Identities involving binomial-coefficients, Bernoulli- and Stirlingnumbers

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Mathematical Miniatures

03 Stirling-matrices St_1 and St_2

Abstract: The Stirling-matrices occur as other initimate and basic relatives of the ZV-(Vandermonde) matrix. Variants transform powerseries to exponentialseries and conversely. Using finite sizes they or their scaled variants give rational approximations to logarithms and exponentials. The most striking property for me is, that they are eigenmatrices of the Bernoullian-matrix G_p, which sums geometric series to zeta-type series of any like powers to any finite number of terms.

Most of the formulae here are heuristic findings (although in the meantime I found most of the formulas in textbooks and online-references). The focus in my recent study was primarily at the binomial- and the Gp-matrix; but I expect to understand more details of these matrices when analyzing the Stirling-matrices intensely.

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1. Definitions/ Identities

1.1. Introduction

The following matrices are defined here:

*(**1.1.1.**) St_1: lower triangular matrix of Stirling-numbers 1'st kind*

The definition for Stirling-numbers of first kind is the expansion of coefficients of *x* after expansion of the product

1 for row r=0
(x-1)(x-2)(x-3)...(x-r) for a row r>0

St1

1	.	.	.	.	.
-1	1	.	.	.	.
2	-3	1	.	.	.
-6	11	-6	1	.	.
24	-50	35	-10	1	.
-120	274	-225	85	-15	1

example:

$(x-1)(x-2) = 2 - 3*x + 1*x^2$ *// coefficients of row 2*
$(x-1)(x-2)(x-3) = -6 + 11*x - 6*x^2 + 1*x^3$ *// coefficients of row 3*

see [MW-St1],[AS-ST]

*(**1.1.2.**) St_2: lower triangular matrix of Stirling-numbers 2'nd kind*

St2

1	.	.	.	.	.
1	1	.	.	.	.
1	3	1	.	.	.
1	7	6	1	.	.
1	15	25	10	1	.
1	31	90	65	15	1

see [MW-St1],[AS-ST]

An explicit expanding of the recurrence-relation give the following identities, which reflect the columns of ***St₂***: *(note, that the indexing of rows (n) and columns is 1-based here)*:

$$\left\{ \begin{matrix} n \\ 2 \end{matrix} \right\} = \frac{\frac{1}{1}(2^{n-1}-1^{n-1})}{0!}$$

$$\left\{ \begin{matrix} n \\ 3 \end{matrix} \right\} = \frac{\frac{1}{1}(3^{n-1}-2^{n-1})-\frac{1}{2}(3^{n-1}-1^{n-1})}{1!}$$

$$\left\{ \begin{matrix} n \\ 4 \end{matrix} \right\} = \frac{\frac{1}{1}(4^{n-1}-3^{n-1})-\frac{2}{2}(4^{n-1}-2^{n-1})+\frac{1}{3}(4^{n-1}-1^{n-1})}{2!}$$

$$\left\{ \begin{matrix} n \\ 5 \end{matrix} \right\} = \frac{\frac{1}{1}(5^{n-1}-4^{n-1})-\frac{3}{2}(5^{n-1}-3^{n-1})+\frac{3}{3}(5^{n-1}-2^{n-1})-\frac{1}{4}(5^{n-1}-1^{n-1})}{3!}$$

Shifted versions

Sometimes these matrices are defined with an additional leading row and column containing zeros except *1* at *[0,0]* (see en.wikipedia.org).

(1.1.3.) *$St_1^{(1)}$: St_1, downshifted one row/column*

This definition can be seen as extension of ***St_1*** by the *(x-0)*-factor:

1	*for row r=0*
(x-0)(x-1)(x-2)(x-3)...(x-(r-1))	*for row r>0*

1	.	.	.	.	.
0	1	.	.	.	.
0	-1	1	.	.	.
0	2	-3	1	.	.
0	-6	11	-6	1	.
0	24	-50	35	-10	1

see [WIKI-St1],[AS-ST]

(1.1.4.) *$St_2^{(1)}$: St_2, downshifted one row/column*

1	.	.	.	.	.
0	1	.	.	.	.
0	1	1	.	.	.
0	1	3	1	.	.
0	1	7	6	1	.
0	1	15	25	10	1

see [WIKI-St2],[AS-ST]

factorial scaled versions fSt_1, fSt_1F, St_2F, fSt_2F

Of special interest ar also the factorial row- and row/column-scaled versions.

(1.1.5.) *$fSt_1 := F^{-1} * St_1$*

1	.	.	.	.	.
-1	1	.	.	.	.
1	-3/2	1/2	.	.	.
-1	11/6	-1	1/6	.	.
1	-25/12	35/24	-5/12	1/24	.
-1	137/60	-15/8	17/24	-1/8	1/120

(1.1.6.) *$fSt_1F := F^{-1} * St_1 * F$*

1	.	.	.	.	.
-1	1	.	.	.	.
1	-3/2	1	.	.	.
-1	11/6	-2	1	.	.
1	-25/12	35/12	-5/2	1	.
-1	137/60	-15/4	17/4	-3	1

They perform the summing to logarithms, if the columns are used as coefficients of a powerseries. (see paragraphs below)

(1.1.7.) *$St_2F := St_2 * F$*

1	.	.	.	.	.
1	1	.	.	.	.
1	3	2	.	.	.
1	7	12	6	.	.
1	15	50	60	24	.
1	31	180	390	360	120

(1.1.8.) *$fSt_2F := F^{-1} * St_2 * F$*

1	.	.	.	.	.
1	1	.	.	.	.
1/2	3/2	1	.	.	.
1/6	7/6	2	1	.	.
1/24	5/8	25/12	5/2	1	.
1/120	31/120	3/2	13/4	3	1

They perform the summing to exponentials (inverse of the summing of ***fSt_1F***), if the columns are used as coefficients of a powerseries. (see paragraphs below)

1.2. Definition in terms of their reciprocity

The matrices St_1 and St_2 (as well as the shifted versions) are also defined by their mutual reciprocity (either St_2 as reciprocal of St_1 or opposite):

(1.2.1.) $St_2 = St_1^{-1}$

$$\sum_{k=0}^{r} St1[r,k] * St2[k,c] = \delta_{r,c} \qquad \text{where } \delta \text{ is the Kronecker-delta}$$

$St_1 * St_2 = I$

$$St1 = \begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 2 & -3 & 1 & . & . & . \\ -6 & 11 & -6 & 1 & . & . \\ 24 & -50 & 35 & -10 & 1 & . \\ -120 & 274 & -225 & 85 & -15 & 1 \end{bmatrix}$$

$$* \qquad St2 = \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 3 & 1 & . & . & . \\ 1 & 7 & 6 & 1 & . & . \\ 1 & 15 & 25 & 10 & 1 & . \\ 1 & 31 & 90 & 65 & 15 & 1 \end{bmatrix} = I = \begin{bmatrix} 1 & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & 1 & . & . & . \\ . & . & . & 1 & . & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & 1 \end{bmatrix}$$

1.3. $St_2^{(1)}$ occurs also as matrix of coefficients of the derivatives of e^{e^x}

If one computes the coefficients of the derivatives of $exp(e^x)$

$f := \quad f(x) = exp(e^x)$

and $\quad z := e^x$

Then

(1.3.1.)

$$\begin{aligned} f &= f *(\ 1\ ) \\ f' &= f *(\ 0 \quad 1\,z) \\ f'' &= f *(\ 0 \quad 1\,z \quad 1\,z^2) \\ f''' &= f *(\ 0 \quad 1\,z \quad 3\,z^2 \quad 1\,z^3) \\ f^{(4)} &= f *(\ 0 \quad 1\,z \quad 7\,z^2 \quad 6\,z^3 \quad 1\,z^4) \\ \ldots\ &\text{etc} \end{aligned}$$

$$St2(1) = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 1 & 1 & . & . & . \\ 0 & 1 & 3 & 1 & . & . \\ 0 & 1 & 7 & 6 & 1 & . \\ 0 & 1 & 15 & 25 & 10 & 1 \end{bmatrix}$$

which means in matrix-notation

(1.3.2.)

$$f * St_2^{(1)} * V(e^x) = [f, f', f'', f''', f^{(4)}, \ldots]\sim$$

$$f * St_2 * e^x\, V(e^x) = [f', f'', f''', f^{(4)}, \ldots]\sim$$

or

(1.3.3.)

$$St_2^{(1)} * V(e^x) = [f, f', f'', f''', f^{(4)}, \ldots]\sim /f$$

1.4. Recursive definitions

Recursive definitions include:

(1.4.1.)
$$St1[r,0] = (-1)^r * r!$$
$$St1[r+1,c] = (-1)^{r-c} * abs((r+1) * St1[r,c]) + abs(St1[r-1,c-1])$$

(1.4.2.)
$$St2[r,0] = 1$$
$$St2[r+1,c] = (c+1) * St2[r,c] + St2[r-1,c-1]$$

(additional remarks: see "details/derivations")

2. Simple relations to other vectors and matrices

2.1. Shifted versions by binomial-transformation

A binomial-transformation performs the shifting of St_2 to $St_2^{(1)}$ resp St_1 to $St_1^{(1)}$ and more generally a shifted binomial-transform $P^{(k)}$ performs the shifting of $St_2^{(k)}$ to $St_2^{(k+1)}$ resp $St_1^{(k)}$ to $St_1^{(k+1)}$.

Shifting of St_2:

(2.1.1.) $$P^{-1} * St_2 = St_2^{(1)}$$

$$P^{-1} * St_2 = St_2^{(1)}$$

$$\begin{matrix} & \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 3 & 1 & . & . & . \\ 1 & 7 & 6 & 1 & . & . \\ 1 & 15 & 25 & 10 & 1 & . \\ 1 & 31 & 90 & 65 & 15 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ -1 & 3 & -3 & 1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ -1 & 5 & -10 & 10 & -5 & 1 \end{bmatrix} & = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 1 & 1 & . & . & . \\ 0 & 1 & 3 & 1 & . & . \\ 0 & 1 & 7 & 6 & 1 & . \\ 0 & 1 & 15 & 25 & 10 & 1 \end{bmatrix} \end{matrix}$$

From the composition of P it also follows that shifting in terms of Bernoulli-numbers:

(2.1.2.) $$G_m * St_2 * {}^dZ(-1)= St_2^{(1)}$$

since

$$\begin{aligned} P^{-1} &= J * P * J \\ &= G_m * J * G_m^{-1} * J \\ &= G_m * G_p^{-1} \\ &= G_m * St_2 * {}^dZ(1)^{-1}\ St_2^{-1} \end{aligned}$$

and

$$\begin{aligned} P^{-1} * St_2 &= G_m * St_2 * {}^dZ(1)^{-1}\ St_2^{-1} * St_2 \\ &= G_m * St_2 * {}^dZ(-1) \\ St_2^{(1)} &= G_m * St_2 * {}^dZ(-1) \end{aligned}$$

Example:

$$G_m * St_2 * {}^dZ(-1)= St_2^{(1)}$$

$$\begin{matrix} & * \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 3 & 1 & . & . & . \\ 1 & 7 & 6 & 1 & . & . \\ 1 & 15 & 25 & 10 & 1 & . \\ 1 & 31 & 90 & 65 & 15 & 1 \end{bmatrix} * diag(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}) \\ \begin{bmatrix} 1 & . & . & . & . & . \\ -1/2 & 1/2 & . & . & . & . \\ 1/6 & -1/2 & 1/3 & . & . & . \\ 0 & 1/4 & -1/2 & 1/4 & . & . \\ -1/30 & 0 & 1/3 & -1/2 & 1/5 & . \\ 0 & -1/12 & 0 & 5/12 & -1/2 & 1/6 \end{bmatrix} & = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 1 & 1 & . & . & . \\ 0 & 1 & 3 & 1 & . & . \\ 0 & 1 & 7 & 6 & 1 & . \\ 0 & 1 & 15 & 25 & 10 & 1 \end{bmatrix} \end{matrix}$$

Shifting of St_1:

(2.1.3.)	$St_1 * P = St_1^{(1)}$

$St_1 * P = St_1^{(1)}$

$$
* \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix} \text{P}
$$

$$
\text{St1} \begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 2 & -3 & 1 & . & . & . \\ -6 & 11 & -6 & 1 & . & . \\ 24 & -50 & 35 & -10 & 1 & . \\ -120 & 274 & -225 & 85 & -15 & 1 \end{bmatrix} = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & -1 & 1 & . & . & . \\ 0 & 2 & -3 & 1 & . & . \\ 0 & -6 & 11 & -6 & 1 & . \\ 0 & 24 & -50 & 35 & -10 & 1 \end{bmatrix} \text{St1(1)}
$$

Again from the composition of ***P*** it also follows that shifting in terms of Bernoulli-numbers:

(2.1.4.)	$G_m * St_2 * {}^dZ(-1) = St_2^{(1)}$

since

$$
\begin{aligned}
P &= P * J * J \\
&= G_p * J * G_p^{-1} * J \\
&= G_p * J * G_p^{-1} * J \\
&= G_p * G_m^{-1} \\
&= St_2 * {}^dZ(1)\, St_2^{-1} * G_m^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
St_1 * P &= St_1 * St_2 * {}^dZ(1)\, St_2^{-1} * G_m^{-1} \\
&= {}^dZ(1)\, St_1 * G_m^{-1} \\
St_1^{(1)} &= {}^dZ(1)\, St_1 * G_m^{-1}
\end{aligned}
$$

Example:

${}^dZ(1)\, St_1 * G_m^{-1} = St_1^{(1)}$

$$
* \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 2 & . & . & . & . \\ 1 & 3 & 3 & . & . & . \\ 1 & 4 & 6 & 4 & . & . \\ 1 & 5 & 10 & 10 & 5 & . \\ 1 & 6 & 15 & 20 & 15 & 6 \end{bmatrix} \text{Gm}^{\wedge}\text{-1}
$$

$$
diag(\begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \\ 1/5 \\ 1/6 \end{bmatrix}) * \begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 2 & -3 & 1 & . & . & . \\ -6 & 11 & -6 & 1 & . & . \\ 24 & -50 & 35 & -10 & 1 & . \\ -120 & 274 & -225 & 85 & -15 & 1 \end{bmatrix} \text{St1} = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & -1 & 1 & . & . & . \\ 0 & 2 & -3 & 1 & . & . \\ 0 & -6 & 11 & -6 & 1 & . \\ 0 & 24 & -50 & 35 & -10 & 1 \end{bmatrix} \text{St1(1)}
$$

The shifting in the limit

A consequence of this is, that repeated application of (analoguously shifted) binomial-matrices approximate to the identity matrix:

$$St_1 * P * P^{(1)} * P^{(2)} * \ldots = I \qquad (2.1.5.)$$

and that means, that the iterated product of shifted binomialmatrices approximate ***St2***.

$$\Pi_{k=0..oo} P^{(k)} = St_2 \qquad (2.1.6.)$$

*** The definition of the shifted Pascal-matrix**

$P^{(n)}$ is the n-positions dowsnshifted version of P

$P^{(n)}[r,c]$ = binomial(r-n,c-n) if r-n>=0

= $\delta_{r,c}$ if r-n<0

where $\delta_{r,c}$ is the Konecker-symbol

Examples:

$P^{(1)}$

1	.	.	.
0	1	.	.
0	1	1	.
0	1	2	1

$P^{(1)\,-1}$

1	.	.	.
0	1	.	.
0	-1	1	.
0	1	-2	1

$P^{(-1)}$

1	.	.	.
2	1	.	.
3	3	1	.
4	6	4	1

...

$P^{(-1)\,-1}$

1	.	.	.
-2	1	.	.
3	-3	1	.
-4	6	-4	1

*** The use of the shifted Pascal-matrix to compute St_2**

Example:

$P*P^{(1)}$

01

1	.	.	.	.	.
1	1	.	.	.	.
1	3	1	.	.	.
1	7	5	1	.	.
1	15	17	7	1	.
1	31	49	31	9	1

$P*P^{(1)}*P^{(2)}$

012

1	.	.	.	.	.
1	1	.	.	.	.
1	3	1	.	.	.
1	7	6	1	.	.
1	15	25	9	1	.
1	31	90	52	12	1

$P*P^{(1)}*P^{(2)}*P^{(3)}$

0123

1	.	.	.	.	.
1	1	.	.	.	.
1	3	1	.	.	.
1	7	6	1	.	.
1	15	25	10	1	.
1	31	90	65	14	1

...

$P*P^{(1)}*P^{(2)} \ldots = St_2$

St2

1	.	.	.	.	.
1	1	.	.	.	.
1	3	1	.	.	.
1	7	6	1	.	.
1	15	25	10	1	.
1	31	90	65	15	1

*** The use of the reciprocal of the shifted Pascal-matrix to compute St_1:**

Conversely, the infinite leftmultiplication-product of shifted inverses of binomialmatrices approximate ***St*₁**.

$$\Pi_{k=oo..0} P^{(k)-1} = St_1 \qquad (2.1.7.)$$

Example

$\ldots *P^{(2)-1}*P^{(1)-1} *P^{-1}= St_1$

St1

1	.	.	.	.	.
-1	1	.	.	.	.
2	-3	1	.	.	.
-6	11	-6	1	.	.
24	-50	35	-10	1	.
-120	274	-225	85	-15	1

...

$P^{(2)-1}*P^{(1)-1} *P^{-1}$

210

1	.	.	.	.	.
-1	1	.	.	.	.
2	-3	1	.	.	.
-6	11	-6	1	.	.
18	-39	29	-9	1	.
-54	135	-126	56	-12	1

$P^{(1)-1} *P^{-1}$

10

1	.	.	.	.	.
-1	1	.	.	.	.
2	-3	1	.	.	.
-4	8	-5	1	.	.
8	-20	18	-7	1	.
-16	48	-56	32	-9	1

2.2. Rowsums of St_1 and St_2, and rightmultiplication with a powerseries

The **rowsums** of St_2 are known as Bell-numbers:

(2.2.1.) $St_2 * V(1) = B$

$$\sum_{c=0}^{r} St2_{r,c} = B_r$$

$$\underbrace{\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 3 & 1 & . & . & . \\ 1 & 7 & 6 & 1 & . & . \\ 1 & 15 & 25 & 10 & 1 & . \\ 1 & 31 & 90 & 65 & 15 & 1 \end{bmatrix}}_{St2} * \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \\ 15 \\ 52 \\ 203 \end{bmatrix}$$

Note: If St_2 is used with a rightmultiplication by a powerseries, this introduces the polynomials called "Bell-polynomials" (see [MW-Bell]).

The **rowsums** of St_1 (except or row $r=0$) are zero, which is obvious since the *first* column of St_2 $(= St_1^{-1})$ is just the 1-vector, as well as from the definition as product of $(x-1)(x-2)\ldots$ when setting $x=1$.

(2.2.2.) $St_1 * V(1) = [1,0,0,0,\ldots]$

$$\sum_{c=0}^{r} St1_{r,c} = 0 \qquad \text{// for } r>0$$

Rightmultiplication by powerseries-vectors produce factorial scaled binomials.

Examples:

various weighted row-sums of St_1 *columns from St_2* *powerseries (or ZV~)*

$$\underbrace{\begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 2 & -3 & 1 & . & . & . \\ -6 & 11 & -6 & 1 & . & . \\ 24 & -50 & 35 & -10 & 1 & . \\ -120 & 274 & -225 & 85 & -15 & 1 \end{bmatrix}}_{St1} * \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 & 3 & 4 \\ 1 & 3 & 1 & 4 & 9 & 16 \\ 1 & 7 & 6 & 8 & 27 & 64 \\ 1 & 15 & 25 & 16 & 81 & 256 \\ 1 & 31 & 90 & 32 & 243 & 1024 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

ZV is a complete set of powerseries whose bases differ by *1* and gives a complete set of powerseries-vectors by rightmultiplication:

(2.2.3.) $St_1 * ZV\sim \; = {}^dF * P\sim$

$$
\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 9 & 16 & 25 & 36 \\ 1 & 8 & 27 & 64 & 125 & 216 \\ 1 & 16 & 81 & 256 & 625 & 1296 \\ 1 & 32 & 243 & 1024 & 3125 & 7776 \end{bmatrix}
$$

$$
\begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 2 & -3 & 1 & . & . & . \\ -6 & 11 & -6 & 1 & . & . \\ 24 & -50 & 35 & -10 & 1 & . \\ -120 & 274 & -225 & 85 & -15 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 9 & 16 & 25 & 36 \\ 1 & 8 & 27 & 64 & 125 & 216 \\ 1 & 16 & 81 & 256 & 625 & 1296 \\ 1 & 32 & 243 & 1024 & 3125 & 7776 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 2 & 3 & 4 & 5 \\ . & . & 2 & 6 & 12 & 20 \\ . & . & . & 6 & 24 & 60 \\ . & . & . & . & 24 & 120 \\ . & . & . & . & . & 120 \end{bmatrix}
$$

2.3. Columnsums (and leftmultiplication with powerseries) of St_2 and fSt_2F

The columnsums of St_2 are all divergent, so we see the summing vector as limit of a powerseries in x, when $x\to 1$:

(2.3.1.) $1/x*V(1/x)\sim * St_2 = S(1/x)\sim$

St2

$$
\begin{matrix}
 & & \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 3 & 1 & . & . & . \\ 1 & 7 & 6 & 1 & . & . \\ 1 & 15 & 25 & 10 & 1 & . \\ 1 & 31 & 90 & 65 & 15 & 1 \end{bmatrix} \\
\begin{bmatrix} 1/x & 1/x\hat{}2 & 1/x\hat{}3 & 1/x\hat{}4 & 1/x\hat{}5 & 1/x\hat{}6 \end{bmatrix} & * & = \begin{bmatrix} s0 & s1 & s2 & s3 & s4 & s5 \end{bmatrix}
\end{matrix}
$$

This can be summed (or is convergent in the leading columns) for $1/x<1$, since the progression in each column is of the order of a geometric series with quotient q, which increases with the column-number c, x must then at least equal q, or must be negative to allow $Euler_p$-summation of an appropriate order.

The following formula for $1/x<1$ seems to be valid for any entry s_c of $S\sim$ in column c:

(2.3.2.)

$$\sum_{r=0}^{oo} \frac{St2_{r,c}}{x^{r+1}} = \prod_{k=1}^{c+1} \frac{1}{x-k} = s_c(\frac{1}{x})$$
$$= \frac{1}{x-1} * \frac{1}{x-2} \dots * \frac{1}{x-(c+1)} \qquad \text{//where } x \neq 1,2,3,\dots c$$

or

$$\sum_{r=0}^{oo} St2_{r,c} x^{r+1} = \prod_{k=1}^{c+1} \frac{x}{1-kx} = s_c(x) \qquad \text{//where } 1/x \neq 1,2,3,\dots c$$

(see also [AS-ST])

Note, that in *(2.3.2)* the productterms in the denominators are just the rowsums of $St_1 * V(x)$:

(2.3.3.)

$$t_r(x) = (St_1 * V(x))_r$$
$$s_c(\frac{1}{x}) = \frac{1}{t_{c+1}(x)}$$

Example: $x=-2$, $S = [-2/3,\ 2^2/(3*5),\ -2^3/(3*5*7),\ 2^4/(3*5*7*9)\dots]$
$= [-2/3,\ 4/15,\ -8/105,\ 16/945,\dots.]$

Note: if x equals a positive integer $1<=x<=c$, we have division by zero, and the value is not defined.

2.4. fSt_2F : column-sums

When the columns of the factorial scaled version of St_2, fSt_2F are summed (leftmultiplied by the summing-vector $V(1)\sim$), this equals:

(2.4.1.)

$$V(1)\sim * fSt_2F \quad = e^d V(e-1)\sim$$

$$\lim_{r\to oo} \sum_{r=0}^{oo} S2_{r,c} \frac{c!}{r!} = e * (e-1)^c \qquad \text{// for a fixed column } c$$

Example:

$$V(1)\sim * fSt_2F = e\, V(e-1)\sim$$

$$* \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1/2 & 3/2 & 1 & . & . & . \\ 1/6 & 7/6 & 2 & 1 & . & . \\ 1/24 & 5/8 & 25/12 & 5/2 & 1 & . \\ 1/120 & 31/120 & 3/2 & 13/4 & 3 & 1 \end{bmatrix}$$

$$\lim r\text{->}oo \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} e \end{bmatrix} * \begin{bmatrix} e1\text{^}0 & e1\text{^}1 & e1\text{^}2 & e1\text{^}3 & e1\text{^}4 & e1\text{^}5 \end{bmatrix} \; {}^{*)}$$

*) *e1 is written for e-1 = exp(1)-1*

The obvious generalization of $V(1)$ in (2.4.1) into a powerseries $V(x)$ gives the transformation of a powerseries in x into a powerseries in e^x-1 (although with a cofactor of e^x):

(2.4.2.)

$$V(x)\sim * fSt_2F \quad = e^x * V(e^x-1)\sim$$

$$\lim_{r\to oo} \sum_{r=0}^{oo} x^r * S2_{r,c} \frac{c!}{r!} = e^x * (e^x-1)^c \qquad \text{//for a fixed column } c$$

(for a further smooting of the result see second next paragraph)

Example:

$$V(x)\sim * fSt_2F = e^x * V(e^x-1)$$

$$* \begin{bmatrix} 1 & . & . & . \\ 1 & 1 & . & . \\ 1/2 & 3/2 & 1 & . \\ 1/6 & 7/6 & 2 & 1 \\ 1/24 & 5/8 & 25/12 & 5/2 \\ 1/120 & 31/120 & 3/2 & 13/4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & x & x\text{^}2 & x\text{^}3 & x\text{^}4 & x\text{^}5 \end{bmatrix} = \begin{bmatrix} e\text{^}x* \end{bmatrix} * \begin{bmatrix} (e\text{^}x-1)\text{^}0 & (e\text{^}x-1)\text{^}1 & (e\text{^}x-1)\text{^}2 & (e\text{^}x-1)\text{^}3 \end{bmatrix}$$

Another variant: column-sums in zf St_2 FZ

A further variant can be given by introducing the zeta-series as additional cofactors.

Define the further scaled version of $\boldsymbol{St_2}$:

$$zfSt_2FZ = {}^dZ(1) * fSt_2F * {}^dZ(-1)$$

Then

(2.4.3.)

$$V(x)\sim * zfSt_2FZ \qquad = (e^x-1) * V(e^x - 1)\sim$$

$$\lim_{r->oo} \sum_{r=0}^{oo} x^r * S2_{r,c} \frac{(c+1)!}{(r+1)!} \qquad = \left(e^x - 1\right)^{c+1}$$

is a plain and smooth transformation of a powerseries in x into one in $(e^x - 1)$.

Example:

$$x\ V(x) \sim * zfSt_2FZ \ = (e^x - 1)\ V(e^x - 1))$$

zfSt2FZ

1	.	.	.	.	.
1/2	1	.	.	.	.
1/6	1	1	.	.	.
1/24	7/12	3/2	1	.	.
1/120	1/4	5/4	2	1	.
1/720	31/360	3/4	13/6	5/2	1

[x x^2 x^3 x^4 x^5 x^6] * = [e1^1 e1^2 e1^3 e1^4 e1^5 e1^6]

where e1 means $e^x - 1$

and using the shifted version of $\boldsymbol{St_2}$, and the factorial scaled version $\boldsymbol{fSt_2^{(1)}F}$, this is even more simple:

Example:

$$V(x) \sim * f(St_2^{(1)})F \ = V(e^x - 1)$$

fSt2(1)F

1	.	.	.	.	.
0	1	.	.	.	.
0	1/2	1	.	.	.
0	1/6	1	1	.	.
0	1/24	7/12	3/2	1	.
0	1/120	1/4	5/4	2	1

[1 x^1 x^2 x^3 x^4 x^5] * = [1 e1^1 e1^2 e1^3 e1^4 e1^5]

where e1 means $e^x - 1$

Double-sum of fSt_2, focusing the Bell-numbers

Eq. *(2.4.2)* with the factorial term *c!* rearranged to the rhs

$$V(x)\sim * fSt_2F = e^x * V(e^x - 1)\sim$$
$$V(x)\sim * fSt_2F * F^{-1} = e^x * V(e^x - 1)\sim * F^{-1}$$
$$V(x)\sim * fSt_2 = e^x * V(e^x - 1)\sim * F^{-1}$$

and then again be summed over then columns:

$$V(x)\sim * fSt_2 * V(1) = e^x * V(e^x - 1)\sim * F^{-1} * V(1)$$
$$= e^x * (1 + (e^x-1)^1/1! + (e^x-1)^2/2! + (e^x-1)^3/3! + ...)$$
$$= e^x * exp(e^x - 1)$$

and since the rowsums of fSt_2 are the Bell-numbers scaled by the reciprocal factorial:

$$fSt_2 * V(1) = F^{-1} * (St_2 *V(1)) = F^{-1} * B$$

we get the result involving the Bell-numbers:

(2.4.4.)
$$\lim \sum_{c=0}^{oo} \sum_{r=0}^{oo} S2_{r,c} \frac{x^r}{r!} = \lim \sum_{r=0}^{oo} \frac{x^r}{r!} \sum_{c=0}^{oo} S2_{r,c} =$$
$$\lim \sum_{r=0}^{oo} x^r \frac{B_r}{r!} = e^x e^{e^x - 1} = e^{e^x + (x-1)}$$

(2.4.5.) *and*
$$\sum_{r=0}^{oo} \frac{B_r}{r!} = e^e$$

Double-sum of zf St_2

The double sum over all columns of **zfSt_2** is, after rearranging the *(c+1)!* -term to the rhs in *(2.4.4)*:

(2.4.6.) $$\lim_{r->oo} \sum_{r=0}^{oo} x^r * S2_{r,c} \frac{1}{(r+1)!} = \frac{(e^x-1)^{c+1}}{(c+1)!} \qquad \text{// for a column } c$$

Summing over all columns:

(2.4.7.) *lhs-double-sum in (2.4.5)*

$$\lim \sum_{c=0}^{oo} \sum_{r=0}^{oo} x^r S2_{r,c} \frac{1}{(r+1)!} = \lim \sum_{r=0}^{oo} \frac{x^r}{(r+1)!} \sum_{c=0}^{oo} S2_{r,c}$$

$$= \lim \sum_{r=0}^{oo} x^r \frac{B_r}{(r+1)!}$$

(2.4.8.) *rhs-sum in (2.4.5)*

$$\sum_{c=0}^{oo} \frac{(e^x-1)^{c+1}}{(c+1)!} = exp(e^x-1)-1 \qquad = e^{e^x-1}-1$$

This gives

(2.4.9.) *V(x)~ * zfST2 *V(1)* $\quad = exp(e^x - 1) - 1$

$$\lim \sum_{c=0}^{oo} \sum_{r=0}^{oo} x^r S2_{r,c} \frac{1}{(r+1)!} =$$

$$\lim \sum_{r=0}^{oo} x^r \frac{B_r}{(r+1)!} = e^{e^x-1}-1$$

and

(2.4.10.) $$\sum_{r=0}^{oo} \frac{B_r}{(r+1)!} = e^{e-1}-1$$

An extension using the Bell-numbers is if not only the column-sums are considered, but the whole set of binomially weighted column-sums, say by right-multiplication by the Pascalmatrix P. Then the first column gives the same result as before, but we have more columns

Right-multiplication with the Pascalmatrix gives a matrix containing the Bell-numbers.

(2.4.11.) $St^{(1)}{}_2 * P = B$

$$\sum_{c=0}^{r} St2_{r,c} = B_r$$

P:

1	.	.	.	.	.
1	1	.	.	.	.
1	2	1	.	.	.
1	3	3	1	.	.
1	4	6	4	1	.
1	5	10	10	5	1

St2_1:

1	.	.	.	.	.
0	1	.	.	.	.
0	1	1	.	.	.
0	1	3	1	.	.
0	1	7	6	1	.
0	1	15	25	10	1

B:

1	.	.	.	.	.
1	1	.	.	.	.
2	3	1	.	.	.
5	10	6	1	.	.
15	37	31	10	1	.
52	151	160	75	15	1

The factorial-scaled version of ***B***

$$fBF = {}^dF^{-1} * B * {}^dF$$

performs summing of powerseries in x to the double-exponential $exp(exp(x)-1)$:

$$V(x)\sim * fBF = exp(exp(x)-1) * V(exp(x)-1)\sim$$

Example:

(2.4.12.) $St^{(1)}{}_2 * P = B$

$$\sum_{c=0}^{r} St2_{r,c} = B_r$$

fBF:

1	.	.	.	.	.
1	1	.	.	.	.
1	3/2	1	.	.	.
5/6	5/3	2	1	.	.
5/8	37/24	31/12	5/2	1	.
13/30	151/120	8/3	15/4	3	1

[1 x x^2 x^3 x^4 x^5] * fBF = [e^ex1] * [1 ex1 ex1^2 ex1^3 ex1^4 ex1^5]

where ex1 means exp(x)-1

2.5. Columnsums (and leftmultiplication with powerseries) of St_1 and fSt_1F

If the previous St_2-related formulae are extended by postmultiplication by St_1 we get the inverse summation-expressions for the variants of St_1:

St1

Since

*-1*V(-1)*St2 = [-1/2, 1/6, -1/24,...] = Fac(-1)* J*

it follows from the reciprocal-relation of St2 and St1 that

*-1*V(-1)*St2*St1 = [-1/2, 1/6, -1/24,...]*St1 = Fac(-1)* J *St1*

and

*Fac(-1)~ * J * St1 = -1 V(-1)*

and -for instance- using the first column of $(J*St_1)$:

$$
\begin{aligned}
&[1/2!,1/3!,1/4!,...] * [0!,1!,2!,3!,...] \\
&= \Sigma_{k=0..oo}\ k!/(k+2)! \\
&= \Sigma_{k=0..oo}\ (1/(1*2) + 1/(2*3) + 1/(3*4)...) \\
&= 1
\end{aligned}
$$

which is a known result.

For the next column we have the entries in $(J*St_1)$:

$$- [0!*0,\ 1!*(1),\ 2!*(1+1/2),\ 3!*(1+1/2+1/3),...]$$

Just the harmonic numbers as cofactors of the coefficients in the first column. So the previous sum-notation can be reused:

$$
\begin{aligned}
&= - \Sigma_{k=1..oo}\ k!/(k+2)!\ (1+1/2+1/3+...1/k) \\
&= - \ (1/(1*2)*0 + 1/(2*3)(1/1) + 1/(3*4)(1/1+1/2) + 1/(4*5)(1/1+1/2+1/3) + ...) \\
&= - 1/2 - \ (1/(3*4)(1/2) + 1/(4*5)(1/2+1/3) + \ ...) \\
&= - 1/2 - \ (1/(2*3*4) + 1/(3*4*5) + ... \\
&\qquad\qquad + 1/(4*5)(1/2) + 1/(5*6)(1/2+1/3) + \ ...) \\
&= -1
\end{aligned}
$$

zfSt₁FZ

Define

$$zfSt_1FZ = {}^dZ(1) * fSt_1F * {}^dZ(-1)$$

Then

(2.5.1.)
$$x\, V(x)\sim * zfSt_1FZ = \log(1+x) * V(\log(1+x))\sim$$
$$\lim_{r->oo} \sum_{r=0}^{oo} x^r * S1_{r,c} \frac{(c+1)!}{(r+1)!} = \log(1+x)^{c+1}$$

transforms a powerseries in x into a powerseries of the logarithm of $(1+x)$.

Example:

$$x\, V(x) \sim * zfSt_1FZ = \log(1+x)\, V(\log(1+x))$$

zfSt1FZ

$$[\, x \quad x^2 \quad x^3 \quad x^4 \quad x^5 \quad x^6 \,] * \begin{bmatrix} 1 & . & . & . & . & . \\ -1/2 & 1 & . & . & . & . \\ 1/3 & -1 & 1 & . & . & . \\ -1/4 & 11/12 & -3/2 & 1 & . & . \\ 1/5 & -5/6 & 7/4 & -2 & 1 & . \\ -1/6 & 137/180 & -15/8 & 17/6 & -5/2 & 1 \end{bmatrix} = [\, l1^1 \quad l1^2 \quad l1^3 \quad l1^4 \quad l1^5 \quad l1^6 \,]$$

where l1 means log(1+x)

Proof:

(2.5.2.) $zfSt_1FZ = zfSt_2FZ^{-1}$

since

$${}^dZ(1)* F^{-1} * St_2 * F * {}^dZ(-1) \quad * \quad {}^dZ(1)* F^{-1} * St_1 * F * {}^dZ(-1)$$
$$= {}^dZ(1)* F^{-1} * St_2 * St_1 * F * {}^dZ(-1)$$
$$= {}^dZ(1)* F^{-1} * I * F * {}^dZ(-1)$$
$$= I$$

From this and right-multiplication of *(2.5.3.)* by ***zfSt₁FZ***

$$y\, V(y)\sim * zfSt_2FZ = (e^y-1) * V(e^y-1)\sim$$
$$y\, V(y)\sim * zfSt_2FZ * zfSt_1FZ = (e^y-1) * V(e^y-1)\sim * zfSt_1FZ$$
(2.5.4.)
$$y\, V(y)\sim = (e^y-1) * V(e^y-1)\sim * zfSt_1FZ$$

Replacing e^y-1 by x and y by $\log(1+y)$ gives *(2.5.1)*.

Again a simpler form occurs with the shifted version of ***St₁***:

Example:

$$V(x) \sim * fSt_1^{(1)}F = V(\log(1+x))$$

fSt1(1)F

$$[\, 1 \quad x^1 \quad x^2 \quad x^3 \quad x^4 \quad x^5 \,] * \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & -1/2 & 1 & . & . & . \\ 0 & 1/3 & -1 & 1 & . & . \\ 0 & -1/4 & 11/12 & -3/2 & 1 & . \\ 0 & 1/5 & -5/6 & 7/4 & -2 & 1 \end{bmatrix} = [\, 1 \quad l1^1 \quad l1^2 \quad l1^3 \quad l1^4 \quad l1^5 \,]$$

where l1 means log(1+x)

---

3. Relations to other matrices

3.1. St_2 and St_1 as compositions of binomially -weighted sums of zetaseries

Composition of St_2

St_2 occurs in the matrix-multiplication $P^{-1} * ZV$ in its factorial scaled version St_2F

(3.1.1.) $P^{-1} * ZV = St_2F\sim$

$$P^{-1} * ZV = St_2F\sim$$

$$* \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 \\ 1 & 3 & 9 & 27 & 81 & 243 \\ 1 & 4 & 16 & 64 & 256 & 1024 \\ 1 & 5 & 25 & 125 & 625 & 3125 \\ 1 & 6 & 36 & 216 & 1296 & 7776 \end{bmatrix}$$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ -1 & 3 & -3 & 1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ -1 & 5 & -10 & 10 & -5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 3 & 7 & 15 & 31 \\ . & . & 2 & 12 & 50 & 180 \\ . & . & . & 6 & 60 & 390 \\ . & . & . & . & 24 & 360 \\ . & . & . & . & . & 120 \end{bmatrix}$$

or, sometimes even given as definition of St_2 in the sense of a generation-function (see *[AS-ST]*):

(3.1.2.) $St_2\sim = F^{-1} * P^{-1} * ZV$

$$St_2\sim = F^{-1} * P^{-1} * ZV$$

$$* \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 \\ 1 & 3 & 9 & 27 & 81 & 243 \\ 1 & 4 & 16 & 64 & 256 & 1024 \\ 1 & 5 & 25 & 125 & 625 & 3125 \\ 1 & 6 & 36 & 216 & 1296 & 7776 \end{bmatrix}$$

$$diag(\begin{bmatrix} 1 \\ 1 \\ 1/2 \\ 1/6 \\ 1/24 \\ 1/120 \end{bmatrix}) * \begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ -1 & 3 & -3 & 1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ -1 & 5 & -10 & 10 & -5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 3 & 7 & 15 & 31 \\ . & . & 1 & 6 & 25 & 90 \\ . & . & . & 1 & 10 & 65 \\ . & . & . & . & 1 & 15 \\ . & . & . & . & . & 1 \end{bmatrix}$$

Composition of St_1

The converse is also true, at least for the case of any finite dimension. (for infinite dimension the inverse of the Vandermondematrix ZV is not defined)

For any finite size the following is valid:

(3.1.3.) $ZV^{-1} * P = fSt_1\sim$

Example, size = 6

$$ZV^{-1} * P = fSt_1\sim$$

$$* \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -15 & 20 & -15 & 6 & -1 \\ -87/10 & 117/4 & -127/3 & 33 & -27/2 & 137/60 \\ 29/6 & -461/24 & 31 & -307/12 & 65/6 & -15/8 \\ -31/24 & 137/24 & -121/12 & 107/12 & -95/24 & 17/24 \\ 1/6 & -19/24 & 3/2 & -17/12 & 2/3 & -1/8 \\ -1/120 & 1/24 & -1/12 & 1/12 & -1/24 & 1/120 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ . & 1 & -3/2 & 11/6 & -25/12 & 137/60 \\ . & . & 1/2 & -1 & 35/24 & -15/8 \\ . & . & . & 1/6 & -5/12 & 17/24 \\ . & . & . & . & 1/24 & -1/8 \\ . & . & . & . & . & 1/120 \end{bmatrix}$$

3.2. St_2 and St_1 form an eigensystem of the bernoullian matrix G_p

With the diagonal eigenvaluematrix of reciprocals of natural numbers ${}^dZ(1)$ they form the eigensystem of the bernoullian-matrix G_p. G_p is called "bernoulian" since it contains the bernoulli-numbers in its first column and is also a simple column-scaled version of the matrix **BN**, which contains the coefficients of the Bernoulli-polynomials (see chapter Matrix G_p and G_m for more detailed discussion of this)

(3.2.1.) $St_2 * {}^dZ(1) * St_1 = G_P$

$$\begin{bmatrix} 1&.&.&.&.&.\\ 1&1&.&.&.&.\\ 1&3&1&.&.&.\\ 1&7&6&1&.&.\\ 1&15&25&10&1&.\\ 1&31&90&65&15&1 \end{bmatrix} * diag\left(\begin{bmatrix}1\\1/2\\1/3\\1/4\\1/5\\1/6\end{bmatrix}\right) * \begin{bmatrix} 1&.&.&.&.&.\\ -1&1&.&.&.&.\\ 2&-3&1&.&.&.\\ -6&11&-6&1&.&.\\ 24&-50&35&-10&1&.\\ -120&274&-225&85&-15&1 \end{bmatrix} = \begin{bmatrix} 1&.&.&.&.&.\\ 1/2&1/2&.&.&.&.\\ 1/6&1/2&1/3&.&.&.\\ 0&1/4&1/2&1/4&.&.\\ -1/30&0&1/3&1/2&1/5&.\\ 0&-1/12&0&5/12&1/2&1/6 \end{bmatrix}$$

3.3. Some useful consequences arising from the eigensystem-decomposition of G_p

Since we have an eigensystem with a very simple eigenvalue-diagonal-matrix, multiplication of the Stirlingmatrices by G_p leaves them "nearly invariant" aside of the scaling of rows(St_1) and columns (St_2) by the incremented row/columnnumber:

(3.3.1.) ${}^dZ(-1) * St_1 * G_P = St_1$

$$diag\left(\begin{bmatrix}1\\2\\3\\4\\5\\6\end{bmatrix}\right) * \begin{bmatrix} 1&.&.&.&.&.\\ -1&1&.&.&.&.\\ 2&-3&1&.&.&.\\ -6&11&-6&1&.&.\\ 24&-50&35&-10&1&.\\ -120&274&-225&85&-15&1 \end{bmatrix} * \begin{bmatrix} 1&.&.&.&.&.\\ 1/2&1/2&.&.&.&.\\ 1/6&1/2&1/3&.&.&.\\ 0&1/4&1/2&1/4&.&.\\ -1/30&0&1/3&1/2&1/5&.\\ 0&-1/12&0&5/12&1/2&1/6 \end{bmatrix} = \begin{bmatrix} 1&.&.&.&.&.\\ -1&1&.&.&.&.\\ 2&-3&1&.&.&.\\ -6&11&-6&1&.&.\\ 24&-50&35&-10&1&.\\ -120&274&-225&85&-15&1 \end{bmatrix}$$

and

(3.3.2.) $G_p * St_2 * {}^dZ(-1) = St_2$

$$\begin{bmatrix} 1&.&.&.&.&.\\ 1/2&1/2&.&.&.&.\\ 1/6&1/2&1/3&.&.&.\\ 0&1/4&1/2&1/4&.&.\\ -1/30&0&1/3&1/2&1/5&.\\ 0&-1/12&0&5/12&1/2&1/6 \end{bmatrix} * \begin{bmatrix} 1&.&.&.&.&.\\ 1&1&.&.&.&.\\ 1&3&1&.&.&.\\ 1&7&6&1&.&.\\ 1&15&25&10&1&.\\ 1&31&90&65&15&1 \end{bmatrix} * diag\left(\begin{bmatrix}1\\2\\3\\4\\5\\6\end{bmatrix}\right) = \begin{bmatrix} 1&.&.&.&.&.\\ 1&1&.&.&.&.\\ 1&3&1&.&.&.\\ 1&7&6&1&.&.\\ 1&15&25&10&1&.\\ 1&31&90&65&15&1 \end{bmatrix}$$

3.4. Summation of G_p and BN by leftmultiplication with the factorial-vector

From the definition of bernoulli-numbers the identity of the first column of the result is known; the others can be computed using derivatives:

(3.4.1.)

$$ {}^{r}(F^{-1}) * G_p * {}^{d}Z(-1) = {}^{r}(F^{-1}) * e/(e-1) $$
$$ {}^{r}(F^{-1}) * BN = {}^{r}(F^{-1}) * e/(e-1) $$

$$ G_p = \begin{bmatrix} 1 & . & . & . & . & . \\ 1/2 & 1/2 & . & . & . & . \\ 1/6 & 1/2 & 1/3 & . & . & . \\ 0 & 1/4 & 1/2 & 1/4 & . & . \\ -1/30 & 0 & 1/3 & 1/2 & 1/5 & . \\ 0 & -1/12 & 0 & 5/12 & 1/2 & 1/6 \end{bmatrix} * diag(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}) $$

$$ \lim_{r\to oo} \begin{bmatrix} 1 & 1 & 1/2 & 1/6 & 1/24 & 1/120 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1/2 & 1/6 & 1/24 & 1/120 \end{bmatrix} * \begin{bmatrix} e/(e-1) \end{bmatrix} $$

Expressed in terms of ***BN***, the matrix of coefficients of the Bernoulli-polynomials this is

(3.4.2.) $$ {}^{r}(F^{-1}) * BN = {}^{r}(F^{-1}) * e/(e-1) $$

$$ BN = \begin{bmatrix} 1 & . & . & . & . & . \\ 1/2 & 1 & . & . & . & . \\ 1/6 & 1 & 1 & . & . & . \\ 0 & 1/2 & 3/2 & 1 & . & . \\ -1/30 & 0 & 1 & 2 & 1 & . \\ 0 & -1/6 & 0 & 5/3 & 5/2 & 1 \end{bmatrix} $$

$$ \lim_{r\to oo} \begin{bmatrix} 1 & 1 & 1/2 & 1/6 & 1/24 & 1/120 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1/2 & 1/6 & 1/24 & 1/120 \end{bmatrix} * \begin{bmatrix} e/(e-1) \end{bmatrix} $$

Rewriting the factorial scaling as similarity-scaled matrices:

(3.4.3.)

$$ V(1)\sim * (f\, G_p\, Z\, F) = e/(e-1) * V(1)\sim $$
$$ V(1)\sim * (f\, BN\, F) = e/(e-1) * V(1)\sim $$

Generally:

(3.4.4.)

$$ V(x)\sim * (f\, G_p\, Z\, F) = x\, V(x)\sim * \; e^x/(e^x - 1) $$
$$ V(x)\sim * (f\, BN\, F) = x\, V(x)\sim * \; e^x/(e^x - 1) $$

The reciprocal expression can also be written:

(3.4.5.) $e/(e-1) * {}^{r}(F^{-1}) * BN^{-1} = {}^{r}(F^{-1})$

$$
* \begin{bmatrix} 1 & . & . & . & . & . \\ -1/2 & 1 & . & . & . & . \\ 1/3 & -1 & 1 & . & . & . \\ -1/4 & 1 & -3/2 & 1 & . & . \\ 1/5 & -1 & 2 & -2 & 1 & . \\ -1/6 & 1 & -5/2 & 10/3 & -5/2 & 1 \end{bmatrix} \quad \text{BN\^{}}
$$

$$
\lim_{r\to oo} \begin{bmatrix} e/(e-1) \end{bmatrix} \begin{bmatrix} 1 & 1 & 1/2 & 1/6 & 1/24 & 1/120 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1/2 & 1/6 & 1/24 & 1/120 \end{bmatrix}
$$

Generally:

(3.4.6.) $x\, V(x)\sim * \ (f\, BN^{-1}\, F) \ = \ (e^{x} - 1)/e^{x} \ * V(x)\sim$

and for $x=1$ the special simple identity involving the factorial scaled ***BN***-similar matrix ***fBN F*** occurs:

(3.4.7.) $e/(e-1) * V(1)\sim * f\, BN\, F = V(1)\sim$

$$
* \begin{bmatrix} 1 & . & . & . & . & . \\ 1/2 & 1 & . & . & . & . \\ 1/12 & 1/2 & 1 & . & . & . \\ 0 & 1/12 & 1/2 & 1 & . & . \\ -1/720 & 0 & 1/12 & 1/2 & 1 & . \\ 0 & -1/720 & 0 & 1/12 & 1/2 & 1 \end{bmatrix} \quad \text{¡BN!}
$$

$$
\lim_{r\to oo} \begin{bmatrix} e/(e-1) \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}
$$

where the columns in ***f BN F*** are simple shifts of the first column.

4. Details and some discussions

St_1: expansion in terms of a powerseries

From the definition we have for a row r:

$(x-1)(x-2)(x-3)\ldots(x-r)$

This gives the expansion in terms of powers of x:

$f_r(x) = x^r - (1+2+\ldots+r)x^{r-1} + (1*2+(1+2)*3 + (1+2+3)*4 + \ldots()*r)x^{r-2} +-\ldots(-1)^r*r! * x^0$

First note the recursion rule, example

$(x-1)(x-2)(x-3)\ldots(x-r)$
$(x-1)(x-2)(x-3)\ldots(x-r+1)$

Scaling with factorials:

$1/3! * (x-1)(x-2)(x-3)..= (x/1 - 1)(x/2 - 1)(x/3 - 1)..$

The factorials can be extracted and the terms reordered for increasing powers of x:

$$\begin{aligned}
&f_r(x)=(-1)^r r!\left(1\right)\\
&+(-1)^{r-1}r!\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{r}\right)x\\
&+(-1)^{r-1}r!\left(\frac{1}{1*2}+\frac{1}{1*3}+\ldots+\frac{1}{1*r}+\frac{1}{2*3}+\ldots+\frac{1}{2*r}+\ldots+\frac{1}{(r-2)*r}+\frac{1}{(r-1)*r}\right)x^2\\
&\ldots\\
&-r!\left(\frac{1}{2*3*..*r}+\frac{1}{1*3*..*r}+\ldots+\frac{1}{1*2*3*..*(r-1)}\right)x^{r-1}\\
&+r!\left(\frac{1}{r!}\right)x^r
\end{aligned}$$

and in the parentheses we have the harmonic numbers of the c'th order.

The factorial rowscaling $fSt_1 = {}^dF^{-1} * St_1$ cancels the r! term in each row; so we have:

$$\begin{aligned}
&\frac{f_r(x)}{r!}=(-1)^r\left(1\right)\\
&+(-1)^{r-1}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{r}\right)x\\
&+(-1)^{r-2}\left(\frac{1}{1*2}+\frac{1}{1*3}+\ldots\ldots+\frac{1}{(r-2)*r}+\frac{1}{(r-1)*r}\right)x^2\\
&\ldots\\
&-\left(\frac{1}{2*3*..*r}+\frac{1}{1*3*..*r}+\ldots+\frac{1}{1*2*3*..*(r-1)}\right)x^{r-1}\\
&+\left(\frac{1}{r!}\right)x^r
\end{aligned}$$

Column-sums.

If we compute the column-sums, that means first, we add the terms of like powers of *x* of each row, so the entries with the same column-index.

Second it means, that the rows are taken as the powerseries in *1*, so *x* is replced by *1* in the formula; and to describe the column-sum now as the limit of a powerseries instead (in *y*, for instance, for *y->1*) we introduce a power of *y* for each row now.

For the first column, *c=0*, this means to add the alternating series

$s_0(y) = 1 - 1y + 1y^2 - 1y^3 \dots$

with the powers of *y*, which gives

$s_0(y) = 1/(1+y)$

and has the obvious convergent, oscillating divergent and divergent cases according to the rules for the geometric series.

For the sum $s_1(y)$ of column *c=1* we add alternating the subsequent harmonic numbers; that means formally:

$$s_1(y) = (1)y - \left(1+\frac{1}{2}\right)y^2 + \left(1+\frac{1}{2}+\frac{1}{3}\right)y^3 - \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)y^4 - \dots + \dots$$

$$= 1\left(y - y^2 + y^3 - y^4 \dots\right) - \frac{1}{2}\left(y^2 + y^3 - y^4 \dots\right) + \frac{1}{3}\left(y^3 - y^4 \dots\right) - \dots + \dots$$

$$= 1\frac{y}{1+y} - \frac{1}{2}\frac{y^2}{1+y} + \frac{1}{3}\frac{y^3}{1+y} - \dots + \dots$$

$$= \frac{1}{1+y}\left(\frac{y}{1} - \frac{y^2}{2} + \frac{y^3}{3} - \dots + \dots\right)$$

$$= \frac{log(1+y)}{1+y}$$

which also has its known convergent and divergent cases.

fSt1

The column c=2 $\begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 1 & -3/2 & 1/2 & . & . & . \\ -1 & 11/6 & -1 & 1/6 & . & . \\ 1 & -25/12 & 35/24 & -5/12 & 1/24 & . \\ -1 & 137/60 & -15/8 & 17/24 & -1/8 & 1/120 \end{bmatrix}$

$$s_1(y)=\left(\frac{1}{2}\right)y^2-\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{2}\frac{1}{3}\right)y^3+\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{2}\frac{1}{3}+\frac{1}{2}\frac{1}{4}+\frac{1}{3}\frac{1}{4}\right)y^4+-\dots$$

$$=\left(\frac{y^2}{2}-\frac{y^3}{3}+\frac{y^4}{4}-\dots\right)(1-y+y^2-\dots)$$

$$-\frac{1}{2}\left(\frac{y^3}{3}-\frac{y^4}{4}+-\dots\right)(1-y+y^2-\dots.)$$

$$+\frac{1}{3}\left(\frac{y^4}{4}-\frac{y^5}{5}+-\right)(1-y+y^2-\dots.)$$

.....

$$(1-y+y^2-\dots)\left(\left(\frac{y^2}{2}-\frac{y^3}{3}+\frac{y^4}{4}-\dots\right)-\frac{1}{2}\left(\frac{y^3}{3}-\frac{y^4}{4}\right)+\frac{1}{3}\left(\frac{y^4}{4}-\frac{y^5}{5}-\right)-\dots\right)$$

$$\frac{1}{1+y}\left(\left(\frac{y^2}{2}-\frac{y^3}{3}+\frac{y^4}{4}-\dots\right)-\frac{1}{2}\left(\frac{y^3}{3}-\frac{y^4}{4}\right)+\frac{1}{3}\left(\frac{y^4}{4}-\frac{y^5}{5}-\right)-\dots\right)$$

$$\left(\frac{y}{1}+\frac{y^2}{2}+\frac{y^3}{3}+\frac{y^4}{4}\right)^2=\frac{y^2}{1}+\frac{y^4}{4}+\frac{y^6}{9}+\frac{y^8}{16}$$

$$+2\left(\left(\frac{1}{1}\frac{1}{2}\right)y^3+\left(\frac{1}{1}\frac{1}{3}\right)y^4+\left(\frac{1}{1}\frac{1}{4}+\frac{1}{2}\frac{1}{3}\right)y^5+\left(\frac{1}{1}\frac{1}{5}+\frac{1}{2}\frac{1}{4}\right)y^6+\dots\right)$$

5. Citations

[Adamchik] http://www.cs.cmu.edu/~adamchik/articles/stirling.pdf , Pg 8

In this example, Adamchik uses the unsigned version of Stirling-numbers 1'st kind

Let us begin with the simple example

$$\sum_{k=1}^{\infty} \left[{k \atop 2} \right] \frac{1}{k!\, k}$$

Using the integral representation (15) and changing the order of summation and integration, we get

$$\sum_{k=1}^{\infty} \left[{k \atop 2} \right] \frac{1}{k!\, k} = \int_0^1 \frac{\pi^2 t - 6\,\mathrm{Li}_2(t)}{6\,t\,(1-t)}\, dt = \zeta(3)$$

From this identity one would expect the pattern to remain unchanged and so that:

$$\mathrm{G}_{p,1} = \sum_{k=1}^{\infty} \left[{k \atop p} \right] \frac{1}{k!\, k} = \zeta(p+1) \tag{19}$$

In my matrix-notation it means:

$V(1)\sim * {}^dZ(1)\sim * J * fSt_1 = [\ \zeta(1),\ \zeta(2),\ \zeta(3),\ldots]$

Pg 8:

extensively studied in [7]. It was shown there, for example, that

$$\sum_{k=1}^{\infty} \left[{k \atop p} \right] \frac{z^k}{k!\, k} = \zeta(p+1) + \sum_{k=0}^{p} \frac{(-1)^{k-1}}{k!}\,\mathrm{Li}_{p+1-k}(1-z)\,\log^k(1-z) \tag{22}$$

$V(z)\sim * {}^dZ(1)\sim * {}^dJ * fSt_1 = [\ \zeta(1)+f(0,z),\ \zeta(2)+f(1,z),\ \zeta(3)+f(2,z),\ldots]$

where f(c,z) denotes the rhs-sum in (22)

6. References

[Project-Index] http://go.helms-net.de/math/binomial_new/index

[Intro] http://go.helms-net.de/math/binomial_new/00_0_Intro.pdf

[List] http://go.helms-net.de/math/binomial_new/00_1_ListOfMatrices.pdf

[binomialmatrix] http://go.helms-net.de/math/binomial_new/01_1_binomialmatrix.pdf

[signed binomial] http://go.helms-net.de/math/binomial_new/01_2_signedbinomialmatrix.pdf

[Stirlingmatrix] http://go.helms-net.de/math/binomial_new/01_3_stirling.pdf

[Gaussmatrix] http://go.helms-net.de/math/binomial_new/01_5_gaussmatrix.pdf

[GenBernRec] (Generalized Bernoulli-recursion)
http://go.helms-net.de/math/binomial_new/02_2_GeneralizedBernoulliRecursion.pdf

[SumLikePow] (Sums of like powers)
http://go.helms-net.de/math/binomial_new/04_3_SummingOfLikePowers.pdf

[Hasse] http://go.helms-net.de/math/binomial_new/10_2_recihasse.pdf

[Vandermonde] http://go.helms-net.de/math/binomial_new/10_3_InverseVandermonde1.pdf

Projekt ***Bernoulli-numbers***, first versions of the above, contain a ***first rough exploratory*** course but already cover most central topics and contain also the basic material about **Gp** and **Gm** which is still missing in the above list:

[Bernoulli] http://go.helms-net.de/math/binomial_new/bernoulli_en.pdf

[Summation] http://go.helms-net.de/math/binomial_new/pmatrix.pdf

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[AS-St] Abramowitz, M. and Stegun, I. A. (Eds.). "Stirling Numbers of the First Kind." §24.1.3 in Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover, p. 824, 1972.
Online-copy at http://www.convertit.com/Go/ConvertIt/Reference/AMS55.ASP

[Wiki-St1] http://en.wikipedia.org/wiki/Stirling_numbers_of_the_first_kind

[Wiki-St2] http://en.wikipedia.org/wiki/Stirling_numbers_of_the_second_kind

[MW-St1] http://mathworld.wolfram.com/StirlingNumberoftheFirstKind.html

[MW-St2] http://mathworld.wolfram.com/StirlingNumberoftheSecondKind.html
http://mathworld.wolfram.com/BellPolynomial.html

[Adamchik] Victor Adamchik:
On Stirling numbers and Euler sums (1996)
http://www.cs.cmu.edu/~adamchik/articles/stirling.pdf

Gottfried Helms