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10-1 Does for any *r* and *n* hold: $\Sigma_{k=1..n} k^r = (n+1)^r$ an Erdös-problem

Abstract: For this problem no solution is known. Here I present an approach, which gives a stong hint for the impossibility of the identity, and possibly by further analysis from here proof may be derived

Contents:

For an intro about the conventions of notation and naming of basic-matrices see [intro]
http://go.helms-net.de/math/binomial/00_0_intro.pdf

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1. Definitions/ Identities

1.1. Representation for the sum

Sums of like powers were considered by Jakob Bernoulli, who found the coefficients β (later named after him), with which such sums can be expressed as polynomials.

$$\sum_{k=1}^{n} k^r = \sum_{j=0}^{r} \frac{\beta_{r-j}}{j+1}\binom{r}{j} n^{r+1} \qquad \text{// where } \beta_1 = +1/2$$

Example:

$$\begin{aligned}
1+2^4+3^4+4^4+5^4 &= 5*\left(\frac{\beta_4}{1}1+\frac{\beta_3}{2}4*5+\frac{\beta_2}{3}6*5^2+\frac{\beta_1}{4}4*5^3+\frac{\beta_0}{5}1*5^4\right)\\
&= 5*\left(-\frac{1}{30}1+\frac{0}{2}4*5+\frac{1}{3*6}6*5^2+\frac{1}{8}4*5^3+\frac{1}{5}1*5^4\right)\\
&= 5*\left(-\frac{1}{30}+\frac{1}{3}*5^2+\frac{1}{2}*5^3+\frac{1}{5}*5^4\right)\\
&= \left(-\frac{1}{6}+\frac{1}{3}5^3+\frac{3}{2}5^4\right)=\frac{1}{6}(-1+125(2+9*5))\\
979 &= \frac{1}{6}(-1+125*47)=\frac{1}{6}(-48+126*47)=-8+21*47=979
\end{aligned}$$

The coefficients that Jakob Bernoulli found, can be represented in a matrix G_p, and the summation can then be expressed as a matrixmultiplication

$$G_p * n\, V(n) = V(1)+V(2)+...+V(n)$$

Example

(1.1.1.) $\quad Gp * n\, V(n) = SU(n)$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1/2 & 1/2 & . & . & . & . \\ 1/6 & 1/2 & 1/3 & . & . & . \\ 0 & 1/4 & 1/2 & 1/4 & . & . \\ -1/30 & 0 & 1/3 & 1/2 & 1/5 & . \\ 0 & -1/12 & 0 & 5/12 & 1/2 & 1/6 \end{bmatrix} * \begin{bmatrix} 5 \\ 25 \\ 125 \\ 625 \\ 3125 \\ 15625 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \\ 55 \\ 225 \\ 979 \\ 4425 \end{bmatrix} = \begin{bmatrix} 1\text{^}0 & +2\text{^}0 & +3\text{^}0 & +4\text{^}0 & +5\text{^}0 \\ 1\text{^}1 & +2\text{^}1 & +3\text{^}1 & +4\text{^}1 & +5\text{^}1 \\ 1\text{^}2 & +2\text{^}2 & +3\text{^}2 & +4\text{^}2 & +5\text{^}2 \\ 1\text{^}3 & +2\text{^}3 & +3\text{^}3 & +4\text{^}3 & +5\text{^}3 \\ 1\text{^}4 & +2\text{^}4 & +3\text{^}4 & +4\text{^}4 & +5\text{^}4 \\ 1\text{^}5 & +2\text{^}5 & +3\text{^}5 & +4\text{^}5 & +5\text{^}5 \end{bmatrix}$$

where the sum for the exponent r is in the r'th row of the result.

1.2. Representation for the r'th power of n+1

On the other hand, the binomial-matrix ***P***, rightmultiplied with a vector ***V****(n)* transforms this vector into ***V****(n+1)*:

Example

(1.2.1.) $P * V(n) = V(n+1)$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix} * \begin{bmatrix} 1 \\ 5 \\ 25 \\ 125 \\ 625 \\ 3125 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 36 \\ 216 \\ 1296 \\ 7776 \end{bmatrix}$$

1.3. Representation of the Erdös-Problem

The Erdös-problem, for any power r can be written as the question, whether in the differencevector ***D***:

$$G_p * n\ V(n) - P\ V(n) = D$$

in any row r a zero can occur for any n.

Example

(1.3.1.)

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1/2 & 1/2 & . & . & . & . \\ 1/6 & 1/2 & 1/3 & . & . & . \\ 0 & 1/4 & 1/2 & 1/4 & . & . \\ -1/30 & 0 & 1/3 & 1/2 & 1/5 & . \\ 0 & -1/12 & 0 & 5/12 & 1/2 & 1/6 \end{bmatrix} * \begin{bmatrix} 5 \\ 25 \\ 125 \\ 625 \\ 3125 \\ 15625 \end{bmatrix} - \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix} * \begin{bmatrix} 1 \\ 5 \\ 25 \\ 125 \\ 625 \\ 3125 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 19 \\ 9 \\ -317 \\ -3351 \end{bmatrix}$$

Result *Result* *Result*

$$\begin{bmatrix} 5 \\ 15 \\ 55 \\ 225 \\ 979 \\ 4425 \end{bmatrix} - \begin{bmatrix} 1 \\ 6 \\ 36 \\ 216 \\ 1296 \\ 7776 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 19 \\ 9 \\ -317 \\ -3351 \end{bmatrix}$$

In no row in ***D*** is a difference $d_r = 0$, but the nature of the entries is not clear in simpler description than that from one row r the entries are increasing negative, and an expected zero need to occur in an early row.

To see an example, I use the matrix ***ZV*** as complete set of powerseries of ***V****(1)* to ***V****(n)*, and the matrix $_n$***ZV*** as complete set of powerseries *1****V****(1) to n****V****(n)*:

Example:

nZV

1	2	3	4	5	6
1	4	9	16	25	36
1	8	27	64	125	216
1	16	81	256	625	1296
1	32	243	1024	3125	7776
1	64	729	4096	15625	46656

ZV

1	1	1	1	1	1
1	2	3	4	5	6
1	4	9	16	25	36
1	8	27	64	125	216
1	16	81	256	625	1296
1	32	243	1024	3125	7776

Using such a complete matrix instead of single powerseries vectors ***V****(n)* and *n*****V****(n)* also the result is a complete set of ***D***-vectors, call it ***DV***; and we have

$$G_p * nZV - P * ZV = DV$$

Example

Gp*nZV

1	2	3	4	5	6
1	3	6	10	15	21
1	5	14	30	55	91
1	9	36	100	225	441
1	17	98	354	979	2275
1	33	276	1300	4425	12201

P*ZV

1	1	1	1	1	1
2	3	4	5	6	7
4	9	16	25	36	49
8	27	64	125	216	343
16	81	256	625	1296	2401
32	243	1024	3125	7776	16807

and the difference ***DV***:

(1.3.2.)

DV

0	1	2	3	4	5
-1	0	2	5	9	14
-3	-4	-2	5	19	42
-7	-18	-28	-25	9	98
-15	-64	-158	-271	-317	-126
-31	-210	-748	-1825	-3351	-4606

Now the question is: does a zero occur in ***DV*** other than in the top left *2x2*-matrix?

$$Gp * ZV * dZ(-1) - P * ZV = DV$$
$$Gp * ZV * dZ(-1) * ZV^{-1} - P = DV * ZV^{-1} = M$$

M is nearly triangular:

Example:

(1.3.3.)

M1

-1	1	0	0	0	0
-2	-1	1	0	0	0
-6	-11	-3	2	0	0
-24	-72	-66	-12	6	0
-120	-484	-720	-440	-60	24
-720	-3600	-7260	-7200	-3300	-360

and we can ask

does a zero in a row/column of D occur by the following matrix-multiplication for r,c>2:
$$M * ZV = D$$

The rows *r* of ***M*** give the coefficients for polynomials in *x*, $f_r(n)$ if postmultiplied by the powerseriesvector ***V****(n)*; so this is equivalent to the question

does some positive integer n exist, so that the polynomial $f_r(n)$ *, constructed by the coefficients of row r in M, is zero?*

Since ***M*** is not really triangular in this example, let for the following then row-index *r* for ***M*** begin at 1 to have consistency with the polynomial order.

1.4. Definition of the coefficients in M

M=matrix(9,9,r,c,   if(c==1,-1,if(r>=c, -P[r-1,c] + Gp[r-1,c-1])))
*M=matrix(9,9,r,c,(r-1)!*if(c==1,-1,if(r>=c, -P[r-1,c] + Gp[r-1,c-1])))*

1.5. Formal Irreducibility-approach

The first approach would be to see and check, whether we can derive, that the roots of fr(n) are integer or not by application of criteria like the Eisentein-criterion for the first several rows in M / the according polynomials fr(n). For this some rescalings may be useful, for instance to represent monic polynomials:

(1.5.1.) *M =*

-1	1	0	0	0	0
-2	-1	1	0	0	0
-3	-11/2	-3/2	1	0	0
-4	-12	-11	-2	1	0
-5	-121/6	-30	-55/3	-5/2	1
-6	-30	-121/2	-60	-55/2	-3

1.6. An approach using properties of observed real roots of the polynomials

A table for real roots from *k=1 to 63* are given in

http://go.helms-net.de/math/divers/ZerosOfGpFunctions.htm

1.7. Further investigations

First we add a leading row, to get a triangular matrix

M

1	.	.	.	.	.
-1	1	.	.	.	.
-2	-1	1	.	.	.
-6	-11	-3	2	.	.
-24	-72	-66	-12	6	.
-120	-484	-720	-440	-60	24

FtmpM

1	.	.	.	.	.
-1	1	.	.	.	.
-1	-1/2	1/2	.	.	.
-1	-11/6	-1/2	1/3	.	.
-1	-3	-11/4	-1/2	1/4	.
-1	-121/30	-6	-11/3	-1/2	1/5

2. References

[Project-Index] http://go.helms-net.de/math/binomial/index

[Intro] http://go.helms-net.de/math/binomial/intro.pdf

[binomialmatrix] http://go.helms-net.de/math/binomial/01_1_binomialmatrix.pdf

[signed binomial] http://go.helms-net.de/math/binomial/01_2_signedbinomialmatrix.pdf

[Gaussmatrix] http://go.helms-net.de/math/binomial/04_1_gaussmatrix.pdf

[Stirlingmatrix] http://go.helms-net.de/math/binomial/05_1_stirlingmatrix.pdf

[Hasse] http://go.helms-net.de/math/binomial/01_x_recihasse.pdf

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[A066325] http://www.research.att.com/~njas/sequences/A066325

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