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10-1 Does for any r and n hold: $\sum_{k=1..n} k^r = (n+1)^r$ an Erdös-problem

Abstract: For this problem no solution is known. Here I present an approach, which gives a stong hint for the impossibility of the identity, and possibly by further analysis from here proof may be derived

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For an intro about the conventions of notation and naming of basic-matrices see [intro] http://go.helms-net.de/math/binomial/00_0_intro.pdf

Gottfried Helms Version: 05.04.2007 16:43

1. Definitions/ Identities

1.1. Representation for the sum

Sums of like powers were considered by Jakob Bernoulli, who found the coefficients β (later named after him), with which such sums can be expressed as polynomials.

$$\sum_{k=1}^{n} k^{r} = \sum_{i=0}^{r} \frac{\beta_{r-j}}{j+1} {r \choose j} n^{r+1}$$
 // where $\beta_{1} = +1/2$

Example:

$$\begin{aligned} 1 + 2^4 + 3^4 + 4^4 + 5^4 &= 5*\left(\frac{\beta_4}{1}l + \frac{\beta_3}{2}4*5 + \frac{\beta_2}{3}6*5^2 + \frac{\beta_1}{4}4*5^3 + \frac{\beta_0}{5}l*5^4\right) \\ &= 5*\left(-\frac{l}{30}l + \frac{0}{2}4*5 + \frac{l}{3*6}6*5^2 + \frac{l}{8}4*5^3 + \frac{l}{5}l*5^4\right) \\ &= 5*\left(-\frac{l}{30} + \frac{l}{3}*5^2 + \frac{l}{2}*5^3 + \frac{l}{5}*5^4\right) \\ &= \left(-\frac{l}{6} + \frac{l}{3}5^3 + \frac{3}{2}5^4\right) = \frac{l}{6}(-l+125(2+9*5)) \\ &= \frac{l}{6}(-l+125*47) = \frac{l}{6}(-48+126*47) = -8+2l*47 = 979 \end{aligned}$$

The coefficients that Jakob Bernoulli found, can be represented in a matrix G_p , and the summation can then be expressed as a matrixmultiplication

$$G_p * n V(n) = V(1) + V(2) + ... + V(n)$$

Example

(1.1.1.)
$$Gp * n V(n) = SU(n)$$

$$\begin{bmatrix}
1 & . & . & . & . & . \\
1/2 & 1/2 & . & . & . & . \\
1/6 & 1/2 & 1/3 & . & . & . \\
0 & 1/4 & 1/2 & 1/4 & . & . \\
-1/30 & 0 & 1/3 & 1/2 & 1/5 & . \\
0 & -1/12 & 0 & 5/12 & 1/2 & 1/6 & = \begin{bmatrix}
5 \\ 15 \\ 55 \\ 225 \\ 979 \\ 4425 \end{bmatrix} = \begin{bmatrix}
1^{0} & +2^{0} & +3^{0} & +4^{0} & +5^{0} \\
1^{0} & +2^{0} & +3^{0} & +4^{0} & +5^{0} \\
1^{0} & +2^{0} & +3^{0} & +4^{0} & +5^{0} \\
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1^{0} & +2^{0} & +3^{0} & +4^{0} & +4^{0} & +4^{0} & +4^{0} \\
1^{0} & +2^{0} & +3^{0} & +4^{0} & +4^{0} & +4^{0} & +4^{0} & +4^{0} & +4^{0} & +4^{0} \\
1^{0} & +2^{0} & +4^{0} & +4^{0} & +4^{0} & +$$

where the sum for the exponent r is in the r'th row of the result.

1.2. Representation for the r'th power of n+1

On the other hand, the binomial-matrix P, rightmultiplied with a vector V(n) transforms this vector into V(n+1):

Example

(1.2.1.)
$$P * V(n) = V(n+1)$$

$$\begin{bmatrix}
1 & 5 \\ 25 \\ 125 \\ 625 \\ 3125
\end{bmatrix}$$

$$\begin{bmatrix}
1 & . & . & . & . & . \\
1 & 1 & . & . & . & . \\
1 & 2 & 1 & . & . & . \\
1 & 3 & 3 & 1 & . & . \\
1 & 4 & 6 & 4 & 1 & . \\
1 & 5 & 10 & 10 & 5 & 1
\end{bmatrix} = \begin{bmatrix}
1 \\ 6 \\ 36 \\ 216 \\ 1296 \\ 7776\end{bmatrix}$$

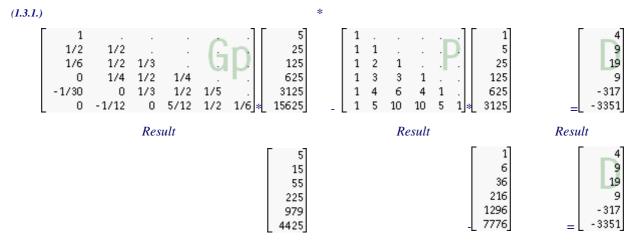
1.3. Representation of the Erdös-Problem

The Erdös-problem, for any power r can be written as the question, whether in the differencevector D:

$$G_p * n V(n) - P V(n) = D$$

in any row r a zero can occur for any n.

Example



In no row in D is a difference $d_r = 0$, but the nature of the entries is not clear in simpler description than that from one row r the entries are increasing negative, and an expected zero need to occur in an early row

To see an example, I use the matrix $\mathbf{Z}\mathbf{V}$ as complete set of powerseries of V(1) to V(n), and the matrix ${}_{n}\mathbf{Z}\mathbf{V}$ as complete set of powerseries 1*V(1) to n*V(n):

Example:

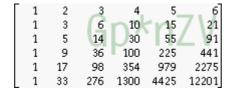
1	2	3	4	5_	6
1	4	9	16	25	36
1	8	27	64	125	216
1	16	81	256	625	1296
1	32	243	1024	3125	7776
1	64	729	4096	15625	46656

1	1	1	1	1	1
1	2	3	4	5	/ \ 6
1	4	9	16	25	36
1	8	27	64	125	216
1	16	81	256	625	1296
_ 1	32	243	1024	3125	7776

Using such a complete matrix instead of single powerseries vectors V(n) and n*V(n) also the result is a complete set of **D**-vectors, call it **DV**; and we have

$$G_p * nZV - P * ZV = DV$$

Example



1	1	1	1	. 1	1
2	3	4	5) 1/6	/\7
4	9	16	25	36	49
8	27	64	125	216	343
16	81	256	625	1296	2401
32	243	1024	3125	7776	16807

and the difference **DV**:

Now the question is: does a zero occur in DV other than in the top left 2x2-matrix?

$$Gp * ZV* dZ(-1) - P * ZV = DV$$

$$Gp * ZV* dZ(-1)*ZV^{I} - P = DV * ZV^{-I} = M$$

M is nearly triangular:

Example:

- 1	1	0	0	0	0
-2	- 1	1	0	0.	0
-6	- 11	- 3	2	0.	0
- 24	-72	-66	-12	6	0
-120	-484	-720	-440	-60	24
-720	-3600	-7260	-7200	-3300	- 360

and we can ask

does a zero in a row/column of D occur by the following matrix-multiplication for r,c>2: M*ZV=D

The rows r of M give the coefficients for polynomials in x, $f_r(n)$ if postmultiplied by the powerseriesvector V(n); so this is equivalent to the question

does some positive integer n exist, so that the polynomial $f_r(n)$, constructed by the coefficients of row r in M, is zero?

Since M is not really triangular in this example, let for the following then row-index r for M begin at 1 to have consistency with the polynomial order.

1.4. Definition of the coefficients in M

```
\begin{split} M &= matrix(9,9,r,c, & if(c == 1,-1,if(r >= c, -P[r-1,c] + Gp[r-1,c-1]))) \\ M &= matrix(9,9,r,c,(r-1)!*if(c == 1,-1,if(r >= c, -P[r-1,c] + Gp[r-1,c-1]))) \end{split}
```

1.5. Formal Irreducibility-approach

The first approach would be to see and check, whether we can derive, that the roots of fr(n) are integer or not by application of criteria like the Eisentein-criterion for the first several rows in M / the according polynomials fr(n). For this some rescalings may be useful, for instance to represent monic polynomials:

1.6. An approach using properties of observed real roots of the polynomials

A table for real roots from k=1 to 63 are given in

http://go.helms-net.de/math/divers/ZerosOfGpFunctions.htm

1.7. Further investigations

First we add a leading row, to get a triangular matrix

2. References

[Project-Index] http://go.helms-net.de/math/binomial/index

[Intro] <u>http://go.helms-net.de/math/binomial/intro.pdf</u>

[binomialmatrix] http://go.helms-net.de/math/binomial/01 1 binomialmatrix.pdf

[signed binomial] http://go.helms-net.de/math/binomial/01_2_signedbinomialmatrix.pdf

[Gaussmatrix] http://go.helms-net.de/math/binomial/04_1_gaussmatrix.pdf
[Stirlingmatrix] http://go.helms-net.de/math/binomial/05_1_stirlingmatrix.pdf

[Hasse] http://go.helms-net.de/math/binomial/01 x recihasse.pdf

[A066325] http://www.research.att.com/~njas/sequences/A066325

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