Gottfried Helms - Univ Kassel
12' 2006 - 05´2007

Summing of like powers

Abstract: The problem of summing of like (positive integer) powers is completely solved by H. Faulhaber's and J. Bernoulli's polynomials.

Here I present a way, using elementary matrix-algebra, how to find such sums in terms of values of the eta- and zeta-function at non-positive arguments. The occuring matrices solve the summing problem and it is immediately obvious, that the numbers, named after J. Bernoulli, are simply the negative zeta-values, scaled by binomials, as they occur in these matrices. One may call the related polynomials "zeta"- or "eta"-polynomials.

Also a seemingly less known property of the zeta/bernoulli-numbers is immediately derivable by this method: to also sum like-powers of negative exponents. I'll add this extension in next version.

Gottfried Helms (for readability updated 8.10.2009 from Vers. 25. Nov. 12)

Keywords: Eta-function Zeta-function, Hurwitz-Zeta-function, Bernoulli numbers, Bernoulli-polynomials, powersums, sums of like powers

Contents

1. The summing problem

1.1. Intro

The problem of summing like powers, for instance for exponent m

$$s_m(n) = 1^m + 2^m + 3^m + ... n^m$$

is completely solved with the introduction of Bernoulli-numbers[1] $ß_k$, arranged in the appropriate Bernoulli-polynomials of degree $m+1$, with the $m+1$'th degree symbolically written as

$$B_{m+1}(n) = ß_{m+1} + a_1 ß_m n + a_2 ß_{m-1} n^2 + ... + ß_0 n^{m+1}$$

where the a_k are binomial-coefficients.

In this article I'll show a way how to arrive at such a formula on another very simple path.

There is also one special interesting aspect involved. I am currently not aware, how Jacob Bernoulli (and Hans Faulhaber) exactly arrived at their results (but see Ed Sandifers note in chap 8.1), but their way to find the coefficients, which perform summing of finitely many like powers seems to have been -at least initially- the heuristic in the empirical results for sums with small exponents and was disconnected from any idea of infinite sum-values like the zeta-/eta-functions, and sometimes one can read characterizations like "*exotic and mystic creatures in the scene of numbers*" - as if they were independent of that zeta-/eta.

The zeta-connection has then be proven several times later, for instance by L. Euler (see chap 9 "references) , by H. Hasse (see [Hasse]) to mention only two.

My proposal seems to be opposite to this (although somehow familiar to the Euler-approach): very naturally the eta/zeta-values at non-positive exponents are introduced as the engine of the summation-process (and one may formulate: only since they cofactor binomials, the resulting coefficients occured to Bernoulli and Faulhaber as a set of individual and characteristic numbers). Bernoulli-numbers, by my approach, are then **essentially** "the appropriate scalings" of integer zeta-values by binomials *(r:1)*, occuring in the second column of the ZETA-matrix. Different generalizations for continuous versions of the Bernoulli-numbers and ~polynomials were proposed (see for instances [Luschny] [Woon]), but the most natural in terms of their property to allow summing like powers is the one, which generalizes them as expressions of the continuous zeta-values.

[Luschny] (2004) made a statement very similar to mine, and it is made very explicite, that the bernoulli-numbers should be seen as scaled zeta-values - but the arguing is starting **from** the bernoulli-numbers (and polynomials) **proceeding to** zeta-values - somehow as the most logical and convenient (re-)definition, which allows then generalization. Here my approach adds some inherent argument: using the zeta-values as the base of all such considerations. (*see footnote (3) next page*)

As a result, this article proposes also "zeta-/eta-polynomials", where the zeta-polynomials can be seen as integrals of the bernoulli-polynomials.

---

[1] *from: Karl Dilcher, Bernoulli-bibliography:* http://www.mscs.dal.ca/~dilcher/bernoulli.html

Bernoulli Numbers

The Bernoulli numbers are among the most interesting and important number sequences in mathematics. They first appeared in the posthumous work *"Ars Conjectandi"* (1713) by Jakob Bernoulli (1654-1705) in connection with sums of powers of consecutive integers (see Bernoulli (1713) or D.E. Smith (1959)). Bernoulli numbers are particularly important in number theory, especially in connection with Fermat's last theorem (see, e.g., Ribenboim (1979)). They also appear in the calculus of finite differences (Nörlund (1924)), in combinatorics (Comtet (1970, 1974)), and in other fields.

Definitions and main properties of Bernoulli numbers can be found in a great number of articles and books listed in this bibliography. Good introductions are given, e.g., in Ireland and Rosen (1982, 1990), Rademacher (1973), and Nörlund (1924). A handy collection of formulas is in Abramowitz and Stegun (1964). Some older books are entirely devoted to Bernoulli numbers; among them are Chistyakov (1895), Nielsen (1923), and Saalschütz (1893). One should, however, be aware of possible differences in notation and indexing, especially in older publications.

The article follows this way of exploring/arguing:

First I recall the binomial-theorem, apply it to a Vandermonde-vector in *x* using the pascal-matrix. This converts then the problem of sums-of-like-powers into one of a geometric series of a matrix. For that matrix-series the shortcut-formula for geometric series can then be used (which is possible only for the ***ETA***-matrix) and serves here as first example.

This approach also proves implicitely, that the values of the Dirichlet's η =*"eta"*-function at non-positive exponents are rational, since they occur as result from a matrix-inversion of finite triangular integer-matrices, and hence also the ζ = *"zeta"*-function-values of same argument - which is another nice feature of this derivation.

For the ***ZETA***-matrix, however, the shortcut-formula for geometric-series cannot be used. But again from the construction of the general arguing it occurs that the entries of the ***ZETA***-matrix are ζ -values cofactored with binomials. Here the limit-problem, where $\zeta(1)$ is involved, is essentially and is specifically considered.

Because everything is based on the $\zeta(\ )/\eta(\ )$-functions, things should be generalizable to positive or fractional exponents, since $\zeta(\ )/\eta(\ )$-values are defined for each complex argument (except $\zeta(1)$) while the notion of Bernoulli-numbers limits itself to the case of non-positive integer exponents in the related $\zeta(\ )$–argument.

Another extensions will be made in the next version of this article: completely analoguosly the ***ZETA/ETA***-matrices can be used to sum negative like powers of consecutive natural numbers – this is simply the re-invention of the *psi*-function. The idea of this is -however sketchy- already described in [*binomialmatrix*].

A remark concerning references: after putting some of the ideas which are covered by this article into the more professional and thus more appropriate keywords, I find a vast amount of articles related to or even discussing them already. Some of them with similar concepts as in my proposal[1] focus the same topics from a different view; for instance my eigenvector-approach, which led to the Faulhaber/Bernoulli-matrix $\boldsymbol{G_p}$, was paralleled in terms of "*invariant sequences under binomial transform*" (see references [*SunZhiHong*], more examples see footnote[2]). The special value of my current text may then lay in that it provides a concise and coherent scenery for that ideas and generalizations. I'll add related items to the bibliography as I'll come across them. (footnote 3[3])

---

[1] Some are from the recent time-period 2004-2007, when also I developed this concept from heuristics in checking systematic matrix-relations not being aware of such articles. I began in 2004 with the first article about the Pascal- and (Bernoulli-like) G_p- matrix for powersums [*Bernoulli_en*]

[2] see for instance in chap 7 "references": Faulhaber's Theorem for Arithmetic Progressions, Chen, William
Explicit inverse of the Pascal matrix plus one: Yang, Sheng-liang & Liu, Zhong-kui
(Zeta-relation) "Ein Summierungsverfahren für die Riemannsche Zeta-Reihe", Hasse, Helmut

[3] in [*Luschny*] we find: *"Die Definitionen sowohl der Bernoulli Funktion B(s) wie der Euler Funktion E(s) basieren auf einer einzigen Funktion, der Hurwitz Zetafunktion. (...)*
(...) Die Definition der Bernoulli Funktion ist $\mathfrak{B}(s) = -2(2\pi)^{-s}\cos(s\pi/2)s!\zeta(s)$
Bei unserer Einführung der Bernoulli und Euler Zahlen haben wir uns vollständig auf die Zetafunktion gestützt und keinerlei motivierende Bemerkungen dazu gemacht. Dieser top-down-approach hat uns zwar schnell die allgemeinen Zusammenhänge aufgezeigt, aber die Frage nach der Adäquatheit ist dabei offen geblieben: Ist der Zusammenhang zwischen den Bernoulli Zahlen (so wie wir sie definiert haben) und der Zetafunktion natürlich?
Deshalb drehen wir jetzt die Blickrichtung um und betrachten, wie wir von den Bernoulli Zahlen (in unserem Sinn) ausgehend zur Zetafunktion bzw. Bernoulli-Funktion gelangen können. Tatsächlich gibt es dafür einen Königsweg, den Satz von Hasse, der unser Vorgehen elementar und anschaulich rechtfertigt.
Ausgangspunkt für den Satz von Hasse ist die explizite Darstellung der Bernoulli-Zahlen von Worpitzky ([11, Formel 36]) (...)"
The motivation of this (re-) definition seems here the convenience, so he asks "*but is the relation between Bernoulli-numbers(as we defined them here) and the Zeta-function also natural?*" Then he discusses, how "*we (can) proceed from the Bernoulli-numbers (in our sense) to Zeta- and Bernoulli-functions*".

1.2. The binomial theorem

The binomial theorem is the statement that

(1.2.1.) $(1+x)^m = 1 + (m:1)\,x + (m:2)\,x^2 + \dots + (m:m-1)\,x^{m-1} + (m:m)\,x^m$

where I introduce the notation $(m:k)$ for the binomial-coefficients *binomial(m,k)* which are different for each m^{th} power Let's write this more explicite and for some exponents, say 0 to 4:

(1.2.2.)
$$(1+x)^0 = (0:0)\,1$$
$$(1+x)^1 = (1:0)\,1 + (1:1)\,x$$
$$(1+x)^2 = (2:0)\,1 + (2:1)\,x + (2:2)\,x^2$$
$$(1+x)^3 = (3:0)\,1 + (3:1)\,x + (3:2)\,x^2 + (3:3)\,x^3$$
$$(1+x)^4 = (4:0)\,1 + (4:1)\,x + (4:2)\,x^2 + (4:3)\,x^3 + (4:4)\,x^4$$

In numbers this is

(1.2.3.)
$$(1+x)^0 = 1\,x^0$$
$$(1+x)^1 = 1\,x^0 + 1\,x$$
$$(1+x)^2 = 1\,x^0 + 2\,x + 1\,x^2$$
$$(1+x)^3 = 1\,x^0 + 3\,x + 3\,x^2 + 1\,x^3$$
$$(1+x)^4 = 1\,x^0 + 4\,x + 6\,x^2 + 4\,x^3 + 1\,x^4$$

where the coefficients at the powers of x form the well known Pascal-triangle.

1.3. The power of the Pascal-matrix

Let this be written as a matrix product of the Pascal-triangle **P** and **V**(x), the Vandermonde-(column-) vector of a variable placeholder x (valid for any finite matrix-dimension) $V(x)=[1,x,x^2,x^3,\dots]$, (I omit the dots for indication of infinite extension here and in the following due to limits of the bitmap-generating-program)

(1.3.1.) $P * V(x) = V(1+x)$

$$\underbrace{\begin{bmatrix} 1 & . & . & . & . \\ 1 & 1 & . & . & . \\ 1 & 2 & 1 & . & . \\ 1 & 3 & 3 & 1 & . \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}}_{P} * \begin{bmatrix} x\char`^0 \\ x\char`^1 \\ x\char`^2 \\ x\char`^3 \\ x\char`^4 \end{bmatrix} = \begin{bmatrix} (1+x)\char`^0 \\ (1+x)\char`^1 \\ (1+x)\char`^2 \\ (1+x)\char`^3 \\ (1+x)\char`^4 \end{bmatrix}$$

This little modification, carefully considered, has his own impact.

Dealing with and thinking of a binomial-transform like

$$(1+x)^m = 1 + (m:1)\,x + (m:2)\,x^2 + \dots + (m:m-1)\,x^{m-1} + (m:m)\,x^m$$

is an important tool from school to determine $(1+x)^m$ in terms of powers of x only. But when put together as a complete, and even infinite, set of transformation-coefficients in the Pascal-triangle, then this is no more simply a collection of an (even infinite) set of transformation rules, but suddenly occurs as a more general entity, let's say "operator", which transforms a whole Vandermonde-vector in x to one of $x+1$. This view of things introduce new qualities.

First, it introduces the vector of powers of x (or "Vandermonde-vector in x") as an individually referrable mathematical object. This includes then, that if we find a certain transformation for x^m, then we have also found one for x^{m+1} - and for all consecutive powers of x. So, for instance, if we find a summing-procedure for $1^m +2^m +3^m +...+x^m$ in a consistent way in this vector-/matrix-context, we also have found one for $1^{m+1} +2^{m+1} +3^{m+1} +...+x^{m+1}$ and also for all consecutive exponents of m.

Second, it introduces **P** like an ***operator***. On a first glance that may not be such an overwhelming news, but note, how natural one would start to think in terms of repetitive application of this operator:

$$P * V(x) = V(1+x)$$
$$P * P * V(x) = V(2+x)$$
$$P * P * P * V(x) = V(3+x)$$
$$...$$
$$P^n * V(x) = V(n+x)$$

and naturally this leads to a notion of powers of **P** which expresses the binomial-theorem at arbitrary natural powers[1].

Also the inverse operation comes immediately into mind, so

$$P^{-1} * V(x) = V(-1+x)$$
$$P^{-2} * V(x) = V(-2+x)$$
$$...$$
$$P^{-n} * V(x) = V(-n+x)$$

and opens the field for more general operations of repeated binomial-transforms, which would hardly be expressible by computing and documenting the appropriate modifications of the individually involved binomial-coefficients.

The Toeplitz-form of a power of P

A useful property, which will be used with the **ETA/ZETA**-matrix, is the Toeplitz-form of the powers of **P**.

Define the Toeplitzmatrix **T**(z) of powers of z

(1.3.2.) $$T(z) = V(z) * V(1/z)\sim$$

where for arbitrary z the matrix **T**(z) looks like:

(1.3.3.) $$T(z) := Toeplitz(z) = \begin{bmatrix} 1 & 1/z & 1/z^2 & 1/z^3 & 1/z^4 & 1/z^5 \\ z & 1 & 1/z & 1/z^2 & 1/z^3 & 1/z^4 \\ z^2 & z & 1 & 1/z & 1/z^2 & 1/z^3 \\ z^3 & z^2 & z & 1 & 1/z & 1/z^2 \\ z^4 & z^3 & z^2 & z & 1 & 1/z \\ z^5 & z^4 & z^3 & z^2 & z & 1 \end{bmatrix}$$

T(z)

Lemma 1.3:
(1.3.4.) *Powers* P^z *of* **P** *are the Hadamardproduct of* **P** *with the Toeplitzmatrix Toeplitz(z)*

$$P^z = {}^dV(z) * P * {}^dV(1/z)$$
$$= (V(z)*V(1/z)\sim) ¤ P \qquad \text{// using "¤" for Hadamard-multiplication}$$
$$= Toeplitz(z) ¤ P$$

Two proofs can be seen in chap (*4. details/proofs*)

---

[1] in [*binomial-matrix*] I even show the consistency of introducing complex powers of **P** by means of its matrix-logarithm

1.4. Geometric series of P

Now, first let's see, what happens if we add the Vandermonde vectors of consecutive arguments into a sum-vector *S*:

(1.4.1.) $$S(1,n) = V(1) + V(2) + V(3) + V(4) + \ldots + V(n)$$

This means, that in each row of the sum-vector **S** we have the sum of like powers from *1* to *n*. And if we had a triangle **X** of coefficients, perhaps similar to that of the Pascal-triangle, such that

$$S(1,n) = X * V(n)$$

then we had solved the initial problem. (but see footnote[1])

An approach, using the newly introduced operator **P** and its powers we could rewrite this as

(1.4.2.) $$S(1,n) = P^0\, V(1) + P^1\, V(1) + P^2\, V(1) + P^3\, V(1) + \ldots + P^{n-1}\, V(1)$$

or, factoring the **V**(*1*)-vector out:

(1.4.3.) $$S(1,n) = (P^0 + P^1 + P^2 + P^3 + \ldots + P^{n-1})\, V(1)$$

This shows a conversion of the original problem, which means: *summing of consecutive numbers with the same exponent,* into one which means: *summing of consecutive powers* or said differently: *of a matrix-polynomial in* **P** or *of the geometric series of the matrix* **P**.

This notation directly allows to set the initial value other than *1* according to the binomial theorem described above:

$$\begin{aligned} S(2,n) &= (P^0 + P^1 + P^2 + P^3 + \ldots + P^{n-2})\ \ V(2) \\ &= (P^0 + P^1 + P^2 + P^3 + \ldots + P^{n-2})\, P\, V(1) && \text{// one factor P extracted from V(2)} \\ &= (P^1 + P^2 + P^3 + P^4 + \ldots + P^{n-1})\ \ V(1) && \text{// the factor P multiplied to each term in parenthese} \\ &= P\ (P^0 + P^1 + P^2 + P^3 + \ldots + P^{n-2})\, V(1) && \text{// factor P extracted as pre-multiplicator} \\ &= P^2\, (P^0 + P^1 + P^2 + P^3 + \ldots + P^{n-2})\, V(0) && \text{// another factor P extracted to "normalize" the Vandermonde vector to V(0)} \end{aligned}$$

and generally:

(1.4.4.)
$$\begin{aligned} S(m,n) &= P^{m-1}\, (P^0 + P^1 + P^2 + P^3 + \ldots + P^{n-m})\, V(1) \qquad \text{or} \\ &= P^m\ \ (P^0 + P^1 + P^2 + P^3 + \ldots + P^{n-m})\, V(0) \end{aligned}$$

This is already very good looking, but for each *n* we had another version of the parenthese, and so this is not the final solution. So we look again for more generalization:

the expression of sum of powers of **P** reminds immediately to the infinite geometric series, and we may ask, what a triangle would occur if this geometric series would be continued to infinity and all powers of **P** would be added.

[1] The final idea is here, to use the difference and the **Y**-matrix (which I call then **ETA** or **ZETA** for alternating or non-alternating summing):

$$S(1,n) = S(1,inf) - S(n+1,inf) = Y *(V(0) - V(n))$$

but first some more introductory remarks are needed.

For a scalar argument the geometric series is a simple formula:

(1.4.5.) $1+x+x^2+x^3+\ldots = (1-x)^{-1}$

which is convergent for $|x|<1$, and can be summed for $x<=-1$ by divergent summation techniques. Generally it has analytic continuation to all x except for $x=1$, where the expression in the parenthese equals zero.

However, we'll see, that this cannot directly be translated into the matrix-formulation, since in the matrix-version, $(\boldsymbol{I} - \boldsymbol{P})$ cannot be inverted. We introduce another step first, which shows the principle of the matrix-formulation, implementing the alternating sum

(1.4.6.) $1-x+x^2-x^3+\ldots = (1+x)^{-1}$

in the next chapter and then come back to the nonalternating version with a workaround using the Toeplitz-matrix-lemma for powers of $\boldsymbol{P}$.

2. Alternating geometric series, "powers of (-P)" : ETA-matrix

The analoguous matrix-formula to *(1.4.5)* for the infinite geometric series of **P** would read as:[1]

$$P^0 + P^1 + P^2 + P^3 + ... = (I - P)^{-1}$$

and would be applicable, if the parenthese would be invertible.

Unfortunately this term, involving **P,** suffers the problem, that all eigenvalues of **P** are just equal to the mentioned exception-case for the scalar version $x=1$, so we cannot proceed. this way.

But we could consider the geometric series of *(-***P***)*, which in turn means to sum like powers with alternating sign, in other words: to compute the $\eta()$-series for each exponent instead of $\zeta()$-series. Then the values for the non-alternating ζ-series could, for instance, be computed by the Eulerian η/ζ-transformation formula.

2.1. The alternating geometric series of P

Applying the idea of an alternating sum of like powers, we had for the finite sum

$$\text{(2.1.1.)} \quad AS(1,n) = V(1) - V(2) + V(3) - V(4) + ... +- V(n). \\ = (P^0 - P^1 + P^2 - P^3 + ... +- P^{n-1})\, V(1)$$

and for the infinite sum

$$\text{(2.1.2.)} \quad AS(1,inf) = V(1) - V(2) + V(3) - V(4) + ... +- ... \\ = (P^0 - P^1 + P^2 - P^3 + ... +- \;)\, V(1)$$

where the alternating sums of each like powers occur in the rows of *AS(1,n)* or *AS(1,inf)*.

These are also the "*alternating-zeta*" or "*eta*"-values of non-positive exponents, each exponent of *eta(-r)* according to its row-index *r*.

For the infinite case this means to apply the summation formula for the geometric series in the following form:

$$\text{(2.1.3.)} \quad AS(1,inf) = (I - (-P))^{-1}\, V(1) \\ = (I + P)^{-1}\, V(1)$$

where the parenthese-term is now invertible.

We call the (provisorial) matrix, which occurs as inverse of the parenthese-term as $\mathbf{ETA}_1$

$$\text{(2.1.4.)} \quad ETA_1 = (I - (-P))^{-1}$$

This matrix is already much interesting. Its top left segment is

$$ETA_1 = (I - (-P))^{-1} \quad \begin{bmatrix} 1/2 & . & . & . & . \\ -1/4 & 1/2 & . & . & . \\ 0 & -1/2 & 1/2 & . & . \\ 1/8 & 0 & -3/4 & 1/2 & . \\ 0 & 1/2 & 0 & -1 & 1/2 \end{bmatrix} \text{ETA1}$$

[1] General proof for applicability of geometric-series formula will be inserted later

By its construction it gives, when right-multiplied with the Vandermonde-vector **V***(1)* (which contains *1* only and performs simply a summing over each row) the vector of *eta*-values at nonpositive integer exponents in the result-vector, call it **H**:

$$H = AS(1,inf) = ETA_1 * V(1)$$

(2.1.5.) $ETA_1 * V(1) = H$

$$\underbrace{\begin{bmatrix} 1/2 & . & . & . & . \\ -1/4 & 1/2 & . & . & . \\ 0 & -1/2 & 1/2 & . & . \\ 1/8 & 0 & -3/4 & 1/2 & . \\ 0 & 1/2 & 0 & -1 & 1/2 \end{bmatrix}}_{ETA1} * \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/2 \\ 1/4 \\ 0 \\ -1/8 \\ 0 \end{bmatrix}}_{H}$$

We may smooth up things a bit, and replace **V***(1)* by **P******V***(0)* and get the final matrix **ETA** according to

(2.1.6.) $AS(1,inf) = (P^0 - P^1 + P^2 - P^3 + \dots +- \quad)P\ V(0)$
$\qquad = P * (P^0 - P^1 + P^2 - P^3 + \dots +- \quad)\ V(0)$

(2.1.7.) $ETA = P * (I + P)^{-1}$

$$ETA = P\ (I - (-P))^{-1} \qquad \underbrace{\begin{bmatrix} 1/2 & . & . & . & . & . \\ 1/4 & 1/2 & . & . & . & . \\ 0 & 1/2 & 1/2 & . & . & . \\ -1/8 & 0 & 3/4 & 1/2 & . & . \\ 0 & -1/2 & 0 & 1 & 1/2 & . \\ 1/4 & 0 & -5/4 & 0 & 5/4 & 1/2 \end{bmatrix}}_{ETA}$$

and again the alternating sums of like powers in each row of **AS**:

(2.1.8.) $AS(1,inf) = ETA * V(0)$

Example:

$$\underbrace{\begin{bmatrix} 1/2 & . & . & . & . \\ 1/4 & 1/2 & . & . & . \\ 0 & 1/2 & 1/2 & . & . \\ -1/8 & 0 & 3/4 & 1/2 & . \\ 0 & -1/2 & 0 & 1 & 1/2 \end{bmatrix}}_{ETA} * \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/2 \\ 1/4 \\ 0 \\ -1/8 \\ 0 \end{bmatrix}}_{H}$$

Explicitely

$$= \begin{bmatrix} 1\hat{}0 & -2\hat{}0 & +3\hat{}0 & -4\hat{}0 & \dots \\ 1\hat{}1 & -2\hat{}1 & +3\hat{}1 & -4\hat{}1 & \dots \\ 1\hat{}2 & -2\hat{}2 & +3\hat{}2 & -4\hat{}2 & \dots \\ 1\hat{}3 & -2\hat{}3 & +3\hat{}3 & -4\hat{}3 & \dots \\ 1\hat{}4 & -2\hat{}4 & +3\hat{}4 & -4\hat{}4 & \dots \end{bmatrix}$$

2.2. Solution for the alternating summing-problem

Now we have two small steps more to go.

Consider, that we start that sum at a higher *n* instead of *n=1*, such that

$$AS(1,inf) = ETA * V(0)$$

becomes

(2.2.1.) $$\begin{aligned} AS(3,inf) &= ETA * V(2) \\ &= P * (P^0 - P^1 + P^2 - P^3 +\dots +-)\ V(2) \\ &= V(3) - V(4) + V(5) - V(6) + \dots -+ \dots \end{aligned}$$

and we have

$$AS(3,inf) = ETA * V(2) \qquad * \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \end{bmatrix} \qquad \text{(Explicitely:)}$$

$$\underbrace{\begin{bmatrix} 1/2 & . & . & . & . \\ 1/4 & 1/2 & . & . & . \\ 0 & 1/2 & 1/2 & . & . \\ -1/8 & 0 & 3/4 & 1/2 & . \\ 0 & -1/2 & 0 & 1 & 1/2 \end{bmatrix}}_{ETA} = \begin{bmatrix} 1/2 \\ 5/4 \\ 3 \\ 55/8 \\ 15 \end{bmatrix} = \begin{bmatrix} 3\hat{}0 & -4\hat{}0 & +5\hat{}0 & -6\hat{}0 & \dots \\ 3\hat{}1 & -4\hat{}1 & +5\hat{}1 & -6\hat{}1 & \dots \\ 3\hat{}2 & -4\hat{}2 & +5\hat{}2 & -6\hat{}2 & \dots \\ 3\hat{}3 & -4\hat{}3 & +5\hat{}3 & -6\hat{}3 & \dots \\ 3\hat{}4 & -4\hat{}4 & +5\hat{}4 & -6\hat{}4 & \dots \end{bmatrix}$$

The last step is, to substract the two results

(2.2.2.) $AS(1,inf) = ETA * V(0) = V(1) - V(2) + V(3) - V(4) + V(5)-V(6)+ \dots -+ \dots$

(2.2.3.) $AS(3,inf) = ETA * V(2) = \qquad V(3) - V(4) + V(5)-V(6)+ \dots -+ \dots$

$AS(1,2) = ETA *(V(0) - V(2))$

By this we get the formula for the alternating sums of like powers (for each exponent):

(2.2.4.) $$\begin{aligned} AS(1,2) &= AS(1,inf) - AS(3,inf) \\ &= ETA * (V(0) - V(2)) \\ &= V(1) - V(2) \end{aligned}$$

Example:

$$ETA * (V(0) - V(2)) = V(1) - V(2) \qquad * \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1/2 & . & . & . & . \\ 1/4 & 1/2 & . & . & . \\ 0 & 1/2 & 1/2 & . & . \\ -1/8 & 0 & 3/4 & 1/2 & . \\ 0 & -1/2 & 0 & 1 & 1/2 \end{bmatrix}}_{ETA} = \begin{bmatrix} 0 \\ -1 \\ -3 \\ -7 \\ -15 \end{bmatrix}$$

Generalized this is for the alternating sum for each nonnegative integer power

Result 2.2:

(2.2.5.)	$AS(1,inf) = ETA * V(0)$	$= [\eta(0), \eta(-1), \eta(-2), ...]\sim$
(2.2.6.)	$AS(1,n) = ETA * (V(0) - (-1)^{n-1} V(n))$	$= V(1) - V(2) + ... + (-1)^{n-1} V(n)$

and even more generalized for any contiguous segment of alternating sums of like powers:

(2.2.7.)	$AS(a,b) = ETA * ((-1)^{a-1} V(a-1) - (-1)^{b-1} V(b-1))$

which is the final result of this derivation.

It is a general solution for determining the alternating sums of like powers by means of a simple fixed polynomial for each exponent.

2.3. "Eta"-polynomials

We may define "*eta*-polynomials" this way, which perform the alternating summing of like powers.

Definition of the m'th eta-polynomial in x

(2.3.1.) $$eta_m(x) = (-1)^x \sum_{c=0}^{m} \left(\eta(-(m-c)) \binom{m}{c} x^c \right)$$

$eta_0(x) = (-1)^x * (\quad \eta(0)$
$eta_1(x) = (-1)^x * (\quad \eta(-1) + 1\,\eta(0)\,x$
$eta_2(x) = (-1)^x * (\quad \eta(-2) + 2\,\eta(-1)\,x + 1\,\eta(0)\,x^2$
$eta_3(x) = (-1)^x * (\quad \eta(-3) + 3\,\eta(-2)\,x + 3\,\eta(-1)\,x^2 + 1\,\eta(0)\,x^3$
$eta_4(x) = (-1)^x * (\quad \eta(-4) + 4\,\eta(-3)\,x + 6\,\eta(-1)\,x^2 + 4\,\eta(-1)\,x^3 + 1\,\eta(0)\,x^4$

...

then

(2.3.2.) $$eta_m(0) = \eta(-m) = \sum_{k=1}^{oo} (-1)^{k-1} k^m = 1^m - 2^m + 3^m - ...$$

(2.3.3.) $$eta_m(n) = \quad = \sum_{k=n+1}^{oo} (-1)^{k-1} k^m = (-1)^n * ((n+1)^m - (n+2)^m + (n+3)^m - ...)$$

and the alternating sum of m'th like powers $as_m(n)$ from 1^m to n^m is:

(2.3.4.) $$as(n) = eta_m(0) - eta_m(n) = \sum_{k=1}^{n} (-1)^{k-1} k^m = 1^m - 2^m + ... + (-1)^{n-1} n^m$$

Examples:

(2.3.5.) $as_1(2) = 1 - 2 = -1$
$= eta_1(0) - eta_1(2) = \eta(-1) - (\eta(-1) + \eta(0)*2) = - \eta(0)*2 = -1$

(2.3.6.) $as_1(4) = 1 - 2 + 3 - 4 = -2$
$= eta_1(0) - eta_1(4) = \eta(-1) - (\eta(-1) + \eta(0)*4) = - \eta(0)*4 = -2$

(2.3.7.) $as_3(3) = 1 - 8 + 27 = 20$
$= eta_3(0) - eta_3(3)$
$= \eta(-3) - (-1)^3 (\eta(-3) + 3\,\eta(-2)*3 + 3\,\eta(-1)*9 + \eta(0)*27)$
$= \quad 2*-1/8 \quad + 9 *0 \quad +27/4 \quad + 27/2$
$= \quad 13/2 \quad + 27/2 = 40/2 = 20$

2.4. Resume: the power of the abstraction

We see in the previous paragraphs a simple derivation for the solution for the problem of the alternating sums of like powers.

The approach was induced by the higher abstraction of the elementary binomial-rules for generating powers of $(x+1)$ from powers of x. Collecting all these rules (and the resulting coefficients) into a matrix introduced the possibility to see the original problem of summing of zeta-/eta-series in terms of a sum of a geometric series by a notion of iterated application of the binomial theorem via the binomial- or Pascal-matrix and their powers.

The approach seems very natural to me, and may be generalized to other problems, involving Bernoulli- or Stirling-numbers to mention only two. In my collection of articles about "*Identities involving binomial-coefficients, Bernoulli- and Stirlingnumbers"* I compiled a lot of similar identities to have the tools to experiment with some of these questions.

The underlying idea of these all is to see the matrices as operators acting on formal powerseries:

- preferably not changing their character,
- or if changing, then in a useful way, for instance to convert them to a formal series of logarithms or exponentials or the like.

3. non-alternating geometric series,"powers of P": ZETA-matrix

3.1. Notes

The same process cannot be applied one-to-one to the zeta-version. The reason, as I stated above, is the impossibility of the application of the formula for the infinite geometric series using **P**, due to the impossibility of inversion of

$$(I - P)^{-1}$$

That there is in fact a matrix, which performs that nonalternating summation may then be surprising. The most significant difference to an expected analogy to the eta-matrix occurs, in that

* the matrix is not exactly triangular
  (but has an additional subdiagonal above the principal diagonal)
* the infinite value $\zeta(1)$ must be handled.

But completely analoguously to the **ETA**-version, the sum of like powers is then determined by the difference

(3.1.1.)	$S(1,n) = ZETA * (V(0) - V(n))$

The submatrix, ignoring the first column of **ZETA**, is already a known entity; it is exactly the matrix of coefficients, which Hans Faulhaber and Jacob Bernoulli described (also using the Bernoulli-number β_1 as $\beta_1 = +1/2$), and I'd described this submatrix already in the chapter **Gp** in the initial article on properties of the pascalmatrix [bernoulli]

3.2. zeta-values as sums of entries of the geometric series of P

Completely analoguously to the ansatz for the **ETA**-matrix we formulate the using the geometric series of **P**:

(3.2.1.)	$S(1,inf) = V(1) + V(2) + V(3) + ...$ $= (P^1 + P^2 + P^3 + ... +- \ ) V(0)$

Since we cannot use the shortcut-formula for the geometric series, we employ here the Lemma concerning the structure of entries in powers of **P**. Remember *Lemma 1.3 (see chap.1.3)*:

Lemma 1.3:
*Powers z of **P** are the Hadamardproduct of **P** with the Toeplitzmatrix **T**(z)*

$$P^z = T(z) ¤ P = Toeplitz(z) ¤ P$$

The Hadamard-product of **P** with the following sum of Toeplitz-matrices shall then be the solution for **ZETA**:

(3.2.2.)	$ZETA = lim (T(1) + T(2) + T(3) + ...) ¤ P$

The entries of **ZETA** shall then analoguously to the **ETA**-matrix[1] be:

(3.2.3.)	$ZETA_{r,c} = (r:c)*\zeta(-(r-c))$	*(a:b) denotes the binomial-coefficient*

Caveat: at a first glance, this seems to be then simply

```
ZETA =
    [ ζ( 0)*( 0 0)        .                .              .      ...]
    [ ζ(-1)*( 1 0)    ζ( 0)*(1  1)         .              .      ...]
    [ ζ(-2)*( 2 0)    ζ(-1)*(2  1)    ζ( 0)*(2 2)         .      ...]
```

but by the sum of the Toeplitz-matrices we get in the first upper subdiagonal entries which involve the sums up the infinite *ζ(1)* - and this must explicitely be considered.

3.3. Considering the entries which sum up to ζ(1)

If we reconsider the binomial-matrix as containing *(r:c)* at ***every*** entry, then in this first ***upper subdiagonal*** we have entries of *(r:r+1)*. These are numerically zero in all cases, but only because of the limit of the binomial-formula:

(3.3.1.)
$$(r:r+1) = \frac{r!}{(r+1)!(r-(r+1))!} = \frac{r!}{(r+1)!}\frac{1}{(-1)!} = \frac{1}{r+1}\frac{1}{(-1)!}$$

which is zero for all entries. The reason is the infinity at *(-1)! = gamma(0)*

The corresponding entries in ***ZETA*** have *ζ(1)* as cofactors for those *(r:r+1),* here marked yellow:

```
ZETA =
    [ ζ( 0)*( 0:0)    ζ( 1)*( 0:1)         .               .        ...]
    [ ζ(-1)*( 1:0)    ζ( 0)*( 1:1)    ζ( 1)*( 1:2)         .        ...]
    [ ζ(-2)*( 2:0)    ζ(-1)*( 2:1)    ζ( 0)*( 2:2)    ζ( 1)*( 2:3)  ...]
    [ ζ(-3)*( 3:0)    ζ(-3)*( 3:1)    ζ(-1)*( 3:2)    ζ( 0)*( 3:3)  ...]
    ...
```

which are conventionally undefined expressions. The composition looks also more explicte like:

1ζ(-0)	-1				
1 ζ(-1)	1 ζ(-0)	-1/2			
1 ζ(-2)	2 ζ(-1)	1 ζ(-0)	-1/3		
1 ζ(-3)	3 ζ(-2)	3 ζ(-1)	1 ζ(-0)	1-/4	
1 ζ(-4)	4 ζ(-3)	6 ζ(-2)	4 ζ(-1)	1 ζ(-0)	...

If we set the ratio of the two infinities

Proposal 3.3:(*see* definition for zeta/gamma ratio)

(3.3.2.)	$\lim_{x\to 0} \zeta(1-x) / \Gamma(x) = -1 = -\beta_0$	*//β_0 the first bernoulli number*

then we have the working model of the **ZETA**-matrix.

[1] Pari/GP: ZETA= matrix(n,n,r,c, \
    if(r==c-1,   -1/r,   \
      if(r>=c, -zetafrac(-r+c)*binomial(r-1,c-1))))

3.4. numerical display of ZETA

The remaining upper-right entries are still zero, since all other zeta-values are finite and cannot cancel the infinities in the denominators of the binomials:

(3.4.1.)

$$ZETA = \begin{bmatrix} -1/2 & -1 & 0 & 0 & 0 & 0 \\ -1/12 & -1/2 & -1/2 & 0 & 0 & 0 \\ 0 & -1/6 & -1/2 & -1/3 & 0 & 0 \\ 1/120 & 0 & -1/4 & -1/2 & -1/4 & 0 \\ 0 & 1/30 & 0 & -1/3 & -1/2 & -1/5 \\ -1/252 & 0 & 1/12 & 0 & -5/12 & -1/2 \end{bmatrix}$$

3.5. Solution for the non-alternating summing problem

The matrix **ZETA** performs the non-alternating summation in the same form like the **ETA**-matrix above:

Example

(3.5.1.) $ZETA * (V(0) - V(2)) \quad = (V(1) + V(2) + V(3) + V(4) + \ldots) - (V(3) + V(4) + \ldots) = V(1) + V(2)$

$$* \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \\ 32 \end{bmatrix}$$

$$\begin{bmatrix} -1/2 & -1 & 0 & 0 & 0 & 0 \\ -1/12 & -1/2 & -1/2 & 0 & 0 & 0 \\ 0 & -1/6 & -1/2 & -1/3 & 0 & 0 \\ 1/120 & 0 & -1/4 & -1/2 & -1/4 & 0 \\ 0 & 1/30 & 0 & -1/3 & -1/2 & -1/5 \\ -1/252 & 0 & 1/12 & 0 & -5/12 & -1/2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 9 \\ 17 \\ 33 \end{bmatrix}$$

Note, that because of subtraction with **V**(0) involved, the first column of **ZETA** is irrelevant and thus the first row of the **V**()-vectors, so we may write this in a shorter form:

(3.5.2.) $^{(-)}ZETA * (0 * V(0) - 2 * V(2)) = V(1)+V(2) = S(1,2)$

$$* \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 8 \\ 16 \\ 32 \\ 64 \end{bmatrix}$$

$$\begin{bmatrix} -1 & . & . & . & . & . \\ -1/2 & -1/2 & . & . & . & . \\ -1/6 & -1/2 & -1/3 & . & . & . \\ 0 & -1/4 & -1/2 & -1/4 & . & . \\ 1/30 & 0 & -1/3 & -1/2 & -1/5 & . \\ 0 & 1/12 & 0 & -5/12 & -1/2 & -1/6 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 9 \\ 17 \\ 33 \end{bmatrix}$$

The deletion of the first row of a Vandermonde-vector **V**(n) means just the multiplication by n, so n * **V**(n) performs this deletion (or simply shifting by one exponent).

The zero-vector can be omitted, and signs can be adapted and the remaining **ZETA**-submatrix, (which I called in my first studies of this "**Gp"**), is then the working example for the summing-problem:

Result 3.5:

(3.5.3.)	$ZETA * V(0)$	$= [\zeta(0), \zeta(-1), \zeta(-2),...]\sim$
(3.5.4.)	$ZETA * (V(0) - V(n))$	$= V(1) + V(2) + ... + V(n)$
(3.5.5.)	$ZETA * (V(m-1) - V(n))$	$= V(m) + V(m+1) + ... + V(n)$
and		
(3.5.6.)	$ZETA * (V(n-1) - V(n))$	$= V(n)$

Short forms:

(3.5.7.)	$G_p * n\, V(n)$	$= V(1) + V(2) + ... + V(n)$
and		
(3.5.8.)	$G_p * (n\, V(n) - (n-1)\, V(n-1))$	$= V(n)$

Example

$$G_p * 2 * V(2) = V(1) + V(2)$$
$$G_p * n * V(n) = V(1) + V(2) + ... + V(n)$$

$$\underbrace{\begin{bmatrix} 1 & . & . & . & . & . \\ 1/2 & 1/2 & . & . & . & . \\ 1/6 & 1/2 & 1/3 & . & . & . \\ 0 & 1/4 & 1/2 & 1/4 & . & . \\ -1/30 & 0 & 1/3 & 1/2 & 1/5 & . \\ 0 & -1/12 & 0 & 5/12 & 1/2 & 1/6 \end{bmatrix}}_{Gp} * \begin{bmatrix} 2 \\ 4 \\ 8 \\ 16 \\ 32 \\ 64 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 9 \\ 17 \\ 33 \end{bmatrix}$$

Here we have the matrix **Gp** as the exact version of the Faulhaber/Bernoulli-matrix of coefficients for summing of like powers[1].

---

[1] I called this matrix **Gp** in my first article about Bernoulli-numbers, where this matrix occured (but with apriori knowledge of the Bernoulli-numbers, precisely: from context of the existence of two versions of sequences of Bernoulli-numbers differing in β_1, having either *-1/2* or *+1/2* as values).

3.6. "Zeta"-polynomials

We may define "*zeta*-polynomials" completely analoguous to the *eta*-polynomials, which then perform the (non-alternating) summing of like powers.
Only -additionally- we have to consider the first upper subdiagonal, which is related to the $\zeta(1)$-expressions.

(3.6.1.)
Definition of the m'th zeta-polynomial in x

$$zeta_m(x) = \sum_{c=0}^{m+1}\left(\zeta(-(m-c))\binom{m}{c}x^c\right) \quad // \text{ where } \zeta(1)*(m:m+1) = -1/(m+1)$$

(3.6.2.) or

$$zeta_m(x) = \sum_{c=0}^{m}\left(\zeta(-(m-c))\binom{m}{c}x^c\right) - \frac{1}{m+1}x^{m+1}$$

note the perfect match with the original J.Bernoulli-consideration(inverse signs), setting $\beta_k/k = -\zeta(1-k)$

Examples:

$zeta_0(x) = \zeta(0) \quad -1 \quad x$
$zeta_1(x) = \zeta(-1) + 1\,\zeta(0)\,x \quad - 1/2 \quad x^2$
$zeta_2(x) = \zeta(-2) + 2\,\zeta(-1)\,x + 1\,\zeta(0)\,x^2 \quad - 1/3\,x^3$
$zeta_3(x) = \zeta(-3) + 3\,\zeta(-2)\,x + 3\,\zeta(-1)\,x^2 + 1\,\zeta(0)\,x^3 \quad - 1/4\,x^4$
$zeta_4(x) = \zeta(-4) + 4\,\zeta(-3)\,x + 6\,\zeta(-1)\,x^2 + 4\,\zeta(-1)\,x^3 + 1\,\zeta(0)\,x^4 \quad - 1/5\,x^5$
...

then (using the notation of the Hurwitz-zeta-function):

(3.6.3.) $$zeta_m(0) = \zeta(-m,1) = \sum_{k=1}^{oo} k^m = 1^m + 2^m + 3^m + \dots$$

(3.6.4.) $$zeta_m(n) = \zeta(-m,n+1) = \sum_{k=n+1}^{oo} k^m = (n+1)^m + (n+2)^m + (n+3)^m + \dots .$$

where the latter is the Hurwitz-zeta-function $\zeta(-m,n)$ for each row m (only my row-indices m have inverse sign to the Hurwitz-zeta-exponent, this may be a bit confusing...).

The (non-alternating) sum of m'th like powers $s_m(n)$ from 1^m to n^m is:

(3.6.5.)

$$s(n) = zeta_m(0) - zeta_m(n) = \sum_{k=1}^{n} k^m = 1^m + 2^m + \dots + n^m$$
$$= \zeta(-m,1) - \zeta(-m,n+1) \quad // \text{ the Hurwitz-zeta-function}$$

Examples:

$s_1(2) = 1 + 2 = 3$
$= zeta_1(0) - zeta_1(2) = \zeta(-1) - (\zeta(-1) + \zeta(0)*2 - 1/2*4) = -\zeta(0)*2 + 2 = 3$

$s_1(4) = 1 + 2 + 3 + 4 = 10$
$= zeta_1(0) - zeta_1(4) = \zeta(-1) - (\zeta(-1) + \zeta(0)*4 - 1/2*16) = -\zeta(0)*4 + 8 = 10$

$s_3(3) = 1 + 8 + 27 = 36$
$= zeta_3(0) - zeta_3(3)$
$= \zeta(-3) - (\zeta(-3) + 3\,\zeta(-2)*3 + 3\,\zeta(-1)*9 + \zeta(0)*27 \quad - 1/4*81)$
$= \quad - 9*0 \quad +27/12 \quad + 27/2 \quad + 81/4$
$= \quad 9/4 \quad + 27/2 + 81/4 = 45/2 + 27/2 = 36$

4. Summing of like powers with negative exponent

4.1. Intro

Completely analoguously to the summing with positive exponents the summing can be performed with negative exponents.

Note for this, that the inverse of the binomial-matrix P^{-1} performs "shifting" when left-multiplied with a Vandermonde-vector. (*For a **proof** see for instance* [*binomialmatrix*])

Example:

$$\lim 1/2\ V(1/2)\sim * P^{-1} = 1/3 * V(1/3) \sim \quad \begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ -1 & 3 & -3 & 1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ -1 & 5 & -10 & 10 & -5 & 1 \end{bmatrix} \dots$$

P^-1

$$\begin{bmatrix} 1/2 & 1/4 & 1/8 & 1/16 & 1/32 & 1/64 \end{bmatrix} \dots \quad \begin{bmatrix} 1/3 & 1/9 & 1/27 & 1/81 & 1/243 & 1/729 \end{bmatrix} \dots$$

The reason, why I call this "shifting", may be better understood, if on the lhs we document a whole set of powerseries (in terms of the Vandermonde-vectors, which I usually call ZV_r) and on the lhs the whole set of results:

Example[1]:

$$\lim 1/x\ V(1/x)\sim * P^{-1} = 1/(x+1) * V(1/(x+1)) \sim \quad \begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ -1 & 3 & -3 & 1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ -1 & 5 & -10 & 10 & -5 & 1 \end{bmatrix} \dots$$

P^-1

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1/2 & 1/4 & 1/8 & 1/16 & 1/32 & 1/64 \\ 1/3 & 1/9 & 1/27 & 1/81 & 1/243 & 1/729 \\ 1/4 & 1/16 & 1/64 & 1/256 & 1/1024 & 1/4096 \\ 1/5 & 1/25 & 1/125 & 1/625 & 1/3125 & 1/15625 \\ 1/6 & 1/36 & 1/216 & 1/1296 & 1/7776 & 1/46656 \end{bmatrix} \dots \quad \begin{bmatrix} 1/2 & 1/4 & 1/8 & 1/16 & 1/32 & 1/64 \\ 1/3 & 1/9 & 1/27 & 1/81 & 1/243 & 1/729 \\ 1/4 & 1/16 & 1/64 & 1/256 & 1/1024 & 1/4096 \\ 1/5 & 1/25 & 1/125 & 1/625 & 1/3125 & 1/15625 \\ 1/6 & 1/36 & 1/216 & 1/1296 & 1/7776 & 1/46656 \\ 1/7 & 1/49 & 1/343 & 1/2401 & 1/16807 & 1/117649 \end{bmatrix}$$

ZV ZVr

Analoguously to chapt. *1*, to repeat this operation means:

$$1 \quad * V(1)\sim \ * P^{-1} = 1/2\ V(1/2)\sim$$
$$1/2 * V(1/2)\sim * P^{-1} = 1/3\ V(1/3)\sim$$

...

and generally

$$V(1)\sim * P^{-n} = 1/(n+1) * V(1/(n+1))\sim \qquad (4.1.1.)$$

[1] Note: In the following examples I'll omit the ellipses, which conventionally denote infinite extension, for brevity

4.2. Sums of like powers of reciprocals: basic notations

The summing of a contiguous interval of reciprocals is denoted as $\mathbf{S_R}(a,b)$

(4.2.1.) $$\mathbf{S_R}(a,b) = \frac{1}{a}V\left(\frac{1}{a}\right) + \frac{1}{a+1}V\left(\frac{1}{a+1}\right) + \frac{1}{a+2}V\left(\frac{1}{a+2}\right) + \ldots + \frac{1}{b}V\left(\frac{1}{b}\right)$$

where in the r'th row of $\mathbf{S_R}(a,b)$ are the scalar sums

(4.2.2.) $$S_R(a,b)_r = \frac{1}{a^{r+1}} + \frac{1}{(a+1)^{r+1}} + \frac{1}{(a+2)^{r+1}} + \ldots + \frac{1}{b^{r+1}}$$

Examples:

The r'th harmonic number $h_r(b)$ is

(4.2.3.) $$h_r(b) = S_R(1,b)_r$$

For $\mathbf{S_R}(1,inf)$ we have in the row r the zeta-value:

(4.2.4.) $$\zeta(r+1) = S_R(1,inf)_r = 1 + \frac{1}{2^{r+1}} + \frac{1}{3^{r+1}} + \frac{1}{4^{r+1}} + \ldots$$

Thus we get again a geometric series, where in each row r of the result is $\zeta(r+1)$,

(4.2.5.) $$S_R(1,inf)\sim = V(1)\sim + 1/2\ V(1/2)\sim + 1/3\ V(1/3)\sim + \ldots$$

can be written as

(4.2.6.) $$S_R(1,inf)\sim = V(1)\sim * (P^{-0} + P^{-1} + P^{-2} + \ldots).$$

and we could express this with the shortcut formula for geometric series in $\mathbf{P}^{-1}$

(4.2.7.) $$S_R(1,inf)\sim = V(1)\sim * (I - P^{-1})^{-1}$$

if the parenthese were invertible.

We simply face the same problem as we did when trying to sum the alternating and non-alternating zeta-series with negative exponents, only that the inverse of $\mathbf{P}$ is involved instead of $\mathbf{P}$. But since the diagonal of $\mathbf{P}$ and $\mathbf{P}^{-1}$ are the same, we have the same problem (and solution) here.

4.3. Solution for the alternating sum of like powers of reciprocals

Define the sum vector for the alternating vector-series

(4.3.1.) $AS_R(1,inf) = V(1) - 1/2\ V(1/2) + 1/3\ V(1/3) - \ldots$

Then, by construction, we'll have in any element of $AS_R(1,inf)$ the according value of the *eta()*-function $\eta()$:

(4.3.2.) $AS_R(1,inf)_{[r]} = \eta(1+r)$

Replacing $1/n\ V(1/n)\sim$ by $V(1)\sim * P^{1-n}$ and factoring out of $V(1)\sim$ we have also:

$$AS_R(1,inf)\sim = V(1)\sim * (P^{-0} - P^{-1} + P^{-2} - ..).$$

The parenthese on the rhs can be computed, using the geometric-series-formula, by

$$AS_R(1,inf)\sim = V(1)\sim * (I + P^{-1})^{-1}$$

We may provisorially call the parenthese-term ETA_2:

(4.3.3.) $ETA_2 = (I + P^{-1})^{-1}$

and see, that this identical to our ***ETA***-matrix.

Proof: recall the definition of ***ETA*** from *(2.???)*

$$ETA = P * (I + P)^{-1}$$

put the first P into the parenthese and multiply out:

$$\begin{aligned} ETA &= ((I + P)*P^{-1})^{-1} \\ &= (P^{-1} + I)^{-1} \\ &= ETA_2 \end{aligned}$$

End of proof

So, analoguously to the summing property of ***ETA*** in chap 2 we have

(4.3.4.)

$$\begin{aligned} AS_R(1,inf)\sim &= V(1)\sim * ETA \\ &= [\eta(1), \eta(2), \eta(3),\ldots] \end{aligned}$$

Beginning at a different starting-value we have the (alternating) Hurwitz-zeta-values in AS_R:

(4.3.5.)

$$\begin{aligned} AS_R(a,inf)\sim &= (-1)^{a-1}\ V(a)\sim * ETA \\ &= [\ \eta(1,a), \eta(2,a), \eta(3,a), \ldots\ ] \end{aligned}$$

The difference of two such partial infinite sums provide the alternating sums of a contiguous segment of like powers of reciprocals:

(4.3.6.) $AS_R(a,b)\sim = AS_R(a,inf)\sim - AS_R(b+1,inf)\sim$

$$= \left(\frac{(-1)^{a-1}}{a} V\left(\frac{1}{a}\right) - \frac{(-1)^b}{b+1} V\left(\frac{1}{b+1}\right) \right) \sim * ETA$$

In conventional notation this is - for the first column $c=0$ - of the result:

$$\sum_{k=a}^{b}(-1)^k\frac{1}{k}=\sum_{k=0}^{oo}\left[\left(\frac{(-1)^{a-1}}{a^{k+1}}-\frac{(-1)^{b-1}}{b^{k+1}}\right)\eta(-k)\right] \qquad \text{// with } c=0 \tag{4.3.7.}$$

Extended to the other columns, using the appropriate binomial-factors:

$$\sum_{k=a}^{b}(-1)^k\frac{1}{k^{c+1}}=\sum_{k=c}^{oo}\left[\left(\frac{(-1)^{a-1}}{a^{k+1}}-\frac{(-1)^{b-1}}{b^{k+1}}\right)\binom{k}{c}\eta(c-k)\right] \qquad \text{// with } c\geq 0 \tag{4.3.8.}$$

Notes:

1) To confirm the result numerically, we need techniques of divergent summation, since the sums for each column diverge; but using for instance Euler-summation the result can be verified easily.

2) If a finite und b infinite, we have the alternating Hurwitz-Zetas in the columns of the result (*see (...)*).

3) If $a=1$, and b finite we have the r'th alternating harmonic numbers $ah_r(b)$, the sums from $1/1$ up to $1/b^r$

$$[V(1)-(-1)^{b-1}V(1/b)]\sim * ETA = [ah_1(b), ah_2(b), ah_3(b), \ldots] \tag{4.3.9.}$$

4.4. Solution for the non-alternating sum of like powers of reciprocals

Define the sum vector for the vector-series

$$S_R(1,inf) = V(1) + 1/2\, V(1/2) + 1/3\, V(1/3) + \ldots \tag{4.4.1.}$$

Then, by construction, we'll have in any element of $\mathbf{S_R}(1,inf)$ the according value of the $zeta()$-function $\zeta()$:

$$S_R(1,inf)_{[r]} = \zeta(1+r) \tag{4.4.2.}$$

Replacing $1/n\ \mathbf{V}(1/n)\sim$ by $\mathbf{V}(1)\sim * \mathbf{P}^{1-n}$ and factoring out of $\mathbf{V}(1)\sim$ we have also:

$$S_R(1,inf)\sim = V(1)\sim * (P^{-0} + P^{-1} + P^{-2} + \ldots).$$

The parenthese on the rhs can be computed, using the Toeplitz-formula, by

$$S_R(1,inf)\sim = \lim V(1)\sim * [P^{-1} \text{☼} \sum_{k=0}^{oo} T(k)] \tag{4.4.3.}$$

The result for $S_R(1,inf)$ is *(see proof for identity zeta2))*

$$S_R(1,inf)\sim = \lim V(1)\sim * (-ZETA) \tag{4.4.4.}$$

The results for $S_R(a,inf)$ and $S_R(1,b)$ and $S_R(a,b)$ are then accordingly

$$S_R(a,inf)\sim = \lim 1/a\, V(1/a)\sim * (-ZETA) = [\gamma, \zeta(2), \zeta(3), \ldots] \tag{4.4.5.}$$

$$S_R(1,b-1)\sim = \lim [\ V(1) - 1/b\, V(1/b)]\sim * (-ZETA) \tag{4.4.6.}$$

$$S_R(a,b-1)\sim = \lim [1/a\, V(1/a) - 1/b\, V(1/b)]\sim * (-ZETA) \tag{4.4.7.}$$

I verified the result numberically using Euler-summation of order *4.7* with dimension *n=64* of the matrices to three decimal places using Pari/GP:

Example:

*lim V(1)~ * (-ZETA)*
=[gamma, ζ(2), ζ(3),...]

$$\begin{bmatrix} 1/2 & 1 & 0 & 0 & 0 & 0 \\ 1/12 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 1/6 & 1/2 & 1/3 & 0 & 0 \\ -1/120 & 0 & 1/4 & 1/2 & 1/4 & 0 \\ 0 & -1/30 & 0 & 1/3 & 1/2 & 1/5 \\ 1/252 & 0 & -1/12 & 0 & 5/12 & 1/2 \end{bmatrix} \text{-Z}$$

...

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \ldots \quad \begin{bmatrix} 0.57722 & 1.6449 & 1.2020 & 1.0821 & 1.0359 & 1.0139 \end{bmatrix} \ldots$$

direct computation using Pari/GP: $\begin{bmatrix} 0.57722 & 1.6449 & 1.2021 & 1.0823 & 1.0369 & 1.0173 \end{bmatrix}$

So, analoguously to the summing property of ***ZETA*** in chap 3 we have

(4.4.8.) $S_R(1,inf)\sim \; = V(1)\sim * ZETA$
$= [\; gamma\, ,\, \zeta(2),\, \zeta(3),\ldots]$

Beginning at a different starting-value we have the Hurwitz-zeta-values in $\boldsymbol{S_R}$:

(4.4.9.) $S_R(a,inf)\sim \; = V(a)\sim * ZETA$
$= [\; \zeta(1,a),\, \zeta(2,a),\, \zeta(3,a),\, \ldots\,]$

The difference of two such partial infinite sums provide the sums of a contiguous segment of like powers of reciprocals:

(4.4.10.) $S_R(a,b)\sim = S_R(a,inf)\sim - S_R(b+1,inf)\sim$

$$= \left(\frac{1}{a} V\left(\frac{1}{a}\right) - \frac{1}{b+1} V\left(\frac{1}{b+1}\right)\right) \sim * ZETA$$

In conventional notation this is - for the first column *c=0* - of the result:

(4.4.11.) $$\sum_{k=a}^{b} \frac{1}{k} = \sum_{k=0}^{oo} \left[\left(\frac{1}{a^{k+1}} - \frac{1}{b^{k+1}} \right) \zeta(-k) \right] \qquad // \text{ with } c=0$$

Extended to the other columns, using the appropriate binomial-factors (where the normed term at *ζ(1)*-positions are added)

(4.4.12.) $$\sum_{k=a}^{b} \frac{1}{k^{c+1}} = \sum_{k=c-1}^{oo} \left[\left(\frac{1}{a^{k+1}} - \frac{1}{b^{k+1}} \right) \binom{k}{c} \zeta(c-k) \right] \qquad // \text{ with } c>0$$

Notes:

1) To confirm the result numerically, we need techniques of divergent summation, since the sums for each column diverge; however Euler-summation does not suffice here.

2) If *a* finite und *b* infinite, we have the Hurwitz-Zetas in the columns of the result.

3) If *a=1*, and *b* finite we have the *r*'th harmonic numbers $h_r(b)$, the sums from *1/1* up to $1/b^r$

(4.4.13.) $(V(1)- V(1/b))\sim * ZETA = [h_1(b), h_2(b) ,h_3(b), ...]$

5. Generalizations

5.1. Different arithmetic progressions

(not yet ready. It mean simply using powers of P in the formula for geometric series)

5.2. Fractional exponents

(not yet ready. It mean simply using fractional powers of P in the formula for geometric series, also P is no more triangular)

5.3. Resume

(not yet ready)

6. Proofs and details

6.1. Proof: the z'th power of P is the Hadamard-product of P and T(z)

(6.1.1.) *Definition:*
$$V(a)* V(1/a)\sim = Toeplitz(a) := T(a)$$

Lemma 6.1.1: The entries of P^a are the entries of the Hadamard-product P ☼ $Toeplitz(a)$.

Proof 6.1.1:

From the binomial-theorem it follows,. that

(6.1.2.) $P * V(a) = P * [1,a,a^2,a^3,...]\sim = [1,a+1,(a+1)^2,(a+1)^3,...]\sim$

where the n'th row of the rhs is (writing $bi()$ for $binomial()$)

$(1+a)^n = a^0\ bi(n, 0)+ a^1\ bi(n, 1) + ...\ a^{n-1}\ bi(n, n-1) + a^n\ bi(n, n)$

But we may also write

$(1+a)^n = a^n\ (1+1/a)^n$

and this is, using the n'th row of **P**:

(6.1.3.) $a^n\ (1 +1/a)^n \quad = a^n\ (a^0\ bi(n,0)+ a^{-1}\ bi(n,1) + ...\ a^{1-n}\ bi(n,n-1) + \ a^{-n}\ bi(n,n))$
$\qquad\qquad = a^n\ P_{n,*} * V(1/a)$

This is formally identical for each row in **P**, so we may write the leading a^n as left-multiplication with the diagonal matrix $^d\mathbf{V}(a)$:

(6.1.4.) $P * V(a) = {}^dV(a) * P * V(1/a)$

Rewriting the rightmost column-vector $\mathbf{V}(1/a)$ as product of $^d\mathbf{V}(1/a)*\mathbf{V}(1)$:

(6.1.5.) $P * V(a) = ({}^dV(a) * P * {}^dV(1/a)) \ * V(1)$

Now the cofactors of **P** on the rhs rescale the entries of **P** by a^{r-c}, which can then be seen as the Hadamard-product of **P** with the Toeplitzmatrix $\mathbf{T}(a)$.

On the other hand,

(6.1.6.) $P * V(a) = P^a * V(1)$

so also, combining *(4.1.5)* and *(4.1.6)*, it is :

(6.1.7.) $P^a = ({}^dV(a) * P * {}^dV(1/a))$

The entries $P^a{}_{[r,c]}$ of $P^a = {}^dV(a)*P*{}^dV(1/a)$ are

(6.1.8.)	$P^a{}_{[r,c]} = a^r * P_{[r,c]}*a^{-c} = P[r,c]*a^{r-c}$

and thus can be written as a hadamard-product of **P** with the Toeplitzmatrix of *a :*

(6.1.9.)	$P^a = P$ ☼ $Toeplitz(a)$

End of Proof.

Second Proof for Lemma 6.1.1

Corollary 6.1.2:

> **(6.1.10.)**
> *The pascal-matrix can be seen as matrix-exponential of a purely subdiagonal-matrix containing the natural numbers.*

Example:

$$L = {}^{SubDiag1}Z(-1) \qquad \begin{bmatrix} . & . & . & . & . & . \\ 1 & . & . & . & . & . \\ . & 2 & . & . & . & . \\ . & . & 3 & . & . & . \\ . & . & . & 4 & . & . \\ . & . & . & . & 5 & . \end{bmatrix} \; L$$

$$P = exp(L) \qquad exp\left(\begin{bmatrix} . & . & . & . & . & . \\ 1 & . & . & . & . & . \\ . & 2 & . & . & . & . \\ . & . & 3 & . & . & . \\ . & . & . & 4 & . & . \\ . & . & . & . & 5 & . \end{bmatrix}\right) = \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix} \; P$$

Proof: this follows immediately from the exponential-series for L.
see for instance in [binomialmatrix]

Corollary 6.1.3:

> **(6.1.11.)**
> *A power of the exponential is the exponential of a multiple of the logarithm.*

Proof: this is a known property valid also for invertible matrices.

Restatement of Lemma 6.1.1:

> **(6.1.12.)**
> *The n'th power of P contains the original binomial-entries cofactored with powers of n, in terms of a Hadamard-product of P with a Toeplitzmatrix formed from powers of n.*

In matrix-description: (where ☼ denotes the elementwise ("Hadamard")-product)

> **(6.1.13.)** $P^n = {}^dV(n) * P * {}^dV(1/n) = P$ ☼ $(V(n)* V(1/n)\sim) = P$ ☼ $Toeplitz(n)$

Proof 6.1.2:

If we use the logarithm **L** and multiply it by an arbitrary complex scalar factor *s* we have initially:

$$\begin{bmatrix} . & . & . & . & . \\ s & . & . & . & . \\ . & 2*s & . & . & . \\ . & . & 3*s & . & . \\ . & . & . & 4*s & . \end{bmatrix}$$

The cofactor *s* arrives at consecutive powers for consecutive subdiagonals, when expanding the exponential-series, and does not affect the constructed binomial-coefficients.

The matrix exponential is then finally

(6.1.14.) $P^s = P$ ☼ $Toeplitz(s)$
$\quad = {}^dV(s) * P * {}^dV(1/s)$

$$\begin{bmatrix} 1 & . & . & . & . \\ s\ *1 & 1 & . & . & . \\ s\text{^}2*1 & s\ *2 & 1 & . & . \\ s\text{^}3*1 & s\text{^}2*3 & s\ *3 & 1 & . \\ s\text{^}4*1 & s\text{^}3*4 & s\text{^}2*6 & s\ *4 & 1 \end{bmatrix}$$

which is also the Hadamardproduct of the Pascal-matrix **P** with the product (**V**$(s)* V(1/s)\sim$) so that *lemma 4.1.1* is proven and we have:

> **(6.1.15.)** $P^s := P^s_{r,c} = (r:c)*s^{r-c}$ $\qquad$ // = 0 if c>r

End of proof 6.1.2

6.2. Formal description of the entries of ETA /ZETA

Corrolary 6.2.1: the sum of $P^* V(a) + P^* V(b)$ is

$$P\ V(a) + P\ V(b) =$$
$$P^a * V(1) + P^b * V(1) =$$
$$(P^a + P^b) * V(1) =$$
$$= [P ☼ (Toeplitz(a) + Toeplitz(b))] * V(1)$$

Lemma 6.2.1: The infinite sum of $P\ V(0) + P\ V(1) + P\ V(2) + \dots$ is:

(6.2.1.)
$$P * \Sigma_{k=0,inf} V(k) = (P ☼ \Sigma_{k=0,inf} Toeplitz(k)) * V(1)$$

conditional on summability of (possibly divergent) sums for any entry

Result 6.2.1 :

the entries $\mathbf{ETA}_{r,c}$ *of the matrix* **ETA** *are the alternating sums of powers, thus* $\eta()$*-values at nonpositive exponents, cofactored by binomial coefficients:*

(6.2.2.)
$$ETA_{r,c} = P_{r,c} * \eta(-(r-c))$$
$$= (r:c) * \eta(-(r-c))$$

Result 6.2.2 :

the entries **ZETAr,c** *of the matrix* **ZETA** *are the sums of like powers, thus* $\zeta()$*-values at nonpositive exponents, cofactored by binomial coefficients:*

(6.2.3.)
$$ZETA_{r,c} = P_{r,c} * \zeta(-(r-c))$$
$$= (r:c) * \zeta(-(r-c))$$

The entries of the first upper subdiagonal are therefore -1/c, where c is the column-number, due to the setting:(see chap (4.3))

$$\zeta(1)/(-1)! = -1$$
and
$$\zeta(1)/(-1)!*1/c = -1/c.$$

Proof 6.2.1:

According to *Lemma 6.2.1*, the entries of **ZETA/ETA** can be described as binomials cofactored with the non-/alternating sums of the according entries of all involved Toeplitzmatrices, which form then the $\zeta() / \eta()$-function-values with row/col-specific exponents.

Since in each entry of the n'th power of **P** is

$$P^n_{[r,c]} = P_{[r,c]} * n^{r-c}$$

the non-/alternating sum of all powers for $n=1..inf$ in each entry of **ZETA/ETA** is

$$ZETA \quad \lim P*(P^0+P^1+P^2+\dots)_{r,c} = P_{r,c} * \Sigma_{k=1..inf}\ k^{r-c}$$
$$= P_{r,c} * \zeta(-(r-c)) \qquad // \text{ where } r,c \ge 0$$
$$= -1/c \ = \zeta(1)/(-1)!\ /c \qquad // \text{ where } c=r+1$$

$$ETA \quad \lim P*(P^0-P^1+P^2-\dots)_{r,c} = P_{r,c} * \Sigma_{k=1..inf} (-1)^{k-1}\ k^{r-c}$$
$$= P_{r,c} * \eta(-(r-c))$$

whose divergent sums are known, or can be evaluated by Ramanujan- or Euler-summation to the appriorate $\zeta()$ or $\eta()$-values.

End of proof

6.3. Ratio of ζ(1) and gamma(0)

I don't have an authoritative reference for the definition yet, but some hints, which I'm trying to extend. Here is a source from a discussion in usenet:

6.3.1. Usenet-discussion (1): sci.math

In article <f2c0fs$5ee$0...@news.t-online.com>, Gottfried Helms wrote:

> Hi -
> in the context of binomials I came across this substitution as meaningful, or say more precisely
> that it would be meaningful to set
> zeta(1) / (-1)! = 1
>
> Is this appropriate also in other contexts? (and if, then to argue why, would be helpful too)
> Gottfried Helms

Laurent series, Zeta(1+x) = 1/x + ...
Gamma(x) = 1/x + ...

so the ratio converges to 1 as x -> 0

That is probably what is meant.
--
G. A. Edgar http://www.math.ohio-state.edu/~edgar/

6.3.2. Wikipedia: Riemann-Zeta-function, "Laurent series"

http://en.wikipedia.org/wiki/Riemann_zeta_function

"The Riemann zeta function is meromorphic with a single pole of order one at s = 1. It can therefore be expanded as a Laurent series about s = 1; the series development then is

$$\zeta(s) = \frac{1}{s-1} + \gamma_0 + \gamma_1(s-1) + \gamma_2(s-1)^2 + \cdots.$$

The constants here are called the Stieltjes constants and can be defined as (...)"

6.3.3. usenet-discussion (2) : de.sci.mathematik

Von: Peter Luschny (venice@luschny.de)
Datum: 23.02.2007 09:55
Message-ID: <erma65$9j7$1@online.de>

Schau auf dieses harmlose Funktiönchen

*g(x) = -x*sin(Pi*x)*GAMMA(-x)/Pi*

und betrachte es für Werte *x < 0*. Im wesentlichen die Funktion *sin(Pi*x)*, nur dass sie immer stärker oszilliert wenn *x -> -oo* geht.
An den Stellen *n=-1,-2,-3,...* hat sie natürlich den Wert *0*.

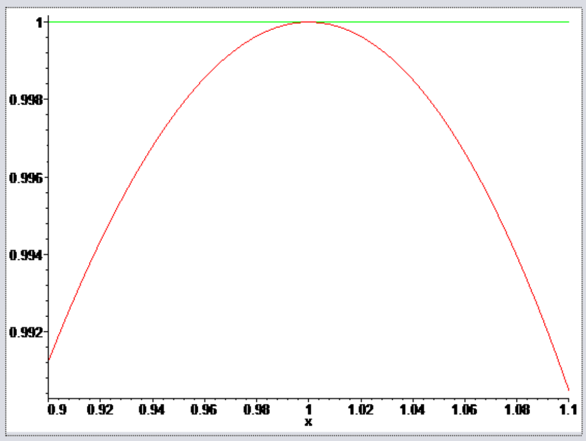
Danach betrachte *Zeta(x)* für *x<0*. Ein kleines Regenwürmchen, das sich um die *x*-Achse schlängelt. Total harmlos.

Also ist auch *Zeta(x)*g(x)* total harmlos überall auf der negativen Achse. Ok so weit?

Na gut, aber *Zeta(x)*g(x) = Zeta(x)/Gamma(x)*. Fertig.

Weil *1/GAMMA(x) = -x*sin(Pi*x)*GAMMA(-x)/Pi* .

Ersetze jetzt das Wort 'harmlos' durch, etwa, 'stetig' und du fragst nicht mehr nach eps/delta.



-Zeta(x)/(-x)! for x=0.9..1.1

http://peter.luschny.googlepages.com/zetafactorial

6.4. Proof for identity $ZETA_2$ = - ZETA

Starting at eq. *(4.7.4.3)*:

(3.7.4.3) $$S_R(1,inf)\sim \quad = \lim V(1)\sim * [P^{-1} ☼ \sum_{k=0}^{oo} T(k)]$$

where

$$P^{-1} = J * P * J = {}^dV(-1) * P * {}^dV(-1) = P ☼ T(-1)$$

We may provisorially call the bracketed term $\mathbf{ZETA_2}$:

(6.4.1.) $$ZETA_2 := ZETA_{2\ r,c} = (-1)^{r-c} \text{binomial}(r,c)*\zeta(c-r)$$

and see, that this identical to our ***ZETA***-matrix.

Proof: Put the first P into the parenthese and multiply out:

$$\begin{aligned} ZETA_2 &= P^{-1} ☼ (T(0) + T(1) + T(2) + T(3) + ...) \\ &= P^{-1} ☼ T(0) + P^{-1} ☼ (T(1) + T(2) + T(3) + ...) \\ &= I + J P J ☼ (T(1) + T(2) + T(3) + ...) \\ \textbf{(6.4.2.)} \quad &= I + J [P ☼ (T(1) + T(2) + T(3) + ...)] J \end{aligned}$$

Now recall the definition of ***ZETA*** from *(3.3.3)*

$$ZETA = \lim (T(1) + T(2) + T(3) + ...) ¤ P$$

Then put this into the bracketed term of the previous formula and get

(6.4.3.) $$\begin{aligned} ZETA_2 &= I + J\ ZETA\ J \\ &= I + ZETA ☼ T(-1) \end{aligned}$$

Now at each place, where the Hadamard-multiplication changes the sign of ***ZETA***, there is a value $\mathbf{ZETA}_{r,c}<>0$, and at each place, where it doesn't $\mathbf{ZETA}_{r,c}=0$, except in the diagonal, where $\mathbf{ZETA}_{r,c} = -1/2$. Thus we have to compensate for these diagonal entries by addition of $2*-1/2*I$:

(6.4.4.) $$ZETA ☼ T(-1) = - ZETA - I$$

Thus

(6.4.5.) $$\begin{aligned} ZETA_2 &= I + ZETA ☼ T(-1) \\ &= I - ZETA - I \end{aligned}$$

(6.4.6.) $$ZETA_2 = - ZETA$$

End of proof

7. Loose ends

7.1. extension of ZETA to ZETA$_+$ and its reciprocal

The matrix **ZETA** could not be found by the shortcut-formula for infinite geometric series due to the impossibility of the inversion of *(***I** - **P***)*. Also the inversion of **ZETA** gives uninteresting results, the entries of a **ZETA**$^{-1}$ - matrix are altered by the dimension of **ZETA**.
But it is an interesting experiment, if we prefix **ZETA** with a row and the most reasonable guess for the missing top-left element, call this extended matrix **Z+** .

```
Z+ =
  [ ζ( 1)*(-1 0)        .               .              .        ...]
  [ ζ( 0)*( 0 0)   ζ( 1)*(0  1)         .              .        ...]
  [ ζ(-1)*( 1 0)   ζ( 0)*(1  1)   ζ( 1)*(1 2)          .        ...]
  [ ζ(-2)*( 2 0)   ζ(-1)*(2  1)   ζ( 0)*(2 2)   ζ( 1)*(2 3)     ...]
```

which is numerically (except (ascii-) writing *z1* for *ζ(1)* itself):

(7.1.1.) $$Z_+ = \begin{bmatrix} -z1 & . & . & . & . & . \\ -1/2 & -1 & . & . & . & . \\ -1/12 & -1/2 & -1/2 & . & . & . \\ 0 & -1/6 & -1/2 & -1/3 & . & . \\ 1/120 & 0 & -1/4 & -1/2 & -1/4 & . \\ 0 & 1/30 & 0 & -1/3 & -1/2 & -1/5 \end{bmatrix}$$

where z1 is ζ(1).

The reciprocal of **Z+** occurs numerically as the following Pascal-similar matrix with one additional column and the diagonal subtracted:

(7.1.2.) $$Z_+^{-1} = \begin{bmatrix} -1/z1 & . & . & . & . & . \\ 1/2/z1 & -1 & . & . & . & . \\ -1/3/z1 & 1 & -2 & . & . & . \\ 1/4/z1 & -1 & 3 & -3 & . & . \\ -1/5/z1 & 1 & -4 & 6 & -4 & . \\ 1/6/z1 & -1 & 5 & -10 & 10 & -5 \end{bmatrix}$$

where z1 is ζ(1).

The entries here are independent of the dimension of **Z+**, and interestingly the submatrix with column *1* removed is just *(***I** - **P**$^{-1}$*)*, the ***reciprocal*** of **P** (instead of **P** itself!). Note that the whole column *1* is numerically zero.

If **Z+** and **Z+**$^{-1}$ are taken as eigensystem with the set of eigenvalues, using **J** = *diag(1,-1,1,-1,...)* as described for instance in [binomial] then we get the eigen-composition of **Pj +** , an extended version of **Pj** :

(7.1.3.) $$Z_+ * J * Z_+^{-1} = P_{j+}$$

$$* \begin{bmatrix} -1/z1 & . & . & . & . & . \\ 1/2/z1 & -1 & . & . & . & . \\ -1/3/z1 & 1 & -2 & . & . & . \\ 1/4/z1 & -1 & 3 & -3 & . & . \\ -1/5/z1 & 1 & -4 & 6 & -4 & . \\ 1/6/z1 & -1 & 5 & -10 & 10 & -5 \end{bmatrix}$$

$$\begin{bmatrix} -z1 & . & . & . & . & . \\ -1/2 & -1 & . & . & . & . \\ -1/12 & -1/2 & -1/2 & . & . & . \\ 0 & -1/6 & -1/2 & -1/3 & . & . \\ 1/120 & 0 & -1/4 & -1/2 & -1/4 & . \\ 0 & 1/30 & 0 & -1/3 & -1/2 & -1/5 \end{bmatrix} * J = \begin{bmatrix} 1 & . & . & . & . & . \\ 1/z1 & -1 & . & . & . & . \\ 1/(2*z1) & -1 & 1 & . & . & . \\ 1/(3*z1) & -1 & 2 & -1 & . & . \\ 1/(4*z1) & -1 & 3 & -3 & 1 & . \\ 1/(5*z1) & -1 & 4 & -6 & 4 & -1 \end{bmatrix}$$

7.2. The matrixlogarithm of ETA

Heuristically, the matrix-logarithm has the following structure, which has the interesting property, that a shifting of the exponents of eta occurs: just by taking the matrix-logarithm we introduce $-\eta(1)$ and the first column has one exponent shifted entries compared with **ETA** itself. Also the signs are changed.

$$\lim_{n\to oo} \log(Eta,n)= \begin{bmatrix} -\log2 & . & . & . & . & . \\ -1/2 & -\log2 & . & . & . & . \\ -1/4 & -1 & -\log2 & . & . & . \\ 0 & -3/4 & -3/2 & -\log2 & . & . \\ 1/8 & 0 & -3/2 & -2 & -\log2 & . \\ 0 & 5/8 & 0 & -5/2 & -5/2 & -\log2 \\ -1/4 & 0 & 15/8 & 0 & -15/4 & -3 \\ 0 & -7/4 & 0 & 35/8 & 0 & -21/4 \\ 17/16 & 0 & -7 & 0 & 35/4 & 0 \\ 0 & 153/16 & 0 & -21 & 0 & 63/4 \\ -31/4 & 0 & 765/16 & 0 & -105/2 & 0 \\ 0 & -341/4 & 0 & 2805/16 & 0 & -231/2 \\ 691/8 & 0 & -1023/2 & 0 & 8415/16 & 0 \end{bmatrix}$$

where n is the number of terms in the logarithmic series used

```
Pari/Gp: m=96; bestappr(MLog(1.0*ETA,m),1e4)
         \\ replace diagonal by token -log2
```

(7.2.1.)

proposal:

$$logETA_{r,0} = - \eta(1-(r-c))*binomial(r,c)$$

(7.2.2.)

a recursive definition:

$$logETA_{r,0} = - eta(1-r)$$
$$logETA_{r+1,c+1} = logETA_{r,c} * (r+1)/(c+1) \qquad //for\ r>=c$$

7.3. The matrix-logarithm of $G_p *^d Z(-1)$ (instead of ZETA)

The matrix-logarithm of **ZETA** seems to be uninteresting, since its entries change with the dimension. Since $\mathbf{ZETA}^+$ is triangular, formally a consistent matrix-logarithm is possible here; but again it is uninteresting because of the $\zeta(1)$-infinity in the top-left-element.

What is actually possible and interesting, is to use the matrix-logarithm of the **ZETA**-torso **Gp**, and actually of the column-scaled-version $\mathbf{Gp} *^d \mathbf{Z}(-1)$, which then is the matrix **B** of Bernoulli-polynomials[1], or differently said, the matrix of derivatives of *Zeta*-polynomials.

We get - in rational arithmetic, since the numerators in the logarithmic matrix-series are nilpotent -:

$$LOG=mlog(G_p *^d Z(-1))$$
$$LOG=mlog(\ B\ ) \qquad \begin{bmatrix} 0 & . & . & . & . & . \\ 1/2 & 0 & . & . & . & . \\ -1/12 & 1 & 0 & . & . & . \\ 0 & -1/4 & 3/2 & 0 & . & . \\ 1/120 & 0 & -1/2 & 2 & 0 & . \\ 0 & 1/24 & 0 & -5/6 & 5/2 & 0 \end{bmatrix}$$

[1] (see article [PMATRIX] for a bit more detailed description)

The entries of **LOG** are

(7.3.1.)
Proposal:
$LOG_{r,c} = -\zeta(1-(r-c)) * binomial(r,c)$ // for r>c

7.4. ETA as eigensystem: building a Genocchi-Matrix

If we use the *ETA*-matrix as eigensystem with eigenvalues ${}^{d}\mathbf{Z}(-1) = diag(1,2,3,4,...)$ we get

$$GEN = -2 * ETA * {}^{d}Z(-1) * ETA^{-1}$$

where the Genocchi-numbers are in the first column starting at *2'nd* row and are in the next columns Toeplitz-like arranged again with binomial cofactors:

$$GEN = \begin{bmatrix} -2 & . & . & . & . & . & . & . \\ 1 & -4 & . & . & . & . & . & . \\ -1 & 2 & -6 & . & . & . & . & . \\ 0 & -3 & 3 & -8 & . & . & . & . \\ 1 & 0 & -6 & 4 & -10 & . & . & . \\ 0 & 5 & 0 & -10 & 5 & -12 & . & . \\ -3 & 0 & 15 & 0 & -15 & 6 & -14 & . \\ 0 & -21 & 0 & 35 & 0 & -21 & 7 & -16 \end{bmatrix}$$

whose entries are

(7.4.1.) *Proposal:*
the diagonal:
$GEN[r,c] = -2 * (r+1)$ // for r = c
the lower triangle:
$GEN[r,c] = A036968(r)$ // for r>c = 0
$GEN[r,c] = binomial(r,c) * GEN[r-c,0]$ // for r>c>0

OEIS :

A036968 Genocchi numbers (of first kind): expansion of $2x/(\exp(x)+1)$.

1, -1, 0, 1, 0, -3, 0, 17, 0, -155, 0, 2073, 0, -38227,

7.5. Graphs for ZETA-polynomials /Gp-polynomials

The *m*'th *zeta()*-polynomial, in matrix-notation is

(7.5.1.) $zeta_m(x) = ZETA[m] * V(x)$

or in conventional notation

(7.5.2.) $$zeta_m(x) = \sum_{k=0}^{m+1} \zeta(-k) * \binom{m}{k} x^k = \sum_{k=0}^{m} \left(\zeta(-k) * \binom{m}{k} x^k \right) - \frac{x^{m+1}}{m+1}$$

Omitting the constant term at $k=0$ in the above formula (which means only to ignore a vertical shifting in the graph) and reversing signs, calling that the *gp()*-polynomials, we have in matrix-notation

(7.5.3.) $$gp_m(x) = Gp_{[m]} * x\ V(x)$$

or

(7.5.4.) $$gp_m(x) = \sum_{k=1}^{m} \left(-\zeta(-k) * \binom{m}{k} x^k \right) + \frac{x^{m+1}}{m+1}$$

and this is also, to see the relation to bernoulli-polynomials:

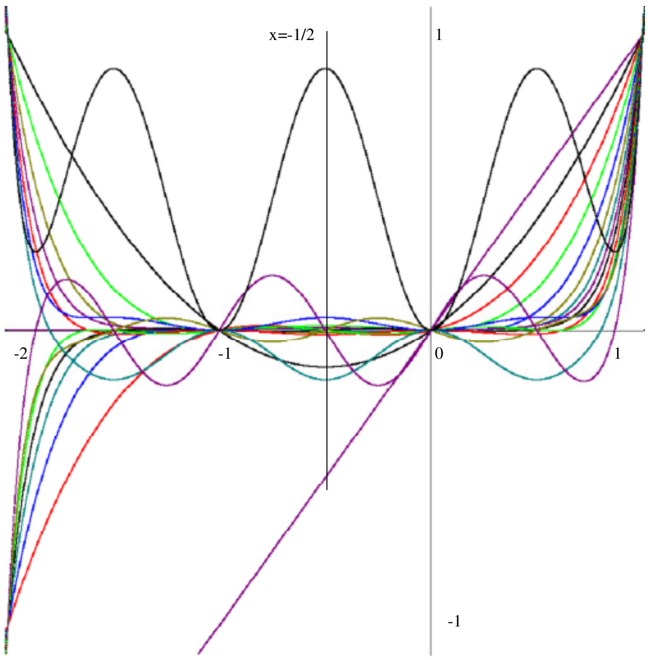
(7.5.5.) $$gp_m(x) = \sum_{k=0}^{m} \left(\binom{m}{k} \beta_{m-k} \frac{x^{k+1}}{k+1} \right)$$

where $gp_m(x)$ can be recognized as the integral of the m'th bernoulli-polynomial $b_m(x)$

Example-graphs:

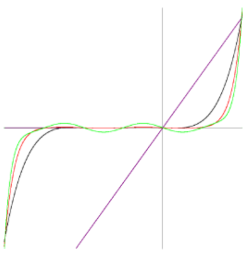
The graph shows the curves for the first 16 *gp()*-polynomials for $x=-2..+1$ $y=-1..1$.

Note the seemingly perfect symmetries about $x=-1/2$, as well as the approximation to a clear *sin()*-type curve in the area

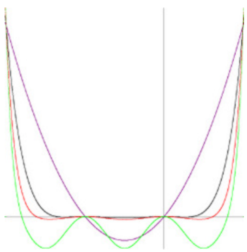


The graphs in 4 groups of gp_m, according to the 4 residuegroups of $m \pmod 4$:

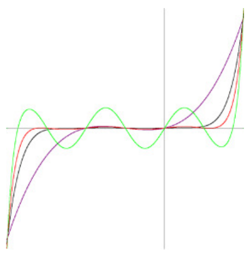
$gp_{0,4,8,12}(x)$



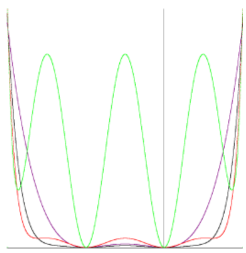
$gp_{1,5,9,13}(x)$



$gp_{2,6,10,14}(x)$



$gp_{3,7,11,15}(x)$



I don't have analyzed the zeros of the *gp*-polynomials besides heuristics so far, but the way of analysis should be similar to that of the zeros of bernoulli-polynomials, for instance in [Vaselov]

8. Citations

8.1. concerning the original finding of Bernoulli-numbers

Jakob Bernoulli in Ars Conjectandi, P.99

Wer aber diese Reihen in Bezug auf ihre Gesetzmässigkeit genauer betrachtet, kann auch ohne umständliche Rechnung die Tafel fortsetzen. Bezeichnet c den ganzzahligen Exponenten irgend einer Potenz, so ist

$$S(n^c) = \frac{1}{c+1} n^{c+1} + \tfrac{1}{2} n^c + \tfrac{1}{2}\binom{c}{1} A n^{c-1}$$

$$+ \tfrac{1}{4}\binom{c}{3} B n^{c-3} + \tfrac{1}{6}\binom{c}{5} C n^{c-5} + \tfrac{1}{8}\binom{c}{7} D n^{c-7} + \cdots,$$

wobei die Exponenten der Potenzen von n regelmässig fort um 2 abnehmen bis herab zu n oder n^2. Die Buchstaben A, B, C, D, . . . bezeichnen der Reihe nach die Coefficienten von n in den Ausdrücken für $S(n^2)$, $S(n^4)$, $S(n^6)$ $S(n^8)$, . . ., nämlich [98]

$$A = \tfrac{1}{6},\ B = -\tfrac{1}{30},\ C = \tfrac{1}{42},\ D = -\tfrac{1}{30},\ \cdots.$$

Eric Weissstein in *http://mathworld.wolfram.com/BernoulliPolynomial.html*

(Abramowitz and Stegun 1972, p. 804), first obtained by Euler (1738). The first few Bernoulli polynomials are

$$B_0(x) = 1$$
$$B_1(x) = x - \tfrac{1}{2}$$
$$B_2(x) = x^2 - x + \tfrac{1}{6}$$
$$B_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x$$
$$B_4(x) = x^4 - 2x^3 + x^2 - \tfrac{1}{30}$$
$$B_5(x) = x^5 - \tfrac{5}{2}x^4 + \tfrac{5}{3}x^3 - \tfrac{1}{6}x$$
$$B_6(x) = x^6 - 3x^5 + \tfrac{5}{2}x^4 - \tfrac{1}{2}x^2 + \tfrac{1}{42}.$$

Bernoulli (1713) defined the polynomials in terms of sums of the *powers* of consecutive integers,

$$\sum_{k=0}^{m-1} k^{n-1} = \frac{1}{n}\left[B_n(m) - B_n(0)\right].$$

Ed Sandifer in MAA-Online, "Bernoulli numbers" (September 2005)

http://www.maa.org/editorial/euler/HowEulerDidIt/Bernoullinumbers.pdf

In modern notation (Bernoulli did not use subscripts, nor did he use Σ for summations or ! for factorials) Bernoulli found that

$$\sum_{k=1}^{n-1} k^p = \sum_{k=0}^{p} \frac{B_k}{k!}\frac{p!}{(p+1-k)!} n^{p+1-k}$$

If n is large and p is small, that means that the left hand side is a sum of a relative large number of relatively small powers, and if we know the necessary Bernoulli numbers then the sum on the right is simpler to evaluate than the sum on the left. Bernoulli himself is said [G+S] to have used this formula to find the sum of the tenth powers of numbers 1 to 1000 in less than eight minutes. The answer is a 32-

8.2. Examples for generalizations of bernoulli-numbers/polynomials

Carlitz, L. "Arithmetic Properties of Generalized Bernoulli Numbers"
Journal für die reine und angewandte Mathematik Bd 202, S 174

1. In a recent paper [4], Leopoldt has defined generalized Bernoulli numbers and polynomials in the following manner. Let f be a fixed integer $\geqq 1$ and $\chi(r)$ a primitive character (mod f). Put

(1.1) $$\sum_{r=1}^{f} \chi(r) \frac{te^{rt}}{e^{ft}-1} = \sum_{n=1}^{\infty} B_\chi^n \frac{t^n}{n!},$$

(1.2) $$\sum_{r=1}^{f} \chi(r) \frac{te^{(r+x)t}}{e^{ft}-1} = \sum_{n=1}^{\infty} B_\chi^n(x) \frac{t^n}{n!},$$

so that

(1.3) $$B_\chi^n(x) = \sum_{r=0}^{n} \binom{n}{r} B_\chi^r x^{n-r} = (B_\chi + x)^n.$$

L. M. Milne-Thomson.. "Two Classes Of Generalized Polynomials" (1932)

Summary.

A certain general class of polynomials is defined and its properties are considered. By specializing the definition in one direction we are led to Bernoulli's polynomials of order n regarded as generalizations of x^ν, by specializing in another direction x^ν leads to Euler's polynomials of order n. The same methods are applied to Hermite's polynomials and thus two new types of polynomial are found.

1. *The ϕ polynomials.*

We define ϕ polynomials $\phi_\nu^{(n)}(x)$ of degree ν and order n by the relation

(1) $$f(t, n)\, e^{xt+g(t)} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \phi_\nu^{(n)}(x),$$

S.C.Woon http://arxiv.org/PS_cache/math/pdf/9812/9812143v1.pdf

Generalization of a relation between the Riemann zeta function and Bernoulli numbers

S.C. Woon ,Trinity College, University of Cambridge, Cambridge CB2 1TQ, UK

December 23, 1998

Abstract

A generalization of a well-known relation between the Riemann zeta function and Bernoulli numbers is obtained. The formula is a new representation of the Riemann zeta function in terms of a nested series of Bernoulli numbers.

Analytically extending the tree-generating operator $(O_L + O_R)$ with (17) in Lemma 1 effectively turns the sequence of B_n into a function $B(s)$ as the analytic continuation of B_n.

□

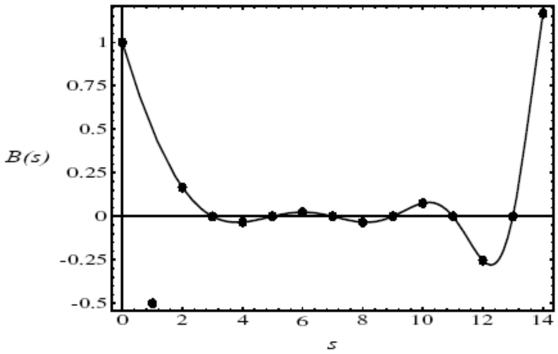


Figure 2: The curve $B(s)$ runs through the points of all (n, B_n) except $(1, B_1)$.

9. References

[Project-Index] http://go.helms-net.de/math/binomial_new/index
[Intro] http://go.helms-net.de/math/binomial_new/00_0_Intro.pdf
[List] http://go.helms-net.de/math/binomial_new/00_1_ListOfMatrices.pdf

[binomialmatrix] http://go.helms-net.de/math/binomial_new/01_1_binomialmatrix.pdf
[signed binomial] http://go.helms-net.de/math/binomial_new/01_2_signedbinomialmatrix.pdf
[Stirlingmatrix] http://go.helms-net.de/math/binomial_new/01_3_stirling.pdf
[Gaussmatrix] http://go.helms-net.de/math/binomial_new/01_5_gaussmatrix.pdf

[GenBernRec] (Generalized Bernoulli-recursion)
http://go.helms-net.de/math/binomial_new/02_2_GeneralizedBernoulliRecursion.pdf

[SumLikePow] (Sums of like powers)
http://go.helms-net.de/math/binomial_new/04_3_SummingOfLikePowers.pdf

[Hasse] http://go.helms-net.de/math/binomial_new/10_2_recihasse.pdf
[Vandermonde] http://go.helms-net.de/math/binomial_new/10_3_InverseVandermonde1.pdf

Projekt ***Bernoulli-numbers***, first versions of the above, contain a ***first rough exploratory*** course but already cover most central topics and contain also the basic material about **Gp** and **Gm** which is still missing in the above list:
[Bernoulli] http://go.helms-net.de/math/binomial_new/bernoulli_en.pdf
[Summation] http://go.helms-net.de/math/binomial_new/pmatrix.pdf

---

[MWBern] Bernoulli-Polynomials Eric Weissstein, mathworld.com
http://mathworld.wolfram.com/BernoulliPolynomial.html

[Matexp] Matrixexponential Alan Edelman & Gilbert Strang, MIT
http://web.mit.edu/18.06/www/pascal-work.pdf

[Toeplitzmatrix] Toeplitz-matrices Wikipedia
http://en.wikipedia.org/wiki/Toeplitz_matrix

[SebGou] Introduction on Bernoulli's numbers Pascal Sebah /Xavier Gourdon
http://numbers.computation.free.fr/Constants/Miscellaneous/bernoulli.html

[Sandifer] How Euler Did It: Bernoulli-numbers Ed Sandifer
http://www.maa.org/editorial/euler/HowEulerDidIt/Bernoullinumbers.pdf

[Luschny] "Sind die Bernoulli-Zahlen falsch definiert" Peter Luschny/ Hermann Kremer
http://www.dsmath.de/archiv/zahlen/BernoulliEuler.pdf

[Kellner] Homepage Bernoulli-org Bernd Kellner
http://www.bernoulli.org

[Woon] Generalization of a relation between the Riemann zeta function S.C.Woon
http://arxiv.org/PS_cache/math/pdf/9812/9812143v1.pdf

[DilBiblio] Bernoulli-bibliography Karl Dilcher
http://www.mscs.dal.ca/~dilcher/bernoulli.html

Further readings:

Zeta-Relation, Basic identities

A succinct method for investigating the sums of infinite series through differential formulae
Leonard Euler (translation J. Bell, 05'2007)
http://arxiv.org/abs/0705.0768v1 (arXiv:0705.0768v1)
(original:) Opera mathematica, Volume 16.
see: Euler Archive at http://www.eulerarchive.org.
This paper is E746 in the Eneström index *(bibliography taken from arXiv-source)*

Ein Summierungsverfahren für die Riemannsche Zeta-Reihe
Helmut Hasse
Mathematische Zeitschrift PERIODICAL VOLUME 32 PAGE 456
http://www-gdz.sub.uni-goettingen.de/cgi-bin/digbib.cgi?PPN266833020_0032

Generalizations

Faulhaber's Theorem for Arithmetic Progressions
William Y.C. Chen, Amy M. Fu, and Iris F. Zhang
http://arxiv.org/PS_cache/math/pdf/0606/0606090v1.pdf

From the bibliography of this article

[2] G. Dattoli, C. Cesarano and S. Lorenzutta, Bernoulli numbers and polynomials from a more general point of view,
Rendiconti di Mathematica, Serie VII, 22 (2002), 193–202.
[3] A. W. F. Edwards, A quick route to sums of powers,
Amer. Math. Monthly,93 (1986), 451–455.
[5] M. D. Hirschhorn, Evaluating $\Sigma^N{}_{n=1}(a + n\ d)^p$, Math. Gaz., to appear.
[7] D. E. Knuth, Johann Faulhaber and sums of powers, Math. Comput., 61 (1993), 277–294.

Generalizations of Bernoulli numbers and polynomials
Qiu-Ming Luo, Bai-Ni Guo, Feng Qi, and Lokenath Debnath
http://www.hindawi.com/GetPDF.aspx?doi=10.1155/S0161171203112070
International Journal of Mathematics and Mathematical Sciences, vol. 2003, no. 59, pp. 3769-3776, 2003

Inversion of (I+P) / Eta-Matrix

Explicit inverse of the Pascal matrix plus one,
Sheng-liang Yang and Zhong-kui Liu
http://www.hindawi.com/GetPDF.aspx?doi=10.1155/IJMMS/2006/90901
International Journal of Mathematics and Mathematical Sciences Volume 2006 (2006)

From the bibliography of this article

[1] L. Aceto and D. Trigiante, The matrices of Pascal and other greats,
The American Mathematical Monthly 108 (2001), no. 3, 232–245.
[2] R. Aggarwala and M. P. Lamoureux, Inverting the Pascal matrix plus one,
The American Mathematical Monthly 109 (2002), no. 4, 371–377.
[4] R. Brawer and M. Pirovino, The linear algebra of the Pascal matrix,
Linear Algebra and Its Applications 174 (1992), 13–23.
[5] G. S. Call andD. J.Velleman, Pascal'smatrices,
The AmericanMathematicalMonthly 100 (1993), no. 4, 372–376.
[6] G.-S. Cheon, A note on the Bernoulli and Euler polynomials,
Applied Mathematics Letters 16 (2003), no. 3, 365–368.
[9] H. M. Srivastava and A´ . Pinte´r, Remarks on some relationships between the Bernoulli and Euler polynomials,
AppliedMathematics Letters 17 (2004), no. 4, 375–380.

Matrix approach

Matrix approach to polynomials 2
Teijo Arponen *14th July 2004*

From the bibliography of this article

[Arp03] T. Arponen. A matrix approach to polynomials.
Lin. Alg. Appl., 359:181-196,2003.
[AT01] L. Aceto and D. Trigiante. The matrices of Pascal and other greats.
Amer.Math. Mon., 108(3):232-245, 2001.
[BP92] R. Brawer and M. Pirovino. The linear algebra of the Pascal matrix.
Lin. Alg. Appl., 174:13-23, 1992.
[CK01] G.-S. Cheon and J.-S. Kim. Stirling matrix via Pascal matrix.

Lin. Alg. Appl.,329:49-59, 2001.
[CV93] G.S. Call and D.J. Velleman. Pascal's matrices.
Amer. Math. Mon., 100(4):372-376, 1993.
[Kal83] D. Kalman. Polynomial translation groups.
Math. Mag., 56(1):23-25, 1983.
[KU87] D. Kalman and A. Ungar.
Combinatorial and functional identities in oneparameter matrices.
Amer. Math. Mon., 94(1):21-35, 1987.
[MS58] N. Macon and A. Spitzbart. Inverses of vandermonde matrices.
Amer. Math. Mon., 65(2):95-100, 1958.

Eigensystems

Invariant sequences under binomial transformation.
Sun, Zhi-Hong
Fibonacci Q. 39, No.4, 324-333 (2001). [ISSN 0015-0517]
http://www.sdstate.edu/ wcsc/http/fibhome.html
(taken from link in: Zbl. Math. 0987.05013)

Bernoulli-polynomials, special analyses

On the real roots of the Bernoulli polynomials and the Hurwitz zeta-function
Vaselov, A.P. ; Ward, J.P. (2002)
http://arxiv.org/abs/math/0205183

Gottfried Helms, 25. Nov. 12