Identities involving binomial-coefficients,
Bernoulli- and Stirlingnumbers

Mathematical Miniatures

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A reciprocal variation of the Hasse-Zeta-summation

Abstract: The following chapter is just a simple attempt to explore the relation, that H HASSE gave in his article "a special summing procedure for zeta-series" in another direction and to see, whether one can find analoguous properties with a more general approach - without any guarantee, that the findings are more than trivialities and reformulations. The following is just exploration, and no special useful or thrilling consequences of the here shown relations are found so far.

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1. Intro

1.1. The HASSE-scheme: zeta-values = Bernoulli-numbers via binomial-matrix.

In his article of 1930 H.HASSE gave a scheme for the summation of zeta-series of negative exponents, which leads to the representation of negative-exponent zeta-values by Bernoulli-numbers.

His introductory observation was (I restate everything in matrix-notation here):

$$Z(1)\sim * Pj * ZV = B\sim$$

where $\mathbf{B}\sim$ is the row-vector of Bernoulli-numbers $\mathbf{B}\sim = [\beta_0, \beta_1, \beta_2, \ldots]$, using $\beta_1 = +1/2$.

This has the meaning, that the columns of ***ZV***, which represent the *zeta*-series with *zero* or negative-integer exponents, added by the column-signed version of the binomial-matrix and then added by the *zeta*-series with exponent *1* are just giving the Bernoulli-numbers in the resulting vector ***B***.

First he states, that

$$Pj * ZV = \Delta$$

with Δ a triangular matrix, allowing to express the zeta-summation later using finitely many terms only:

$$Pj * ZV = \Delta \quad (= \text{"Delta"})$$

$$ZV = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 \\ 1 & 3 & 9 & 27 & 81 & 243 \\ 1 & 4 & 16 & 64 & 256 & 1024 \\ 1 & 5 & 25 & 125 & 625 & 3125 \\ 1 & 6 & 36 & 216 & 1296 & 7776 \end{bmatrix}$$

$$Pj = \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & -1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ 1 & -3 & 3 & -1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix} \qquad = \qquad \text{Delta}\sim = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & -1 & -3 & -7 & -15 & -31 \\ . & . & 2 & 12 & 50 & 180 \\ . & . & . & -6 & -60 & -390 \\ . & . & . & . & 24 & 360 \\ . & . & . & . & . & -120 \end{bmatrix}$$

and then summing/leftmultiplicating of *Delta* (Δ) by the *zeta(1)*-series:, giving the Bernoulli-numbers:

$$Z(1)\sim * \Delta = B$$

$$\text{Delta}\sim = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & -1 & -3 & -7 & -15 & -31 \\ . & . & 2 & 12 & 50 & 180 \\ . & . & . & -6 & -60 & -390 \\ . & . & . & . & 24 & 360 \\ . & . & . & . & . & -120 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 1/6 & 0 & -1/30 & 0 \end{bmatrix}$$

He first shows, that these identities hold generally for all negative integer exponents of the zeta-series and related indexes of the Bernoulli-numbers, thus providing an independent proof for the identity

$$\beta_n / n = - \zeta(1-n) \qquad // \text{for } n>0$$

The more interesting result is, that instead of using only a column of ***ZV***, (which represents a zeta-series with negative integer exponent -n) the same identity holds for any zeta-series with complex exponent *s*. in terms of this matrix-notation:

$$Z(1)\sim * Pj * Z(s) = -s * \zeta(1-s)$$

A detail of this scheme of H.HASSE is, that it gives for the indeterminate expression of $\zeta(1)$ *infinitiy* multiplied by *zero* the evaluation $\zeta(1) * 0 = 1 = \beta_0$

1.2. Another proof of the HASSE-Identity

The following proof is very simple and its only possible disadvantage is, that it depends on another proof, which concerns the composition of the G_p-matrix from the stirling-numbers, (but may be given elsewhere):

$$G_p = St2 * {}^dZ(1) * St2^{-1} = St2 * {}^dZ(1) * St1$$

With the above identity the proof goes as follows:

Theorem (HASSE):

$$Z(1)\sim * Pj * ZV = B \sim$$

Proof:

To begin with that proof, observe, that the *delta*-matrix Δ is a simple row-scaled version of the transpose of the *stirling- kind 2*-matrix ***St2*** (which was discussed in another chapter). Precisely it is

$$J * {}^dFac(1) * St2 \sim = \Delta$$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ . & -1 & . & . & . & . \\ . & . & 1 & . & . & . \\ . & . & . & -1 & . & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & -1 \end{bmatrix}_{J} * \begin{bmatrix} 1 & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & 2 & . & . & . \\ . & . & . & 6 & . & . \\ . & . & . & . & 24 & . \\ . & . & . & . & . & 120 \end{bmatrix}_{n!} * \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 3 & 7 & 15 & 31 \\ . & . & 1 & 6 & 25 & 90 \\ . & . & . & 1 & 10 & 65 \\ . & . & . & . & 1 & 15 \\ . & . & . & . & . & 1 \end{bmatrix}_{St2\sim}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & -1 & -3 & -7 & -15 & -31 \\ . & . & 2 & 12 & 50 & 180 \\ . & . & . & -6 & -60 & -390 \\ . & . & . & . & 24 & 360 \\ . & . & . & . & . & -120 \end{bmatrix}_{Delta}$$

The leftmost zeta-vector can be rewritten as a product

$$Z(1)\sim = E\sim * {}^dZ(1)$$

and we have:

$$E\sim * {}^dZ(1) * Pj * ZV = B \sim$$

The term ***Pj*** * ***ZV*** is equal to a scaled-version of ***St2~*** :

$$Pj * ZV = J * {}^dFac(1) * St2 \sim$$

so we have

$$E\sim * {}^dZ(1) * J * {}^dFac(1) * St2 \sim = B \sim$$

First rewrite this in transposed form to use the more common expressions in this series of articles:

$$St2 * {}^dFac(1) * J * {}^dZ(1) * E = B$$

Now using the Stirling-matrix of first-kind ***St1*** and the fact, that its first column is the sequence of alternating-signed factorials, we can rewrite

$$St1[,0] = J * {}^dFac(1) * E$$
$${}^dFac(-1)*J * St1[,0] = E$$

We now use the fact, that the first column of ***Gp*** is just the vector of Bernoulli-numbers and thus equals ***B***:

$$Gp[,0] = B$$

The latter two identities inserted in the previous:

$$St2 * {}^dFac(1) * J * {}^dZ(1) * {}^dFac(-1)* J * St1[,0] = Gp[,0]$$

Now the product of the middle terms of diagonal-matrices cancel out:

*St2 * ^{d}Fac(1) * J * ^{d}Z(1) * ^{d}Fac(-1)*J * St1[,0] = Gp[,0]* // cancelling J*J = I
*St2 * ^{d}Fac(1) * ^{d}Z(1) * ^{d}Fac(-1) * St1[,0] = Gp[,0]* // cancelling ^{d}Fac(1)* ^{d}Fac(-1) = I
*St2 * ^{d}Z(1) * St1[,0] = Gp[,0]*

and we get the description of the first column of ***G**p* in terms of its eigensystem, which was assumed to be true:

*St2 * ^{d}Z(1) * St1 = Gp*

Thus the HASSE-identity can be extended to:

*St1~ * ^{d}Fac(-1)* J * ^{d}Z(1) * J * ^{d}Fac(1) * St2~ = Gp ~*
*St1~ * ^{d}Fac(-1)* J * ^{d}Z(1) * Pj * ZV = Gp ~*

where the HASSE-identity is expressed by the *first row* of this matrix-equation only.

*(St1~ * ^{d}Fac(-1)* J)[0,] * ^{d}Z(1) * Pj * ZV = (Gp ~)[0,]*
*E~ * ^{d}Z(1) * Pj * ZV = B~*
*Z(1)~ * Pj * ZV = B~*

The HASSE-equation for the *zeta*-value for the negative integer exponents *-m* , meaning $\zeta(-m)$, written in the conventional summation-formula :

$$\sum_{n=0}^{oo} \frac{1}{n+1} \sum_{k=0}^{oo} (-1)^k \binom{n}{k} k^m = \beta_m = -m\zeta(1-m)$$

can thus be extended to a set of identities for *m* and *r*

$$\sum_{n=r}^{oo} \begin{bmatrix} n \\ r \end{bmatrix} \frac{(-1)^n}{n!} \frac{1}{n+1} \sum_{k=0}^{oo} (-1)^k \binom{n}{k} k^m = \frac{\beta_{m-r}}{r+1} \binom{m}{r}$$

$$\sum_{n=r}^{oo} \sum_{k=0}^{oo} \frac{(-1)^{k+n} k^m}{(n+1)!} \begin{bmatrix} n \\ r \end{bmatrix} \binom{n}{k} = \frac{\beta_{m-r}}{r+1} \binom{m}{r}$$

where *[n,r]* indicates the Stirling-number first kind (using the definition in mathworld).

2. The ZV-reciprocal experiment

2.1. Finding a "Binomial/Stirling"-pair for the reciprocal ZV-matrix

The occurence of the triangularity of the above delta-matrix Δ was triggering me, to try, whether one could generate the same scheme with the "right" zeta-series too, that means, that series with the positive exponents for their terms.

If we define a matrix ***ZVr***, which contains the reciprocals of the entries of ***ZV***:

$$ZVr = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1/2 & 1/4 & 1/8 & 1/16 & 1/32 \\ 1 & 1/3 & 1/9 & 1/27 & 1/81 & 1/243 \\ 1 & 1/4 & 1/16 & 1/64 & 1/256 & 1/1024 \\ 1 & 1/5 & 1/25 & 1/125 & 1/625 & 1/3125 \\ 1 & 1/6 & 1/36 & 1/216 & 1/1296 & 1/7776 \end{bmatrix}$$

then it is possible to define a pair of matrices ***PR*** and ***DR*** which both are triangular and reflect an analoguus difference-scheme, as H.HASSE had it introduced:

$PR * ZVr = DR$

$$ZV = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1/2 & 1/4 & 1/8 & 1/16 & 1/32 \\ 1 & 1/3 & 1/9 & 1/27 & 1/81 & 1/243 \\ 1 & 1/4 & 1/16 & 1/64 & 1/256 & 1/1024 \\ 1 & 1/5 & 1/25 & 1/125 & 1/625 & 1/3125 \\ 1 & 1/6 & 1/36 & 1/216 & 1/1296 & 1/7776 \end{bmatrix}$$

$$PR = \begin{bmatrix} 1 & . & . & . & . & . \\ 2 & -2 & . & . & . & . \\ 3 & -12 & 9 & . & . & . \\ 4 & -48 & 108 & -64 & . & . \\ 5 & -160 & 810 & -1280 & 625 & . \\ 6 & -480 & 4860 & -15360 & 18750 & -7776 \end{bmatrix} \qquad DR = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 3/2 & 7/4 & 15/8 & 31/16 \\ . & . & 1 & 11/6 & 85/36 & 575/216 \\ . & . & . & 1 & 25/12 & 415/144 \\ . & . & . & . & 1 & 137/60 \\ . & . & . & . & . & 1 \end{bmatrix}$$

(note, that there are infinitely many scaled versions of this solution)

Now, to have meaningful zeta-values in a new result-row like

$W\sim * DR = Zet(2)$

$$DR = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 3/2 & 7/4 & 15/8 & 31/16 \\ . & . & 1 & 11/6 & 85/36 & 575/216 \\ . & . & . & 1 & 25/12 & 415/144 \\ . & . & . & . & 1 & 137/60 \\ . & . & . & . & . & 1 \end{bmatrix}$$

$$\begin{bmatrix} w0 & w1 & w2 & w3 & w4 & w5 \end{bmatrix} \quad \begin{bmatrix} z(2) & z(3) & z(4) & z(5) & z(6) & z(7) \end{bmatrix}$$

it is obvious, that the w_k-coefficients must be rational compositions of zeta-values, actually for instance $w_0 = \zeta(2)$.

The complete composition of ***W*** results from multiplying $W\sim = Zet(2)\sim * DR^{-1}$

$W\sim = Zet(2)\sim * DR^{-1}$

DR^-1

1	-1	1/2	-1/6	1/24	-1/120
.	1	-3/2	1	-5/12	1/8
.	.	1	-11/6	35/24	-17/24
.	.	.	1	-25/12	15/8
.	.	.	.	1	-137/60
.	.	.	.	.	1

[z(2) z(3) z(4) z(5) z(6) z(7)] [w0 w1 w2 w3 w4 w5]

where (rounded to 4 significant places)

w0 =	1.645
w1 =	-0.4429
w2 =	0.1017
w3 =	-0.01943
w4 =	0.003147
w5 =	-0.0004408

Explicitely this means for the w-values:

$w_0 = \zeta(2)$
$w_1 = -\zeta(2) + \zeta(3)$
$w_2 = 1/2\ \zeta(2) - 3/2\ \zeta(3) + \zeta(4)$
...

Checking for $Z(s)$ with fractional or complex values for s seem to show, that this identity holds for all zeta-exponents (check performed with Pari/GP and $n=16$ as dimension of matrices), for instance:

$\zeta(2+1.5)$	= 1.126733867...	
$W * PR * Z(1.5)$	= 1.126732951...	
$\zeta(2+4.2+2*I)$	= 1.001611861...	- 0.014311654...*I
$W * PR * Z(4.2+2*I)$	= 1.001611864...	- 0.014311649...*I

This strict identity also seems to indicate, that the multiplication-term ***W~ * PR*** may simply produce the terms of the $\zeta(2)$-series itself. A check, using the first 16 rows of ***PR*** only, gives numerical evidence:

$\lim W\sim * PR = Z(2)\sim$

PR

1	.	.	.	.	.
2	-2	.	.	.	.
3	-12	9	.	.	.
4	-48	108	-64	.	.
5	-160	810	-1280	625	.
6	-480	4860	-15360	18750	-7776

[w0 w1 w2 w3 w4 w5] [1.000 0.2500 0.1111 0.06510 -0.1836 7.661]

(rounded to 4 significant places)

This observation, in conventional notation suggests the following identities:

$$\sum_{k=0}^{oo} \left[(k+1) * w_k\right] = 1$$

$$-\sum_{k=1}^{oo} \left[(k+1) * 2^k * \binom{k}{1} * w_k\right] = \frac{1}{2^2}$$

$$\sum_{k=2}^{oo} \left[(k+1) * 3^k * \binom{k}{2} * w_k\right] = \frac{1}{3^2}$$

The zeta-components expanded, taking a useful decomposition of DR^{-1} into

$DR^{-1} = ER * {}^{d}Fac(-1)$

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ . & 1 & -3 & 6 & -10 & 15 \\ . & . & 2 & -11 & 35 & -85 \\ . & . & . & 6 & -50 & 225 \\ . & . & . & . & 24 & -274 \\ . & . & . & . & . & 120 \end{bmatrix} * \begin{bmatrix} 1 & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & 1/2 & . & . & . \\ . & . & . & 1/6 & . & . \\ . & . & . & . & 1/24 & . \\ . & . & . & . & . & 1/120 \end{bmatrix}$$

and using the coefficients of $ER[r,c]$ of row r and column c as $e_{r,c}$ in the following formulae, then w_c is:

$$w_c = \sum_{j=0}^{c} \left(e_{j,c} * \frac{\zeta(2+j)}{c!} \right)$$

and we get for the first $(c=0)$ column of $W * PR$:

$$\sum_{k=0}^{oo} \left[(k+1) * 1^k * \binom{k}{0} * \sum_{j=0}^{k} \left(e_{j,k} * \frac{\zeta(2+j)}{k!} \right) \right] = 1$$

and for the column c :

$$(-1)^c \sum_{k=c}^{oo} \left[(k+1) * (c+1)^k * \binom{k}{c} * \sum_{j=0}^{k} \left(e_{j,k} * \frac{\zeta(2+j)}{k!} \right) \right] = \frac{1}{(c+1)^2}$$

2.2. The structure of the entries of PR

PR can be seen as a Hadamard-product of ***ZV*** and a truncated ***P***-matrix (using the symbol **#** for the Hadamard (elementwise)-multiplication operator):

$$PR = \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & -2 & . & . & . & . \\ 1 & -4 & 9 & . & . & . \\ 1 & -8 & 27 & -64 & . & . \\ 1 & -16 & 81 & -256 & 625 & . \\ 1 & -32 & 243 & -1024 & 3125 & -7776 \end{bmatrix} \# \begin{bmatrix} 1 & . & . & . & . & . \\ 2 & 1 & . & . & . & . \\ 3 & 3 & 1 & . & . & . \\ 4 & 6 & 4 & 1 & . & . \\ 5 & 10 & 10 & 5 & 1 & . \\ 6 & 15 & 20 & 15 & 6 & 1 \end{bmatrix}$$

However, to get it in a clearer relation to the known matrix ***Pj*** and ***ZV*** of the HASSE-scheme, one could also rewrite this as:

*PR = $^dZ(-1)$ * Pj # ZV~*

$$PR = \begin{bmatrix} 1 & . & . & . & . & . \\ . & 2 & . & . & . & . \\ . & . & 3 & . & . & . \\ . & . & . & 4 & . & . \\ . & . & . & . & 5 & . \\ . & . & . & . & . & 6 \end{bmatrix} * \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & -1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ 1 & -3 & 3 & -1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix} \# \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 3 & . & . & . \\ 1 & 4 & 9 & 16 & . & . \\ 1 & 8 & 27 & 64 & 125 & . \\ 1 & 16 & 81 & 256 & 625 & 1296 \end{bmatrix}$$

or using ^{d}***XX****(e)=diag([1^{1+e}, 2^{2+e}, 3^{3+e},...$(k+1)^{k+1+e}$,...])* and an appropriate variation ***ZVX*** of ***ZV***:

*PR = $^dZ(-1)$ * Pj # ZVX~ * $^dXX(-2)$*

$$PR = \begin{bmatrix} 1 & . & . & . & . & . \\ . & 2 & . & . & . & . \\ . & . & 3 & . & . & . \\ . & . & . & 4 & . & . \\ . & . & . & . & 5 & . \\ . & . & . & . & . & 6 \end{bmatrix} * \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & -1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ 1 & -3 & 3 & -1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix} \# \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 4 & 3 & 1 & . & . \\ 1 & 8 & 9 & 4 & 1 & . \\ 1 & 16 & 27 & 16 & 5 & 1 \end{bmatrix} * \begin{bmatrix} 1 & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & 3 & . & . & . \\ . & . & . & 16 & . & . \\ . & . & . & . & 125 & . \\ . & . & . & . & . & 1296 \end{bmatrix}$$

2.3. The reciprocal of PR

The reciprocal ***PR***$^{-1}$ is

$$PR^{-1} = \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & -1 & . & . & . & . \\ 1 & -4/3 & 1/3 & . & . & . \\ 1 & -3/2 & 9/16 & -1/16 & . & . \\ 1 & -8/5 & 18/25 & -16/125 & 1/125 & . \\ 1 & -5/3 & 5/6 & -5/27 & 25/1296 & -1/1296 \end{bmatrix}$$

and this can be decomposed into

*PR^{-1} = Pj # ZVr * $^dZ(-1)$*

$$PR^{-1} = \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & -1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ 1 & -3 & 3 & -1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix} \# \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1/2 & 1/4 & 1/8 & 1/16 & 1/32 \\ 1 & 1/3 & 1/9 & 1/27 & 1/81 & 1/243 \\ 1 & 1/4 & 1/16 & 1/64 & 1/256 & 1/1024 \\ 1 & 1/5 & 1/25 & 1/125 & 1/625 & 1/3125 \\ 1 & 1/6 & 1/36 & 1/216 & 1/1296 & 1/7776 \end{bmatrix} * \begin{bmatrix} 1 & . & . & . & . & . \\ . & 2 & . & . & . & . \\ . & . & 3 & . & . & . \\ . & . & . & 4 & . & . \\ . & . & . & . & 5 & . \\ . & . & . & . & . & 6 \end{bmatrix}$$

2.4. Additional observations about the structure of Delta - matrix Δ and DR

The delta-matrix, occuring by matrix-multiplication ***D*** = Δ = *Pj*ZV* , may be continued by extension of ***ZV*** to negative indexes, which means positive zeta-exponents. This extension can simply be done by premultiplication of ***ZV*** by a diagonal zeta-vector $^{d}Z(m)$ with a positive *m*, which, in matrix-display is then a right-shift of the ***ZV***-matrix by *m* columns The resulting delta-matrix produced by a *m=4*-shift is

$D(4) = Pj * (^{d}Z(4)*ZV)$

D(4) =

1	1	1	1	1	1	1	1	1	1	1	1
15/16	7/8	3/4	1/2	0	-1	-3	-7	-15	-31	-63	-127
575/648	85/108	11/18	1/3	0	0	2	12	50	180	602	1932
5845/6912	415/576	25/48	1/4	0	0	0	-6	-60	-390	-2100	-10206
874853/1080000	12019/18000	137/300	1/5	0	0	0	0	24	360	3360	25200
336581/432000	13489/21600	49/120	1/6	0	0	0	0	0	-120	-2520	-31920

and this can be decomposed into

D(4) =

0!	.	.	.	.	.
.	1!	.	.	.	.
.	.	2!	.	.	.
.	.	.	3!	.	.
.	.	.	.	4!	.
.	.	.	.	.	5!

#

1	1	1	1	1	1	1	1	1
15/16	7/8	3/4	1/2	0	-1	-3	-7	-15
575/1296	85/216	11/36	1/6	0	0	1	6	25
5845/41472	415/3456	25/288	1/24	0	0	0	-1	-10
874853/25920000	12019/432000	137/7200	1/120	0	0	0	0	1
336581/51840000	13489/2592000	49/14400	1/720	0	0	0	0	0

where the numerators and denominators are

numerators:

1	1	1	1	1	1	1	1	1
15	7	3	1	0	-1	-3	-7	-15
575	85	11	1	0	0	1	6	25
46760	1660	50	1	0	0	0	-1	-10
6998824	48076	274	1	0	0	0	0	1
1744835904	1942416	1764	1	0	0	0	0	0

denominators:

1!^4	1!^3	1!^2	1!	1	1	1	1	1
2!^4	2!^3	2!^2	2!	1	1	1	1	1
3!^4	3!^3	3!^2	3!	1	1	1	1	1
4!^4	4!^3	4!^2	4!	1	1	1	1	1
5!^4	5!^3	5!^2	5!	1	1	1	1	1
6!^4	6!^3	6!^2	6!	1	1	1	1	1

and, for instance for column *c=3* (representing the use of **Z***(1)*) we have the summation:

$$\Sigma_{k=0..oo}\, k!/((k+1)*(k+1)!) = \Sigma_{k=0..oo}\, k!/(k!*(k+1)^2) = \Sigma_{k=0..oo}\, 1/(k+1)^2 = \zeta(2)$$

$$\Sigma_{k=0..oo}\, [1,3,11,50,...]_k\, k!/((k+1)*(k+1)!^2) = \Sigma_{k=0..oo}\, [1,3,11,50,...]_k /k! * 1/(k+1)^3 = \zeta(3)$$

$$d(k) = (k+1)!\ \Sigma_{j=1..k+1}\, 1/j$$

$$2*\zeta(3) = \sum_{k=0}^{oo} \frac{[1,3,11,50,...]_k}{k!} \frac{1}{(k+1)^3} = \sum_{k=0}^{oo} \frac{d(k)}{k!} \frac{1}{(k+1)^3}$$

$$d(k) = (k+1)! \sum_{j=1}^{k+1} \frac{1}{j}$$

$$2*\zeta(3) = \sum_{k=0}^{oo} \frac{1}{(k+1)^3} (k+1) \sum_{j=1}^{k+1} \frac{1}{j}$$

$$2*\zeta(3) = \sum_{k=0}^{oo} \frac{1}{(k+1)^2} \sum_{j=1}^{k+1} \frac{1}{j} = \sum_{k=0}^{oo} \frac{1}{(k+1)^2} \left(\frac{1}{1} + \frac{1}{2} + ... + \frac{1}{k+1} \right)$$

3. References

[Project-Index] http://go.helms-net.de/math/binomial/index

[Intro] http://go.helms-net.de/math/binomial/intro.pdf

[binomialmatrix] http://go.helms-net.de/math/binomial/01_1_binomialmatrix.pdf

[signed binomial] http://go.helms-net.de/math/binomial/01_2_signedbinomialmatrix.pdf

[Gaussmatrix] http://go.helms-net.de/math/binomial/04_1_gaussmatrix.pdf

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Gottfried Helms, 13.12.2006