



## 02 Signed Binomial/Pascalmatrix $P_J$

*Abstract: The (column-) signed version of the Pascalmatrix earns special attention due to its many number-theoretical relations. Not only is it near-related to the ubiquitous unsigned Pascalmatrix, but also has it a remarkable eigensystem, which consists of the bernoulli-numbers or alternatively the coefficients of the Euler-polynomials and others, as well as it allows summation of unsigned series which are outside of the region of convergence by replacing them by their alternating version. As the most important may be seen the variants of zeta-summation here.*

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Note: I'm using " $bi(r,c)$ " as a shorthand for " $binomial(r,c)$ " here, where always

$$bi(r,c)=0 \text{ if } c>r$$

reflecting the triangular form of  $P_J$ .

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# 1. Identities

## 1.1. The row- and column-signed versions of the "Pascal"~/ binomialmatrix

The "signed binomial" or "~Pascal" matrix contains just the binomial-coefficients in a lower triangle, column- or rowwise alternating signed:

<p>(1.1.1.) <math>P_J := P_{J,r,c} = (-1)^c * \text{binomial}(r,c)</math> // if <math>r \geq c</math></p> <p>(1.1.2.) <math>= P * J</math></p>	$\begin{bmatrix} 1 & . & . & . & . \\ 1 & -1 & . & . & . \\ 1 & -2 & 1 & . & . \\ 1 & -3 & 3 & -1 & . \\ 1 & -4 & 6 & -4 & 1 \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix}$
<p>(1.1.3.) <math>{}_jP := P_{J,r,c} = (-1)^r * \text{binomial}(r,c)</math> // if <math>r \geq c</math></p> <p>(1.1.4.) <math>= J * P</math></p>	$\begin{bmatrix} 1 & . & . & . & . \\ -1 & -1 & . & . & . \\ 1 & 2 & 1 & . & . \\ -1 & -3 & -3 & -1 & . \\ 1 & 4 & 6 & 4 & 1 \\ -1 & -5 & -10 & -10 & -5 & -1 \end{bmatrix}$

## 1.2. Matrix-multiples with integer powers

A consequence of (1.2.1) and (1.2.2) is, that each even power of  $P_J$  or  ${}_jP$  equals the identity-matrix:

(1.2.1.)  $P_J^{2k} = I$                        ${}_jP^{2k} = I$

Odd powers equal thus  $P_J$  or  ${}_jP$  themselves:

(1.2.2.)  $P_J^{2k+1} = P_J$                        ${}_jP^{2k+1} = {}_jP$

## 1.3. Reciprocal ("inverse" for finite dimension)

$P_J$  and  ${}_jP$  are their own reciprocals.

(1.3.1.)  $P_J = P_J^{-1}$                        $P_J * P_J = P * (J P J) = P P^{-1} = I$

(1.3.2.)  ${}_jP = {}_jP^{-1}$                        ${}_jP * {}_jP = J P * J P = P^{-1} P = I$

$P_J * P_J = I$   
 $\delta_{r,c} = \sum_{k=0..r} (-1)^{r-c} \text{binomial}(r,k) * \text{binomial}(k,c)$   
 where  $\delta_{r,c}$  is the Kronecker delta

\* 
$$\begin{bmatrix} 1 & . & . & . & . \\ 1 & -1 & . & . & . \\ 1 & -2 & 1 & . & . \\ 1 & -3 & 3 & -1 & . \\ 1 & -4 & 6 & -4 & 1 \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & . & . & . & . \\ 1 & -1 & . & . & . \\ 1 & -2 & 1 & . & . \\ 1 & -3 & 3 & -1 & . \\ 1 & -4 & 6 & -4 & 1 \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix} = \begin{bmatrix} 1 & . & . & . & . \\ . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 1 \end{bmatrix}$$

This is the same for the rowwise-signed version  ${}_jP$ .

### 1.4. Matrix-logarithm not available

Since there are zero-entries in the diagonal of  $(P, I)$  and  $(jP, I)$  and thus cannot be inverted, a matrix-logarithm cannot be computed.

### 1.5. Eigensystem-decomposition

$P_j$  and  $jP$  can be decomposed in various eigensystems (besides the trivial scaling of columns). That there are essentially different eigensystems possible is due to the infinite multiplicity of the two eigenvalues  $(1, -1)$ . The vectorspace for each set of eigenvalues can be rotated arbitrarily and can be positioned in numbertheoretical meaningful positions. The diagonalmatrix of eigenvalues is simply the matrix  $J$ , and the eigenmatrices, which I discuss here have generally the structure of a Toeplitzmatrix, hadamard-multiplied with the binomialmatrix  $P$  and possibly column-scaled, if that seems appropriate.

The most natural eigenmatrix seems to be the Faulhaber/Jacob Bernoulli-matrix  $G_p$  which contains (and introduced historically) the Bernoulli-numbers and its signed variant  $G_m$  such that

$$(1.5.1.) \quad P_j = G_p * J * G_p^{-1}$$

$$(1.5.2.) \quad jP = G_m * J * G_m^{-1}$$

Example:

$$G_p * J * G_p^{-1} = P_j$$

$$* \begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 2 & . & . & . & . \\ 1 & -3 & 3 & . & . & . \\ -1 & 4 & -6 & 4 & . & . \\ 1 & -5 & 10 & -10 & 5 & . \\ -1 & 6 & -15 & 20 & -15 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1/2 & 1/2 & . & . & . & . \\ 1/6 & 1/2 & 1/3 & . & . & . \\ 0 & 1/4 & 1/2 & 1/4 & . & . \\ -1/30 & 0 & 1/3 & 1/2 & 1/5 & . \\ 0 & -1/12 & 0 & 5/12 & 1/2 & 1/6 \end{bmatrix} * \text{diag} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & -1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ 1 & -3 & 3 & -1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix}$$

For more details see [Eigensystems of  \$P\_j\$  and  \$jP\$](#)

### 1.6. General (also complex) powers of $P_j$

From the diagonalization of  $P_j$  general powers of  $P_j$  can be determined by computing general powers of  $J$ , and that means, general powers of its entries  $+1$  and  $-1$ . But non-integer powers of  $-1$  are multivalued and there is no obvious criterion, which of the cyclotomic roots and powers of  $1$  and  $-1$  should be used and be placed into the diagonal.

My currently preferred proposition is, to assign the  $r$ 'th powers of the  $s$ 'th primitive complex unit-root into the  $r$ 'th entry of the diagonal, such that

$$(1.6.1) \quad D^s = \text{diag}_{r=0..inf} ( \exp ( r*s * \pi i ) )$$

is assumed and the general  $s$ 'th power of  $P_j$  is:

$$(1.6.2) \quad P_j^s = G_p D^s G_p^{-1}$$

Example:

$$P_j^{1/2} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1/2-1/2*I & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1/6-1/2*I & 1+I & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1/2 & 3/2-1/2*I & -3/2+3/2*I & -I & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1/6+1/2*I & -2*I & 1+3*I & -2-2*I & 1 & \cdot & \cdot & \cdot & \cdot \\ 1/2+1/3*I & -5/2-5/6*I & 5 & -5+5/3*I & 5/2-5/2*I & 1 & \cdot & \cdot & \cdot \end{bmatrix} P_j^{0.5}$$

The structure of  $P_j^{1/2}$  is then a Hadamard-product of

$$P_j^{1/2} = ( {}^dV(I) \odot C ) * P$$

where  $C$  is a Toeplitzmatrix with a repeated/downshifted first column:

Example:

$$C = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1/2-1/2*I & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1/6+1/2*I & -1/2-1/2*I & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1/2*I & 1/6+1/2*I & -1/2-1/2*I & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1/6+1/2*I & -1/2*I & 1/6+1/2*I & -1/2-1/2*I & 1 & \cdot & \cdot & \cdot & \cdot \\ 1/3-1/2*I & -1/6+1/2*I & -1/2*I & 1/6+1/2*I & -1/2-1/2*I & 1 & \cdot & \cdot & \cdot \end{bmatrix} C$$

The extraction of the imaginary factor  $\hat{i}$  may seem to be overcomplicated; the reason for this is, that then all entries of  $C$  have exactly  $1/2 \hat{i}$  as remaining imaginary part.

But anyway - I don't have a specific description for the entries of  $C$ , which were simpler than that by the defining matrixmultiplication, other than this way all columns are only repetitions of the first column.

## 2. Operations with vectors and matrices

### 2.1. Rowsums of $P_J$

The rowsums of  $P_J$  give the first column of  $I$  (which is identical to the powerseriesvector  $V(0)$ ).

Summing expressed as matrix-multiplication (which follows easily from the formulae of the binomial-theorem with powerseries with negative arguments in [binomial matrix]):

(2.1.1.)  $P_J * V(1) = V(0)$       since  $P J * V(1) = P * V(-1) = V(1 - 1) = V(0)$

(2.1.2.)  $\sum_{c=0..r} (-1)^c * bi(r,c) = \delta_{r+c,0}$   
 // where  $\delta$  is the Kronecker-symbol

Example:

$$P_J * V(1) = V(0)$$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & -1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ 1 & -3 & 3 & -1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The rowsums of  $J_P$  are the row-signed rowsums of  $P$  and are not discussed here (see [binomialmatrix](#))

### 2.2. $P_J$ right-multiplication with powerseries (binomial theorem)

The right-multiplication by a powerseries means to apply the binomial-theorem.

(2.2.1.)  $P_J * V(s) = V(1-s)$       // for all complex  $s$

(2.2.2.)  $\sum_{c=0..r} (-1)^c s^c binomial(r,c) = (1-s)^r$

$P_J * V(s) = V(1-s)$

	Generally	Examples
*	$\begin{bmatrix} 1 \\ s \\ s^2 \\ s^3 \\ s^4 \\ s^5 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 9 & 16 & 25 & 36 \\ 1 & 8 & 27 & 64 & 125 & 216 \\ 1 & 16 & 81 & 256 & 625 & 1296 \\ 1 & 32 & 243 & 1024 & 3125 & 7776 \end{bmatrix}$
or		
=	$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & -1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ 1 & -3 & 3 & -1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix} * \begin{bmatrix} 1 \\ 1-s \\ (1-s)^2 \\ (1-s)^3 \\ (1-s)^4 \\ (1-s)^5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 & -4 & -5 \\ 0 & 1 & 4 & 9 & 16 & 25 \\ 0 & -1 & -8 & -27 & -64 & -125 \\ 0 & 1 & 16 & 81 & 256 & 625 \\ 0 & -1 & -32 & -243 & -1024 & -3125 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 & -4 & -5 \\ 0 & 1 & 4 & 9 & 16 & 25 \\ 0 & -1 & -8 & -27 & -64 & -125 \\ 0 & 1 & 16 & 81 & 256 & 625 \\ 0 & -1 & -32 & -243 & -1024 & -3125 \end{bmatrix}$

The right-multiplication mirrors the applied powerseries about the point  $x_0=1/2+0 \hat{i}$ .

:

(2.2.3.)  $P_j * V(1/2+s) = V(1/2-s)$  //for all complex  $s$

(2.2.4.)  $\sum_{c=0..r} (-1)^c \text{binomial}(r,c) * (1/2+s)^c = (1/2-s)^r$

	Example	Generally
	$\begin{bmatrix} 1 \\ 1/2 \\ 1/4 \\ 1/8 \\ 1/16 \\ 1/32 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1/2+s \\ (1/2+s)^2 \\ (1/2+s)^3 \\ (1/2+s)^4 \\ (1/2+s)^5 \end{bmatrix}$
	*	or
$\begin{bmatrix} 1 & . & . & . & . \\ 1 & -1 & . & . & . \\ 1 & -2 & 1 & . & . \\ 1 & -3 & 3 & -1 & . \\ 1 & -4 & 6 & -4 & 1 \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix}$	=	$\begin{bmatrix} 1 \\ 1/2 \\ 1/4 \\ 1/8 \\ 1/16 \\ 1/32 \end{bmatrix}$
		or
		$\begin{bmatrix} 1 \\ 1/2-s \\ (1/2-s)^2 \\ (1/2-s)^3 \\ (1/2-s)^4 \\ (1/2-s)^5 \end{bmatrix}$

Powers of  $P_j$ , right-multiplied by a powerseries are not yet discussed.

### 2.3. right-multiplication with harmonic/zeta-series

The right-multiplication with a zeta-like-series was also already discussed in [\[binomialmatrix\]](#), but shall be shown here again. If the zeta-series with zero and negative exponents are concatenated to a complete matrix  $ZV$ , the resulting matrix  $P_j * ZV = D\sim$  is the factorial-scaled transposed matrix of the Stirling-numbers of 2'nd kind  $St_2$ .

for an integer exponent  $n$

(2.3.1.)  $\sum_{c=0..r} (-1)^c \text{bi}(r,c) * c^n = (-1)^r * r! * St_{2,n,r}$

for any column  $c$  of  $ZV$

(2.3.2.)  $P_j * ZV[,c] = J * {}^dF * St_{2\sim}[,c]$

$\begin{bmatrix} 1 & . & . & . & . \\ 1 & -1 & . & . & . \\ 1 & -2 & 1 & . & . \\ 1 & -3 & 3 & -1 & . \\ 1 & -4 & 6 & -4 & 1 \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix}$	*	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 \\ 1 & 3 & 9 & 27 & 81 & 243 \\ 1 & 4 & 16 & 64 & 256 & 1024 \\ 1 & 5 & 25 & 125 & 625 & 3125 \\ 1 & 6 & 36 & 216 & 1296 & 7776 \end{bmatrix}$
		=
		$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & -1 & -3 & -7 & -15 & -31 \\ . & . & 2 & 12 & 50 & 180 \\ . & . & . & -6 & -60 & -390 \\ . & . & . & . & 24 & 360 \\ . & . & . & . & . & -120 \end{bmatrix}$

The decomposition of  $D\sim$  shows the transpose of  $St_2$ , the triangular matrix of Stirlingnumbers of 2'nd kind.



### 3.3. Eigenmatrices $G_p$ and $G_m$

(3.3.1.1)  $G_p * J * G_p^{-1} = P_j$

$$* \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & 2 & \cdot & \cdot & \cdot & \cdot \\ 1 & -3 & 3 & \cdot & \cdot & \cdot \\ -1 & 4 & -6 & 4 & \cdot & \cdot \\ 1 & -5 & 10 & -10 & 5 & \cdot \\ -1 & 6 & -15 & 20 & -15 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1/2 & 1/2 & \cdot & \cdot & \cdot & \cdot \\ 1/6 & 1/2 & 1/3 & \cdot & \cdot & \cdot \\ 0 & 1/4 & 1/2 & 1/4 & \cdot & \cdot \\ -1/30 & 0 & 1/3 & 1/2 & 1/5 & \cdot \\ 0 & -1/12 & 0 & 5/12 & 1/2 & 1/6 \end{bmatrix} * \text{diag} \left( \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & -1 & \cdot & \cdot & \cdot & \cdot \\ 1 & -2 & 1 & \cdot & \cdot & \cdot \\ 1 & -3 & 3 & -1 & \cdot & \cdot \\ 1 & -4 & 6 & -4 & 1 & \cdot \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix}$$

where , with bernoulli-numbers  $\beta_k$  and  $\beta_1 = +1/2$ :

$$G_p[r,c] = \beta_{r-c} * \text{binomial}(r,c) / (c+1)$$

(3.3.1.2)  $G_m * J * G_m^{-1} = {}_jP$

$$* \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 2 & \cdot & \cdot & \cdot & \cdot \\ 1 & 3 & 3 & \cdot & \cdot & \cdot \\ 1 & 4 & 6 & 4 & \cdot & \cdot \\ 1 & 5 & 10 & 10 & 5 & \cdot \\ 1 & 6 & 15 & 20 & 15 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1/2 & 1/2 & \cdot & \cdot & \cdot & \cdot \\ 1/6 & -1/2 & 1/3 & \cdot & \cdot & \cdot \\ 0 & 1/4 & -1/2 & 1/4 & \cdot & \cdot \\ -1/30 & 0 & 1/3 & -1/2 & 1/5 & \cdot \\ 0 & -1/12 & 0 & 5/12 & -1/2 & 1/6 \end{bmatrix} * \text{diag} \left( \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 2 & 1 & \cdot & \cdot & \cdot \\ -1 & -3 & -3 & -1 & \cdot & \cdot \\ 1 & 4 & 6 & 4 & 1 & \cdot \\ -1 & -5 & -10 & -10 & -5 & -1 \end{bmatrix}$$

where the entries of the first column are bernoulli-numbers  $\beta_k$  and (with the now general convention,

$$\beta_1 = -1/2$$

$$G_m[r,c] = \beta_{r-c} * \text{binomial}(r,c) / (c+1)$$

Defining  $\beta_1 = +1/2$  , then simply

$$G_m = J G_p J$$

See a more extensive discussion of this matrix in Bernoulli-matrix and [Generalized Bernoulli-Recursion](#).

Related to this, also the bernoulli-polynomials, which are only column-scalings of  $G_m$ , can be used as eigenmatrix, where the setting  $\beta_1 = -1/2$  leads to the composite matrix  ${}_jP$ :

Example:

(3.3.1.3)  $BY * J * BY^{-1} = {}_jP$

$$* \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1/2 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1/3 & 1 & 1 & \cdot & \cdot & \cdot \\ 1/4 & 1 & 3/2 & 1 & \cdot & \cdot \\ 1/5 & 1 & 2 & 2 & 1 & \cdot \\ 1/6 & 1 & 5/2 & 10/3 & 5/2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1/2 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1/6 & -1 & 1 & \cdot & \cdot & \cdot \\ 0 & 1/2 & -3/2 & 1 & \cdot & \cdot \\ -1/30 & 0 & 1 & -2 & 1 & \cdot \\ 0 & -1/6 & 0 & 5/3 & -5/2 & 1 \end{bmatrix} * \text{diag} \left( \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 2 & 1 & \cdot & \cdot & \cdot \\ -1 & -3 & -3 & -1 & \cdot & \cdot \\ 1 & 4 & 6 & 4 & 1 & \cdot \\ -1 & -5 & -10 & -10 & -5 & -1 \end{bmatrix}$$

### 3.4. Eigenmatrix VE

Based on the eigenvector  $V(1/2)$  another eigenmatrix can be defined, which has a similar structure:

$$(3.4.1.1) \quad VE := VE_{r,c} = (1/2)^{r-c} * \text{binomial}(r,c) \\ \text{if } r \geq c, \quad \text{else} := 0$$

$$\begin{bmatrix} 1 & . & . & . & . \\ 1/2 & 1 & . & . & . \\ 1/4 & 1 & 1 & . & . \\ 1/8 & 3/4 & 3/2 & 1 & . \\ 1/16 & 1/2 & 3/2 & 2 & 1 \\ 1/32 & 5/16 & 5/4 & 5/2 & 5/2 & 1 \end{bmatrix} \quad VE$$

The inverse of  $VE$  is

$$(3.4.1.2) \quad VE^{-1}_{r,c} = (-1/2)^{r-c} * \text{binomial}(r,c) \quad \text{if } r \geq c, \quad \text{else} := 0$$

$$(3.4.1.3) \quad VE^{-1} = J * VE * J$$

$$\begin{bmatrix} 1 & . & . & . & . \\ -1/2 & 1 & . & . & . \\ 1/4 & -1 & 1 & . & . \\ -1/8 & 3/4 & -3/2 & 1 & . \\ 1/16 & -1/2 & 3/2 & -2 & 1 \\ -1/32 & 5/16 & -5/4 & 5/2 & -5/2 & 1 \end{bmatrix} \quad VE^{-1}$$

$P_j$  can then be seen as composed from

$$(3.4.1.4) \quad VE * J * VE^{-1} = P_j$$

$$\begin{bmatrix} 1 & . & . & . & . \\ -1/2 & 1 & . & . & . \\ 1/4 & -1 & 1 & . & . \\ -1/8 & 3/4 & -3/2 & 1 & . \\ 1/16 & -1/2 & 3/2 & -2 & 1 \\ -1/32 & 5/16 & -5/4 & 5/2 & -5/2 & 1 \end{bmatrix} * \begin{bmatrix} 1 & . & . & . & . \\ -1 & 1 & . & . & . \\ 1 & -1 & 1 & . & . \\ -1 & 1 & -1 & 1 & . \\ 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & . & . & . & . \\ 1 & -1 & . & . & . \\ 1 & -2 & 1 & . & . \\ 1 & -3 & 3 & -1 & . \\ 1 & -4 & 6 & -4 & 1 \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix} \quad P_j$$

$$\begin{bmatrix} 1 & . & . & . & . \\ 1/2 & 1 & . & . & . \\ 1/4 & 1 & 1 & . & . \\ 1/8 & 3/4 & 3/2 & 1 & . \\ 1/16 & 1/2 & 3/2 & 2 & 1 \\ 1/32 & 5/16 & 5/4 & 5/2 & 5/2 & 1 \end{bmatrix} * \text{diag} \left( \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & . & . & . & . \\ 1 & -1 & . & . & . \\ 1 & -2 & 1 & . & . \\ 1 & -3 & 3 & -1 & . \\ 1 & -4 & 6 & -4 & 1 \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix}$$

Since

$$VE = J VE J$$

and

$$P_j = VE * J * (J VE J)$$

we have the surprising result that also

$$(3.4.1.5) \quad VE * J * (J VE J) = VE * VE J = VE^2 J = P_j$$

$$(3.4.1.6) \quad VE^2 = P$$

$$VE^2 = P \quad \begin{bmatrix} 1 & . & . & . & . \\ 1/2 & 1 & . & . & . \\ 1/4 & 1 & 1 & . & . \\ 1/8 & 3/4 & 3/2 & 1 & . \\ 1/16 & 1/2 & 3/2 & 2 & 1 \\ 1/32 & 5/16 & 5/4 & 5/2 & 5/2 & 1 \end{bmatrix} * \begin{bmatrix} 1 & . & . & . & . \\ -1 & 1 & . & . & . \\ 1 & -1 & 1 & . & . \\ -1 & 1 & -1 & 1 & . \\ 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & . & . & . & . \\ 1/2 & 1 & . & . & . \\ 1/4 & 1 & 1 & . & . \\ 1/8 & 3/4 & 3/2 & 1 & . \\ 1/16 & 1/2 & 3/2 & 2 & 1 \\ 1/32 & 5/16 & 5/4 & 5/2 & 5/2 & 1 \end{bmatrix} * \begin{bmatrix} 1 & . & . & . & . \\ 1 & 1 & . & . & . \\ 1 & 2 & 1 & . & . \\ 1 & 3 & 3 & 1 & . \\ 1 & 4 & 6 & 4 & 1 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & . & . & . & . \\ 1 & -1 & . & . & . \\ 1 & -2 & 1 & . & . \\ 1 & -3 & 3 & -1 & . \\ 1 & -4 & 6 & -4 & 1 \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix}$$

This illustrates the identity:

$$\begin{aligned}
 \binom{r}{c} &= \sum_{k=0}^{\infty} \frac{1}{2^{r-k}} \frac{1}{2^{k-c}} \binom{r}{k} \binom{k}{c} = \frac{1}{2^{r-c}} \sum_{k=0}^{\infty} \binom{r}{k} \binom{k}{c} \quad \text{where } \binom{a}{b} = 0 \text{ if } a < b \\
 &= \frac{r!}{c! d!} \frac{1}{2^d} \sum_{k=0}^d \frac{d!}{(d-k)! k!} \quad \text{where } d = r - c \\
 (3.4.1.7) \quad &= \binom{r}{c} * \frac{1}{2^d} \sum_{k=0}^d \binom{d}{k} \\
 &= \binom{r}{c} * \frac{1}{2^d} * 2^d \\
 &= \binom{r}{c}
 \end{aligned}$$

### 3.5. Eigenmatrix ZE

Based on the eigenvector, which is the harmonic series:

$$(3.5.1.1) \quad ZE := ZE_{r,c} = 1/(r-c+1) * \text{binomial}(r,c) \quad \text{if } r \geq c, \quad \text{else } := 0$$

$$(3.5.1.2) \quad P_J * ZE = ZE * J$$

$$\begin{bmatrix}
 1 & . & . & . & . \\
 1/2 & 1 & . & . & . \\
 1/3 & 1 & 1 & . & . \\
 1/4 & 1 & 3/2 & 1 & . \\
 1/5 & 1 & 2 & 2 & 1 \\
 1/6 & 1 & 5/2 & 10/3 & 5/2 & 1
 \end{bmatrix} \text{ ZE}$$

The reciprocal of **ZE** is

$$(3.5.1.3) \quad ZE^{-1} =$$

$$\begin{bmatrix}
 1 & . & . & . & . \\
 -1/2 & 1 & . & . & . \\
 1/6 & -1 & 1 & . & . \\
 0 & 1/2 & -3/2 & 1 & . \\
 -1/30 & 0 & 1 & -2 & 1 \\
 0 & -1/6 & 0 & 5/3 & -5/2 & 1
 \end{bmatrix} \text{ ZE}^{-1}$$

where

$$(3.5.1.4) \quad \text{with } \beta_1 = +1/2 \\
 ZE_{r,c}^{-1} = \beta_{r-c+1} * \text{binomial}(r,c) \quad \text{if } r \geq c, \quad \text{else } := 0,$$

and the row  $r$  of  $ZE^{-1}$  just defines the coefficients of a variant of the bernoulli-polynomial  $B_r(x)$ .

$P_J$  can now be seen as composed from

$$(3.5.1.5) \quad ZE * J * ZE^{-1} = P_J$$

$$* \begin{bmatrix}
 1 & . & . & . & . \\
 -1/2 & 1 & . & . & . \\
 1/6 & -1 & 1 & . & . \\
 0 & 1/2 & -3/2 & 1 & . \\
 -1/30 & 0 & 1 & -2 & 1 \\
 0 & -1/6 & 0 & 5/3 & -5/2 & 1
 \end{bmatrix} \text{ ZE}^{-1}$$

$$\begin{bmatrix} 1 & . & . & . & . \\ 1/2 & 1 & . & . & . \\ 1/3 & 1 & 1 & . & . \\ 1/4 & 1 & 3/2 & 1 & . \\ 1/5 & 1 & 2 & 2 & 1 \\ 1/6 & 1 & 5/2 & 10/3 & 5/2 & 1 \end{bmatrix} * \text{diag} \left( \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 1 & . & . & . & . \\ 1 & -1 & . & . & . \\ 1 & -2 & 1 & . & . \\ 1 & -3 & 3 & -1 & . \\ 1 & -4 & 6 & -4 & 1 \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix}$$

and this describes the identity

$$(3.5.1.6) \quad \binom{r}{c} = (-1)^c \sum_{k=0}^{\infty} t_k \quad \text{where } t_k = \begin{cases} \frac{\beta_{k-c}}{r-k+1} \binom{r}{k} \binom{k}{c} & \text{if } r \geq k \geq c \\ 0 & \text{else} \end{cases}$$

and  $\beta_k$  are the bernoulli-numbers using  $\beta_1 = +1/2$

### 3.6. General powers

## 4. Details/Proofs

### 4.1. Reciprocal ("inverse" for finite dimension)

#### Proof:

First note, that

$$(4.1.1) \quad P_J^{-1} = (P * J)^{-1} = J^1 * P^{-1}$$

$$\text{From (eq. 01 1.2.2)} \quad P^{-1} = J * P * J$$

$$\text{and it is also} \quad J = J^1$$

Then

$$(4.1.2) \quad \begin{aligned} P_J^{-1} &= J^1 * (J * P * J) \\ &= (J^1 * J) * (P * J) \\ &= P * J \\ P_J^{-1} &= P_J \end{aligned}$$

Analogously:

$$(4.1.3) \quad \begin{aligned} {}_J P^{-1} &= (J * P)^{-1} \\ &= P^{-1} * J^1 \\ &= J * P * J * J^1 \\ &= J * P \end{aligned}$$

$$(4.1.4) \quad {}_J P^{-1} = {}_J P$$

End of proof

### 4.2. Eigenmatrices

#### 4.2.1. A basic identity based on the Bernoulli-recursion

The known recursive Bernoulli-identity

$$(1 - \beta)^{[k]} = \beta_k$$

expanded to a system of equations for all Bernoulli-numbers expressed a matrix-identity, where  $B^+$  is the vector of Bernoulli-numbers  $B^+ = [\beta_0, \beta_1, \beta_2, \dots] = [1, 1/2, 1/6, 0, -1/30, \dots]$ .

There is also the second identity

$$-(1 + \beta)^{[k]} = \beta_k$$

where now  $B^-$  is the vector of Bernoulli-numbers  $B^- = [\beta_0, \beta_1, \beta_2, \dots] = [1, -1/2, 1/6, 0, -1/30, \dots]$ .

$$(4.2.1.1) \quad \begin{matrix} \begin{bmatrix} b0 \\ b1 \\ b2 \\ b3 \\ b4 \\ b5 \end{bmatrix} \\ * \end{matrix} \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & -1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ 1 & -3 & 3 & -1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix} = \begin{bmatrix} b0 \\ b1 \\ b2 \\ b3 \\ b4 \\ b5 \end{bmatrix}$$

The sets of Bernoulli-numbers differ only at  $\beta_1$ , and all other odd-indexed Bernoulli-numbers are zero. We may write in vector-notation:

$$B^+ = J * B^-$$

Now assume Toeplitz-matrices based on these bernoulli-vectors,  $T(B^+)$  and  $T(B^-)$ , then

$$T(B^+) = J * T(B^-) * J$$

and their difference is a matrix with units in the first subdiagonal:

$$T(B^+) - T(B^-) = {}^{(-1)}I$$

$$jP * Bm = Bm * J$$

$$P * Bm = J Bm J = Bp$$

$$P * Bm = Bp$$

$$(P-I) Bm = Ix$$

$$Bm = (P-I)^{-1} * Ix$$



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## 5. References

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- [Project-Index] <http://go.helms-net.de/math/binomial/index>
- [Intro] [http://go.helms-net.de/math/binomial/00\\_0\\_intro.pdf](http://go.helms-net.de/math/binomial/00_0_intro.pdf)
- [binomialmatrix] [http://go.helms-net.de/math/binomial/01\\_1\\_binomialmatrix.pdf](http://go.helms-net.de/math/binomial/01_1_binomialmatrix.pdf)
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- [SumLikePow] (Sums of like powers)  
[http://go.helms-net.de/math/binomial/04\\_3\\_SummingOfLikePowers.pdf](http://go.helms-net.de/math/binomial/04_3_SummingOfLikePowers.pdf)
- [GenBernRec] (Generalized Bernoulli-recursion)  
[http://go.helms-net.de/math/binomial/02\\_2\\_GeneralizedBernoulliRecursion.pdf](http://go.helms-net.de/math/binomial/02_2_GeneralizedBernoulliRecursion.pdf)

Projekt **Bernoulli-numbers**, first versions of the above, contain a **first rough exploratory** course but are already cover most topics and contain also the basic material about  $G_p$  and  $G_m$  which is still missing in the above list:

- [Bernoulli] [http://go.helms-net.de/math/binomial/bernoulli\\_en.pdf](http://go.helms-net.de/math/binomial/bernoulli_en.pdf)
- [Summation] <http://go.helms-net.de/math/binomial/pmatrix.pdf>
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Gottfried Helms, 13.12.2006