x)^1 e^{2x} - \frac{1x+3}{3!} (1x)^2 e^{1x}$$

$$fe(x,4) = e^{5x} - \frac{4x+1}{1!} e^{4x} + \frac{3x+2}{2!} (3x)^1 e^{3x} - \frac{2x+3}{3!} (2x)^2 e^{2x} + \frac{1x+4}{4!} (1x)^3 e^{1x}$$

$$\ldots$$

$$fe(x,c) = e^{x(1+c)} + \sum_{k=1}^{c} {}_{z=(c+1-k)*x} (-1)^k \frac{z^k + kz^{k-1}}{k!} e^{z}$$

The partial sums:

$$fe(x,0..0) \quad = e^{1x}$$

$$fe(x,0..1) \quad = e^{2x} - 1xe^{1x}$$

$$fe(x,0..2) \quad = e^{3x} - 2xe^{2x} + \frac{(1x)^2}{2!} e^{1x}$$

$$fe(x,0..3) \quad = e^{4x} - 3xe^{3x} + \frac{(2x)^2}{2!} e^{2x} - \frac{(1x)^3}{3!} e^{1x}$$

$$fe(x,0..4) \quad = e^{5x} - 4xe^{4x} + \frac{(3x)^2}{2!} e^{3x} - \frac{(2x)^3}{3!} e^{2x} + \frac{(1x)^4}{4!} e^{1x}$$

$$\ldots$$

$$fe(x,0..c) = e^{x(1+c)} \sum_{k=0}^{c} \frac{(c+1-k)^k}{k!} (-xe^{-x})^k$$

3.5. Another variant of the E-matrix

If we use the factorially scaled ***E***-matrix the expression of the exponential terms in the column-sums is not optimally smooth. If moreover we use a shift of the matrix down one row, such that in the row *r=0* contains *[1,1/2,1/3,1/4,...]* which is the *r=-1* – row in the Euler-matrix, then the formulae look somehow smoother. Let's call the unrescaled matrix ***E2***.

Here is the top-left edge of the matrix ***E2***:

$$E2 = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ 1 & . & . & . & . & . \\ 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 4 & 1 & . & . & . \\ 1 & 11 & 11 & 1 & . & . \end{bmatrix}$$

and let's call the factorially scaled version ***fE2***

$$fE2 = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ 1 & . & . & . & . & . \\ 1/2 & . & . & . & . & . \\ 1/6 & 1/6 & . & . & . & . \\ 1/24 & 1/6 & 1/24 & . & . & . \\ 1/120 & 11/120 & 11/120 & 1/120 & . & . \end{bmatrix}$$

The entries in the colums are now

	col 0	col 1	col 2	col 3
	1/0!	$(-0+2^{-1})/0!$	$(0-0*2^{-1}+3^{-1})/0!$	$(-0+0*2^{-1}-0*3^{-1}+4^{-1})/0!$
	1/1!	$(-1+2^{0})/1!$	$(0-1*2^{0}+3^{0})/1!$	$(-0+0*2^{0}-1*3^{0}+4^{0})/1!$
	1/2!	$(-2+2^{1})/2!$	$(1-2*2^{1}+3^{1})/2!$	$(-0+1*2^{1}-2*3^{1}+4^{1})/2!$
	1/3!	$(-3+2^{2})/3!$	$(3-3*2^{2}+3^{2})/3!$	$(-1+3*2^{2}-3*3^{2}+4^{2})/3!$
	1/4!	$(-4+2^{3})/4!$	$(6-4*2^{3}+3^{3})/3!$	$(-4+6*2^{3}-4*3^{3}+4^{3})/4!$
	1/5!	$(-5+2^{4})/5!$	$(10-5*2^{4}+3^{4})/4!$	$(-10+10*2^{3}-5*3^{3}+4^{3})/5!$

The columnsums are easily recognized as composed from exponential-series and their derivatives.

colsum 0: exp(1)
colsum 1: $exp(2)*2^{-1} - exp(1)$
colsum 2: $exp(3)*3^{-1} - exp(2) + exp(1)*1/2!$
colsum 3: $exp(4)*4^{-1} - exp(3) + exp(2)*2/2! - exp(1)*1^{2}/3!$
colsum 4: $exp(5)*5^{-1} - exp(4) + exp(3)*3/2! - exp(2)*2^{2}/3! + exp(1)*1^{3}/4!$
colsum 5: $exp(6)*6^{-1} - exp(5) + exp(4)*4/2! - exp(3)*3^{2}/3! + exp(2)*2^{3}/4! - exp(1)*1^{4}/5!$
...

colsum c = s(c)

$$s(c) = \sum_{k=0}^{c} (-1)^k \exp(c+1-k) \frac{(c+1-k)^{k-1}}{k!}$$

$$= -\sum_{k=0}^{c} \exp(c-(k-1)) \frac{((k-1)-c)^{k-1}}{k!}$$

Using that formula for the entries of

$V(1)\sim * fE2 \quad = SE\sim$
$\quad = [s(0), \; s(1), \; s(2), \ldots]$

gives numerically

$= [2.7182818\ldots, 0.976246\ldots, 0.66526\ldots, 0.50001\ldots, 0.40001\ldots, \ldots]$

On the other hand we have

$E2 * V(-1) \quad = {}^dV(2) * H$
$\quad = [\eta(1), 2\,\eta(0), 2^2\,\eta(-1), 2^3\,\eta(-2), \ldots]$

where

$H = column(\eta(1), \eta(0), \eta(-1), \eta(-2), \ldots)$

Then denote the double sum as ds

$V(1)\sim * fE2 * V(-1) = ds$

and ds should be computable by two formulae:

$ds \quad = SE\sim * V(-1)$
$\quad = V(1)\sim {}^dF(-1) * {}^dV(2) * H$

and indeed, using convergence acceleration with Euler-summing we get for ds both ways

$sd \quad = 2.12692801104\ldots$ // sumalt k=0, $(-1)^k*SE[k]$)
$\quad = 2.12692801104\ldots$ // sumalt (k=0, $\eta(1-k)*2^k/k!$)

We have another option to compute this. If we look at the diagonals in

colsum 0: $\exp(1)$
colsum 1: $\exp(2)*2^{-1} - \exp(1)$
colsum 2: $\exp(3)*3^{-1} - \exp(2) + \exp(1)*1/2!$
colsum 3: $\exp(4)*4^{-1} - \exp(3) + \exp(2)*2/2! - \exp(1)*1^2/3!$
colsum 4: $\exp(5)*5^{-1} - \exp(4) + \exp(3)*3/2! - \exp(2)*2^2/3! + \exp(1)*1^3/4!$
colsum 5: $\exp(6)*6^{-1} - \exp(5) + \exp(4)*4/2! - \exp(3)*3^2/3! + \exp(2)*2^3/4! - \exp(1)*1^4/5!$
…

we have a better insight.

Now we just add the columns in the following way with alternating signs:

colsum 0=						e
colsum 1=					$-e^2/2*1$	$-e/1!$
colsum 2=				$e^3/3*1$	$+e^2/2*2^1/1!$	$+e/2!$
colsum 3=			$-e^4/4*1$	$-e^3/3*3^1/1!$	$-e^2/2*2^2/2!$	$-e/3!$
colsum 4=		$e^5/5*1$	$+e^4/4*4^1/1!$	$+e^3/3*3^2/2!$	$+e^2/2*2^3/3!$	$+e/4!$
colsum 5=	$-e^6/6*1$	$-e^5/5*5^1/1!$	$-e^4/4*4^2/2!$	$-e^3/3*3^3/3!$	$-e^2/2*2^4/4!$	$-e/5!$
…	…	…	…	…	…	…
alt.sum=	$=-e^6/6*e^6$	$= e^5/5*e^5$	$=-e^4/4*e^4$	$= e^3/3*e^3$	$=-e^2/2*e^2$	$= e^1/1*e^1$
using $ee=e^2$	$=-ee^6/6$	$= ee^5/5$	$=-ee^4/4$	$= ee^3/3$	$=-ee^2/2$	$= ee$

Each column in the above is convergent because it is an ordinary exponential-series. The sum of the alternating column-sums gives

$$\begin{aligned} ds &= e \quad *(1 + 1/1! + 1/2! + \dots) \\ &\quad - e^2/2 *(1+ 2 + 2^2/2! + 2^3/3! + \dots) \\ &\quad + e^3/3 *(1+ 3 + 3^2/2! + 3^3/3! + \dots) \\ &\quad - \dots + - \dots \\ &= (e^1)^2/1 - (e^2)^2/2 + (e^3)^2/3 - \dots + \dots \\ &= (e^2)^1/1 - (e^2)^2/2 + (e^2)^3/3 - \dots + \dots \end{aligned}$$

writing $ee = e^2$

$$\begin{aligned} &= ee^1/1 - ee^2/2 + ee^3/3 - \dots + \dots \\ &= \log(1 + ee) \\ &= 2.126928011042972496\dots \end{aligned}$$

Using *PkPowSum* (a variant of Euler-summation) with parameter *(1.3,1.2)* directly at the sum-formula I get again

$$ds = 2.126928011043\dots$$

which is the same as the log-expression up to the shown digits.

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[GenBernRec] (Generalized Bernoulli-recursion)
http://go.helms-net.de/math/binomial/02_2_GeneralizedBernoulliRecursion.pdf

Projekt **Bernoulli-numbers**, first versions of the above, contain a ***first rough exploratory*** course but are already cover most topics and contain also the basic material about G_p and G_m which is still missing in the above list:

[Bernoulli] http://go.helms-net.de/math/binomial/bernoulli_en.pdf

[Summation] http://go.helms-net.de/math/binomial/pmatrix.pdf

Gottfried Helms, 22.01.2010 *first version 13.12.2006*

4.1. HAKMEM: ITEM 121 (Gosper):

Consider the triangular array:

```
        1
      1   1
    1   4   1
  1  11  11   1
 1  26  66  26   1
1  57 302 302 57   1
```

This bears an interesting relationship to Pascal's triangle. The 302 in the 4th southeast diagonal and the 3rd southwest one = 4*26 + 3*66. Note that rows then sum to factorials rather than powers of 2. If the nth row of the triangle is dotted with any n consecutive elements of (either) n+1st diagonal of Pascal's triangle, we get the nth Bernoulli polynomial: for n = 5, 1(6,i) + 26(6,i+1) + 66(6,i+2) + 26(6,i+3) + 1(6,i+4) = sum of 5th powers of 1 thru i+5, where (j,i) = BINOMIAL (j+i j).

5. Appendix: Notation for matrices and vectors

Matrices and vectors are understood with infinite size; the indexes for (*row,column*) are denoted as *(r,c)* beginning at zero.

A vector is as default understood as column-vector;
the transpose is indicated by the tilde ("~") as in ***Pari/Gp***,
its use as diagonalmatrix is indicated by a small prefix ^{d}.

Some standardvectors are

the Vandermonde-vector *V(x)*. This is a notation for a ***type*** of vector:

$V(x) = \text{columnvector}_{r=0..inf}(x^r)$

So, for example I write

$V(x)\sim * V(1) = \sum_{r=0..inf} (x^r) = 1/(1-x)$

the factorial vector *F(s)*, often simply used as
diagonalvectors $^dF(1) = F$ and $^dF(-1) = f$ with

$F(s) = \text{columnvector}_{r=0..inf}((r!)^s)$

the dirichlet-vector *Z(s)* ; this is a notation for

$Z(s) = \text{columnvector}_{r=0..inf}(1/(r+1)^s)$

Some standardmatrices are

the lower triangular pascalmatrix ***P***

$P = \text{matrix}_{r,c=0..inf}(\text{binomial}(r,c))$

the matrix of Stirlingnumbers 2nd kind ***S2***:

$S2 = \text{matrix}_{r,c=0..inf}(\{r,c\})$

the matrix of Stirlingnumbers 1st kind ***S1***:

$S1 = \text{matrix}_{r,c=0..inf}(<r,c>)$

and few others.

Most of the matrices I deal with have the property, that multiplication with a vandermonde-vector yields again a vandermonde-vector, so such multiplications can be concatenated and simplified expressed by a matrixpower, if that is defined. So for instance

$P * V(x) = V(x+1)$ // by binomial-theorem
$P*P*P*P * V(x) = P^4*V(x)$
$= V(x+4)$ // by repeated app. of binomial-theorem

I call such matrices "*operator*"(matrices) "*acting on formal powerseries*", because the vector-/matrix manipulations are then short notations for manipulations on the coefficients of formal powerseries. Often a similarity-transformation of a matrix converts it to an operator, for instance

$fS2F = f * S2 * F = {}^dF(1)^{-1} * S2 * {}^dF(1)$

where I assign a short memorizable name to the resulting matrix if this becomes a standard-matrix. Then

$V(x)\sim * fS2F = V(\exp(x)-1) \sim$

by a well known definition, found for instance in M.Abramowitz/I.Stegun, and the new similarity transformed matrix ***fS2F*** is an "operator".