Some special properties of the unsigned and the signed Pascal-Matrix

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$$\begin{bmatrix} 1 & . & . & . & . \\ 1 & 1 & . & . & . \\ 1 & 2 & 1 & . & . \\ 1 & 3 & 3 & 1 & . \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} = exp\left(\begin{bmatrix} . & . & . & . & . \\ 1 & . & . & . & . \\ . & 2 & . & . & . \\ . & . & 3 & . & . \\ . & . & . & 4 & . \end{bmatrix} \right)$$

$$P = exp(L_p)$$

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This article is currently a manuscript-version. Corrections and more references are waiting

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1. Notations and basic matrices/vectors

For the following assume, that all matrices can have infinite dimension, the column-matrices as well as the square-matrices.

I denote integer numbers as *m*,*n*,... real numbers as *x,y,z*, and complex numbers as *s*, where it is not explicitely stated different.

1.1. Basic matrices

Let **I** or $_d$**I** be the identity-matrix, **J** be the column-vector *[1,-1,1,-1,...]~*, and **E** the column-vector of *[1,1,1,1,1,..]~*

Here the ~-sign is the operator for transposing a matrix/vector.

Let **N** be the column-matrix of natural numbers,
Let **Z***(s)* be the column-matrix of *s'th* powers of the reciprocals of natural numbers, say

$\boldsymbol{Z}(-1) = \boldsymbol{N}$

Let **V***(s)* be the column-matrix of powers of *s*, beginning with 1, say

$\boldsymbol{V}(3) = [1,3,3^3,3^3,...]\sim$

Let **P** be the lower triangular pascalmatrix.

1	.	.	.	.	.
1	1	.	.	.	.
1	2	1	.	.	.
1	3	3	1	.	.
1	4	6	4	1	.
1	5	10	10	5	1

P

Also I use to denote the diagonal with entries of a columnvector, like

diagonal(1,-1,1,-1,...]

the prefix-notation with a small *d*, like

$_d$**J** for the diagonal-matrix containing the elements of **J** in the diagonal

and, for instance,

$_{d-1}$**N** the matrix, containing the elements of **N** in the first lower subdiagonal.

0	.	.	.	.	.
1	0	.	.	.	.
0	2	0	.	.	.
0	0	3	0	.	.
0	0	0	4	0	.
0	0	0	0	5	0

d-1N

1.2. Denoting elements of a matrix, like rows, columns and elements:

Let also, allowing more generality, denote the column *n* of a matrix **P** as **P***[*,n]*, the row m of a matrix **P** as **P***[m,*]* and a single element as **P***[m,n]*.

Note, that the indices start at zero, so the first row is **P***[0,*]*, and the topleft element is **P***[0,0]*. (Unfortunately, using Pari/Gp one has to increment this indices, since Pari/Gp uses 1-based indices for matrices and vectors.)

Extraction of ranges of columns/rows and lists of ranges can be expressed by

$start_1..end_1'start_2..end_2'....$

so

P[0..1´4..8]

gives the first two and the 5'th to 9'th rows.

The column *n* of the identity-matrix is sometimes denoted as $\mathbf{I}_n$.for convenience.

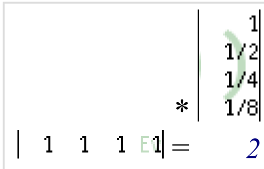
1.3. Denoting concatenation of matrices/vectors

concatenation of columns : **ZV** = **Z**(0) || **Z**(-1) || **Z**(-2) || **Z**(-3)...

concatenation of rows: **ZV** = { **V**(0)~, **V**(1)~, **V**(2)~,...}

1.4. Some simple examples: Summing in matrix-notation

The matrixmultiplication using the vector **E** is the a simple summation procedure. In matrixnotation this looks like, for instance:

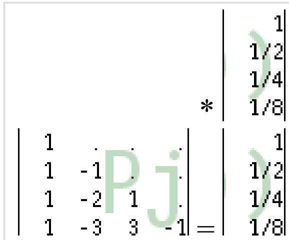


$\Sigma_{k>0} 1/k^2 = E\sim * Z(2) = \zeta(2)$

The summation of the powers of *1/2* is in this matrix-notation

$\Sigma_{k=>0} 1/2^k = E\sim * V(1/2) = 2$

In the following I describe in detail eigenvector properties of the column-signed version of **P**, so I denote this matrix as P_j.



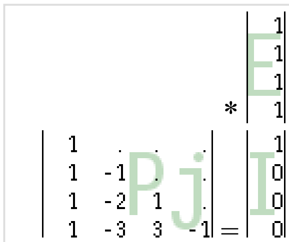
$P_j = P *_d J$

A short glimpse into the eigenvector and into the sumation-chapter: each row of **P***j* summed with **V***(1/2)* gives just the row-entry of **V***(1/2)*:

$P_j * V(1/2) = V(1/2) * 1$

so **V***(1/2)* is obviously an eigenvector of **P**j associated with the eigenvalue *1*.

Another summation:



$P_j * E = I[*,1] = [1,0,0,0,....]\sim$

Caution is needed, if the summation involves divergent series; such a summation -in its direct application- is not defined.

But solutions for dealing with divergent summations are subject of chap 5; in many cases meaningful values can be assigned to such series-summations.

2. The Pascalmatrixces P and Pj

2.1. Properties of the pascalmatrix P

2.1.1. The entries of P

The entries of the Pascalmatrix are simply the binomial-coefficients *choose(r,c)* where *r* denotes the row and *c* denotes the column of such an entry, the indices running from zero.

2.1.2. The Inverse of P

For any finite dimension we find, that the inverse of **P** is simply its signed version:

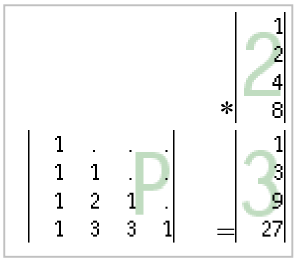
$$P^{-1} = {}_dJ * P * {}_dJ$$

That the inverse is defined even with infinite dimension can be justified, since **P** is a lower triangular matrix, and the elements of its inverse can be iteratively calculated using only already known elements:

when the first element $P^{-1}[0,0]$ is computed, the validity of the computation of the next elements follows by induction.

2.1.3. Right-Multiplication with a Powersum-vector V(x): Applying the binomial theorem

Performing a matrix-multiplication of **P** with a vector **E** or **V**(*n*) means to apply the binomial-theorem.



$$P * E = [1,2,4,8,...]\sim$$
$$P * [1,2,4,8,...]\sim = [1,3,9,27,...]\sim$$

and generally

$$P * V(n) = V(n+1)$$
$$P * V(s) = V(s+1) \qquad \text{// this is also defined for any complex s}$$

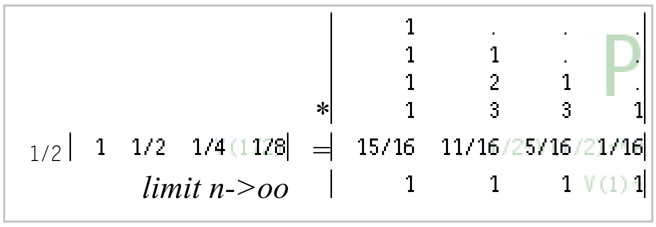
Also from this follows by induction:

$$P^n * V(0) = V(n)$$
$$P^n * V(s) = V(s+n)$$

Using the inverse this also means

$$P^{-n} * V(n) = V(0)$$
$$P^{-n} * V(s) = V(s-n)$$

2.1.4. Leftmultiplication of V(x)~ with P



The leftmultiplication looks like:

$$1/2*V(1/2)\sim * P = [1,1,1,1,...] = E\sim$$

and with elementary means it can be derived for n>0:

$$1/n*V(1/n)\sim * P^{n-1} = E\sim$$

For $n>1$ we have in the multiplication with the first column of P^{n-1} an ordinary geometric series with

$q=(n-1)/n <1$.

For $n=1$ we have $q=0/1 = 0$, which comes out to be $E\sim * [1,0,0,0....]=1$

This can be generalized to real x having $q = 1-1/x$;

for $oo > x >= 1$	this is the convergent geometric series with $1>q>=0$
for $1 > x > 1/2$	we have an convergent alternating series of $0 > q > -1$,
for $1/2> x > 0$	we have divergent alternating series of $-1 > q > -oo$,
for $0 > x > -oo$	we have divergent series $oo > q >= 1$

We could surely use the fact, that for any $q<>1$ the formula for the sum of geometric series $sum=1/(1-q)$ can be used due to known analytic continuation, but this fact cannot be proven in this context. However in the chapter about summation it can be shown, that the divergent alternating series steming from $1/2>x>0$ can be summed to their values by the means of this matrix-approach. See chapter below.

Also, since **V**(x)~ * **P** represents the values of the powersum $f(x)=1+x+x^2+x^3+...$ in the first column of the resulting row-vector, in the following columns we have the values of the derivatives of $f(x)$ at the same point.

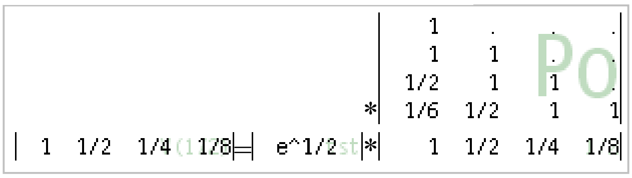
So, for leftmultiplication we have, written with derivatives:

$$V(x)\sim P = [f(x), f'(x), f''(x), f'''(x),...]$$

(short excurs:)

If we define a column-vector **F**, containing the factorials **F**$=[0!,1!,2!,...]\sim$ and define the diagonalmatrix $_d$**F** containing **F** in its diagonal, then we have

$$V(s)\sim *_dF^{-1} * P * _dF = [exp(s),\ s*exp'(s),\ s^2*exp''(s),...]$$
$$= [exp(s),\ s*exp(s),\ s^2*exp(s)...]$$
$$= exp(s)*V(s)\sim$$
$$= e^s * V(s)\sim$$



and

$$E\sim *_dF^{-1}*P*_dF = e * E\sim$$

where e is the euler-constant. Repeating the operations one gets

$$V(s)\sim*(_dF^{-1}*P*_dF)^n = e^s * V(s)\sim* (_dF^{-1}*P*_dF)^{n-1}$$
$$= e^{s*n} *V(s)\sim$$

Since we can assume the $s'th$ power of **P** as usable even in infinite context (see next paragraph), in this expression n can be replaced by any complex number t.

Thus defining a rescaled matrix $P_0 = {}_dF^{-1} *P * {}_dF$, we can compute any complex $t'th$ power and can say:

$$V(s)\sim * P_0{}^t = e^{s*t}*V(s)$$

See chap 5 and 6 for more.

2.1.5. Rightmultiplication of P with its transpose

In an article it was mentioned as a curiousity, that the product of **P** with its transpose gives a symmetric version of **P** where the columns are simply shifted upwards:

```
P*P~  =  | 1,1,1, 1, 1.....|
         | 1,2,3, 4, 5.... |
         | 1,3,6,10,15...  |
         | ...             |
```

Aside from curiousity, this is a property, which shall be of interest later.

2.1.6. Fractional, real and complex powers of P

The common way to define functions like this for square matrices is to decompose the matrix into its eigensystem and apply the desired function to the elements of its eigenvalues, which are the elements of a diagonal-matrix.

Now for **P** an eigensystem cannot be found; the eigenvalues are all 1, but the matrix of eigenvectors could only consist of null-vectors, and there would be no inverse of the eigenmatrix. So this approach is useless for the pascalmatrix **P**.

But there is another solution for this problem: the logarithmic, or exponential-approach.

The pascalmatrix **P** can be expressed as the matrix-exponential of a simple subdiagonal-matrix $\mathbf{L}_p$:

$$P = exp(L_p)$$

Here $\mathbf{L}_p$ is the matrix, where the first principal subdiagonal contains the natural numbers.

```
    |  .    .    .    . ...  |
    |  1    .    .    . ...  |
Lp= |  .    2    .    . ...  |
    |  .    .    3    . ...  |
    |  .    .    .    4 ...  |
      ....
```

Again by induction from the case of finite dimension one can show, that this is meaningful even if infinite dimension is assumed. Here the series-representation of the *exp()*-function is needed, and since any $\mathbf{L}_p$ of finite dimension *n* is nilpotent to the power of *n*, and the values involved are also bounded on each step, the use for infinite dimension can be justified.

From the exponential-representation it is easy to find fractional, real or even complex powers of **P**: simply multiply $\mathbf{L}_p$ with the desired power and perform exponentiation:

$$\mathbf{P}^s = exp(\mathbf{L}p * s)$$

The above equations can then be generalized:

$$\mathbf{P}^s * \mathbf{V}(0) = \mathbf{V}(s)$$
$$\mathbf{P}^{-s} * \mathbf{V}(s) = \mathbf{V}(0)$$

This makes the pascalmatrix a versatile tool for more general questions.

2.2. Properties of the signed Pascalmatrix P_j

The signed pascalmatrix $\mathbf{P}_j$ has even more special properties, which are interesting for numbertheory.

$$P_j = \begin{vmatrix} 1 & . & . & . \\ 1 & -1 & . & . \\ 1 & -2 & 1 & . \\ 1 & -3 & 3 & -1 \end{vmatrix}$$

2.2.1. The inverse of Pj

Let

$$P_j = P * {}_dJ$$

Then some of the above formula get interesting modifications. From

$$P^{-1} = {}_dJ * P * {}_dJ$$

follows, that $\mathbf{P}_j$ is its own inverse:

$$P * P^{-1} = {}_dI = P * {}_dJ * P * {}_dJ \ = (P * {}_dJ) * (P * {}_dJ) = Pj * Pj$$
$$P_j * P_j = {}_dI$$
$$P_j * P_j * P_j^{-1} = {}_dI * P_j^{-1}$$
$$P_j = P_j^{-1}$$

2.2.2. Rightmultiplication with a powersum-vector: application of the binomial-theorem

Considering the binomial-theorem,

$$P_j * V(s) = V(1-s)$$

Proof:

$$P_j * V(s) = {}_dJ*(P^{-1} * V(s)) = {}_dJ\ V(s-1) = V(-s+1)= V(1-s)$$

Special cases:

$$P_j * V(1/2) = V(1/2)$$
$$P_j * V(1/2+s) = V(1/2-s)$$
$$P_j * V(1/2-s) = V(1/2+s)$$

$$\begin{vmatrix} 1 & . & . & . \\ 1 & -1 & . & . \\ 1 & -2 & 1 & . \\ 1 & -3 & 3 & -1 \end{vmatrix} * \begin{vmatrix} 1 \\ 1/2 \\ 1/4 \\ 1/8 \end{vmatrix} = \begin{vmatrix} 1 \\ 1/2 \\ 1/4 \\ 1/8 \end{vmatrix}$$

saying, that leftmultiplication of a powerseries-vector $\mathbf{V}(s)$ by $\mathbf{P}_j$ is symmetric by the vertical line in the complex plane at $x=1/2$.

Note, that in conventional notation such matrix-identities are -in one bunch- a complete set of many known binomial-relations, valid for each row r, where r expresses the exponent, for instance

$$\left(\frac{1}{2}+s\right)^r = \sum_{c=0}^{r}\left((-1)^c\binom{r}{c}*\left(\frac{1}{2}-s\right)^r\right) \qquad \text{// for } r=\text{row-index, } c=\text{col-index, for any row } r$$

2.2.3. Leftmultiplication with a powersum-vector

Since $\mathbf{P}_j$ is only a column-signed version of $\mathbf{P}$, there is nothing new to mention here, except that the resulting vector as the same signing-scheme as $\mathbf{P}_j$ itself.

2.2.4. Eigensystem(s) of P_j

Interestingly $\mathbf{P}_j$, in contrast to **P**, can be decomposed into - even several - eigensystems.

The first example was already given, where $\mathbf{V}(1/2)$ obviously is an eigenvector of $\mathbf{P}_j$:

$$P_j * V(1/2) = V(1/2)*1$$

Here we see, that $\mathbf{V}(1/2)$ is associated with the eigenvalue 1.

Another simple example comes from the definition of the Bernoulli-numbers.

There is the recursive definiton of Bernoulli-numbers:

$$(-1)^n(1+\beta)^n = \beta_n \qquad \text{for } n>0, \text{ initializing } \beta_0 = 1$$

where after expansion of the binomial equation the powers of β have to be replaced by β_k, where k is taken from the exponent of the k~th power (expanding from $k=0$ to $k=n$).

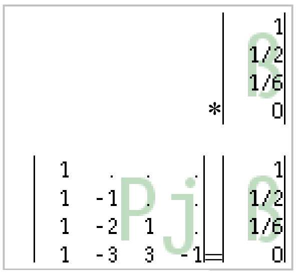
That recursive definition is now precisely the statement:

$$_dJ * P * [\beta_0,\beta_1,\beta_2,...]\sim = [\beta_0,\beta_1,\beta_2,\beta_3,...]\sim$$

So the vector of bernoulli-numbers is by definition an eigenvector of the (signed) pascalmatrix ($_d\mathbf{J}$ **P**).

Relating this conversely to $\mathbf{P}_j$, this goes from:

$$P * [\beta_0,\beta_1,\beta_2,0,\beta_4,0...]\sim = \qquad [\beta_0,-\beta_1,\beta_2,0,\beta_4,0...]\sim$$



to

$$\begin{aligned}
[\beta_0, \beta_1,\beta_2,0,\beta_4,0...]\sim &= P^{-1} \quad [\beta_0,-\beta_1,\beta_2,0,\beta_4,0...]\sim \\
[\beta_0, \beta_1,\beta_2,0,\beta_4,0...]\sim &= {}_dJ\ P\ {}_dJ* \ [\beta_0,-\beta_1,\beta_2,0,\beta_4,0...]\sim \\
{}_dJ\ [\beta_0, \beta_1,\beta_2,0,\beta_4,0...]\sim &= P_J* \quad [\beta_0,-\beta_1,\beta_2,0,\beta_4,0...]\sim \\
[\beta_0,-\beta_1,\beta_2,0,\beta_4,0...]\sim &= P_J* \quad [\beta_0,-\beta_1,\beta_2,0,\beta_4,0...]\sim
\end{aligned}$$

$$\begin{aligned}
P_J* \ [\beta_0,-\beta_1,\beta_2,0,\beta_4,0...]\sim &= [\beta_0,-\beta_1,\beta_2,0,\beta_4,0...]\sim \\
P_J* B_+ &= B_+
\end{aligned}$$

where $\mathbf{B}_+$ means, we use $-\beta_1 = +1/2$ which is the most common setting in this article.

A complete set of eigenvectors containing Bernoulli-numbers

From that a computing strategy for even a whole set of eigenvectors can be derived, shown in the following.

Rearranging $_d\mathbf{J}$

$$\begin{aligned}
P * [\beta_0,\beta_1,\beta_2,...]\sim &= {}_dJ * [\beta_0,\beta_1,\beta_2,\beta_3,...]\sim \\
P * [\beta_0,\beta_1,\beta_2,...]\sim &= \quad [\beta_0,-\beta_1,\beta_2,-\beta_3,...]\sim
\end{aligned}$$

and since the odd indexed bernoulli-numbers are zero, the lhs and rhs are equal except for the second row containing β_1.

Rewriting this, recalling that $\beta_1 = -1/2$ if the above recursive definition of bernoulli-numbers is used, this means (denoting the Bernoulli-vector with **B**)

$$P*B = {}_dJ * B = B + [0,1,0,0,0,...]$$

Subtracting **B** on each side we have

$$(P - I)B = I\,[*,1] = I_1$$

The parenthese-term on the lhs is now the pascalmatrix reduced by its diagonal; the first row is empty here. But from the second row we have a triangular matrix, which is invertible.

This allows to discuss even a complete eigenmatrix-solution for an eigenmatrix **X** of $\mathbf{P}_j$. With a one-row upshifted $(\mathbf{P} - \mathbf{I})^{\uparrow}$ and an extended and accordingly upshifted $\mathbf{I}_1^{\uparrow} = \mathbf{I}$ we can try to solve

$$(P\text{-}I)^{\uparrow} X = I_1^{\uparrow} = I$$
$$X = ((P - I)^{\uparrow})^{-1}$$

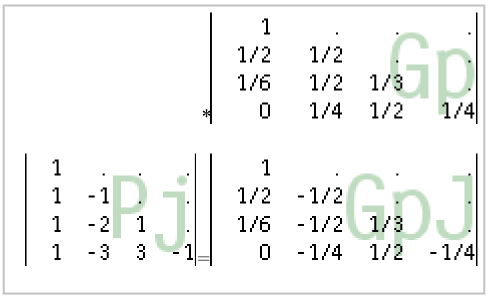
The resulting matrix contains the bernoulli-numbers and introduces many very interesting features, which I shall discuss later in more detail.

I call this matrix the "Gotti-m-matrix" "$\mathbf{G}_m$" using the index "m"(m-inus) as the negative valued Bernoulli-number $\beta_1 = -1/2$ is used. The "Gotti-p-matrix" "$\mathbf{G}_p$", using the positive version $\beta_1=1/2$ is of even more interest and will also be discussed later in detail.

Here it may suffice that

$$P * G_m = G_p$$
$$G_p = {}_dJ * G_m * {}_dJ$$

and concerning the eigensystem



$$P * G_m * {}_dJ = G_p * {}_dJ$$

and

$$P * G_m * {}_dJ = P * ({}_dJ * G_p * {}_dJ) * {}_dJ$$
$$= (P * {}_dJ) * G_p$$
$$= P_j * G_p$$

so

$$P_j * G_p = G_p * {}_dJ$$

and $\mathbf{G}_p$ is an eigenmatrix of $\mathbf{P}_j$. Here are the first six rows/columns of $\mathbf{G}_p$:

$$Gp = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/6 & 1/2 & 1/3 & 0 & 0 & 0 \\ 0 & 1/4 & 1/2 & 1/4 & 0 & 0 \\ -1/30 & 0 & 1/3 & 1/2 & 1/5 & 0 \\ 0 & -1/12 & 0 & 5/12 & 1/2 & 1/6 \end{vmatrix}$$

Different meaningful matrices as eigensystems of Pj

Precisely spoken: there are infinitely many eigensystems possible (even ignoring the arbitrary scaling of columns) of which some are remarkable and exhibit intimate relations to important numbers of number theory.

The simplest exposition of the possibilities to choose a meaningful matrix of eigenvectors is the following.

Since $\mathbf{P}_j$ is triangular, it is obvious, that for any even finite dimension 2n we have n eigenvalues 1 and n eigenvalues of -1 and for each set of equal eigenvalues an associated set of eigenvectors (with arbitrary scaling).

For an eigenvector X associated to 1 it is needed, that

$$P_j * X = X * 1$$

and for a eigenvector Y associated to -1 it is needed that

$$P_j * Y = Y * -1$$

For a complete eigenmatrix Z it is needed that

$$P_j * Z = Z * {}_dJ$$

An attempt to solve this iteratively exhibits, that we have *n* parameters free and the other *n* parameters are determined by the first *n* parameters.

One gets a system for the even and odd indexed columns of $\mathbf{P}_j$:

$$Pj_{even}*A = Pj_{odd}*B$$

where also each second row comes out to be linearly dependent from the previous row. One can now either choose the entries for **A** and compute that for **B** or in opposite direction.

For dimension *n=16* I give an example.

With the two partial matrices, occuring from gaussian elimination according to the even-indexed columns:

P_Even

1	0	0	0	0	0	0	0
0	3	0	0	0	0	0	0
0	0	15	0	0	0	0	0
0	0	0	21	0	0	0	0
0	0	0	0	45	0	0	0
0	0	0	0	0	33	0	0
0	0	0	0	0	0	1365	0
0	0	0	0	0	0	0	45

P_odd

2	0	0	0	0	0	0	0
1	2	0	0	0	0	0	0
-1	10	6	0	0	0	0	0
1	-7	21	6	0	0	0	0
-3	20	-42	60	10	0	0	0
5	-33	66	-66	55	6	0	0
-691	4550	-9009	8580	-5005	2730	210	0
105	-691	1365	-1287	715	-273	105	6

one has to solve

$$P_{even} * A = P_{odd} * B$$
$$B = P_{odd}^{-1} * P_{even} * A$$

and can freely choose parameters for **A**.

Using the gaussian elimination for the odd columns instead:

P_Even

1,	0,	0,	0,	0,	0,	0,	0
-1,	6,	0,	0,	0,	0,	0,	0
1,	-5,	5,	0,	0,	0,	0,	0
-17,	84,	-70,	28,	0,	0,	0,	0
31,	-153,	126,	-42,	9,	0,	0,	0
-691,	3410,	-2805,	924,	-165,	22,	0,	0
5461,	-26949,	22165,	-7293,	1287,	-143,	13,	0
-929569,	4587240,	-3772860,	1241240,	-218790,	24024,	-1820,	120

P_odd

2,	0,	0,	0,	0,	0,	0,	0,
0,	4,	0,	0,	0,	0,	0,	0,
0,	0,	2,	0,	0,	0,	0,	0,
0,	0,	0,	8,	0,	0,	0,	0,
0,	0,	0,	0,	2,	0,	0,	0,
0,	0,	0,	0,	0,	4,	0,	0,
0,	0,	0,	0,	0,	0,	2,	0,
0,	0,	0,	0,	0,	0,	0,	16,

one has to solve

$$P_{even} * A = P_{odd} * B$$
$$A = P_{even}^{-1} * P_{odd} * B$$

One can now choose meaningful sets of numbers for **B**.

Some special interesting ones are:

(1) $A=[1/2, \ 0, \ 0, \ 0 \ \ldots]$ $\quad$ --> $B=[1, \ 1/6, -1/30, \ldots]$
(2) $A=[1/2, \ 1/8, \ 1/32, \ 1/128 \ \ldots]$ $\quad$ --> $B=[1, \ 1/4, \ 1/16, \ldots]$
(3) $A=[1/2, \ 1/4, \ 1/6, \ 1/8 \ \ldots]$ $\quad$ --> $B=[1, \ 1/3, \ 1/5, \ldots]$

(4) $A=[1/2, -1/4, \ 1/2, \ -17/8 \ \ldots]$ $\quad$ <-- $B=[1, \ 0, \ 0, \ ]$

Selection (1) produces the bernoulli-numbers. Selection (2) produces the powerseries of 1/2, selection (3) produces the zeta-series for exponent 1, and selection (4) , where B is set to *[1,0,0,...]*, produces a series of numbers, which is related to the values of the eta-function in the same way, as the bernoulli-numbers are related to the values of the zeta-function.

If we order the selections in a different way, we get an impression of a family of number-theoretical sets-of-numbers, which may be extended systematically to top or to bottom. Here some solutions (where the A and B vectors are correctly re-joined) are given, which have an interesting pattern:

(4) $B \Rightarrow A=[\ 1, \ 1/2, \ 0, -1/4, \ 0, \ 1/2, \ 0, \ -17/8 \ \ldots]$ *eta-related*
(1) $A \Rightarrow B=[\ 1, \ 1/2, \ 1/6, \ 0, -1/30, \ 0, \ , \ \ldots]$ *zeta-related*

(2) $B \Rightarrow A=[$ 1, 1/2, 1/4, 1/8, 1/16, 1/32, ...] *powers of 1/2*
(3) $B \Rightarrow A=[$ 1, 1/2, 1/3, 1/4, 1/5, 1/6 , ...] *reciprocals*
....
(5) $A \Rightarrow B=[$ 1 , 1/2, 1/2, 1/2, 1/2, 1/2 ...] *1/2*

The result of such selections is, that we construct arbitrary, but hopefully meaningful, eigenvectors of **P**j by interleaving the **B** and **A** values.

For selection (1) this means, we get the vector of bernoulli-numbers as eigenvector, which are intimely related to the zeta-function,($\beta_k = - k\, zeta(1-k)$),
for selection (4) this means, we get the vector of "eta-numbers" as eigenvector,
selection (2) gives *V(1/2)* as eigenvector and
selection (3) gives *Z(1)* as eigenvector.

Selection (5) gives [1,1/2,1/2,1/2,...] as eigenvector, which is not discussed yet.

To make this even more profitable, Let's see, how we can construct eigenmatrices as full sets of eigenvectors, based on this selections.

2.2.5. Fractional powers of P_j based on Eigensystem-decomposition

Having an eigensystem of $\mathbf{P}_j$ this allows to compute common functions with $\mathbf{P}_j$ as parameter by applying that function to the entries of its diagonal eigenvalue-matrix, which is ${}_d\mathbf{J}$.

To compute integral powers of $\mathbf{P}_j$ is simple:

$$P_j^{2n} = I$$
$$P_j^{2n+1} = P_j^{2n} * P_j = I * P_j = P_j$$

It is more difficult to make a meaningful decision, what to assume as fractional power.

The first options is, just to use the principal complex fractional power of the entries *1* and *-1* of its eigenvalue-matrix ${}_d\mathbf{J}$, for instance

$${}_dJ^{1/2} = diag[1,i,1,i,...]$$

The second options seems more reasonable: one may assume the p complex roots of $e^{i\,\pi k/p}$ repeating cyclically, so

$${}_dJ^{1/p} = diag\,(e^{i\,\pi\, r/p}) \qquad r=0\,..oo \quad \textit{indicating the row}$$

I did not explore this option very deep; some significant observations using ${}_d\mathbf{J}^{1/2}$ are:

$P_j^{1/2} * V(1/2) = X$,
$P_j^{1/2} * Z(1) \quad = X$,
then X has only real entries

Pj2

1	.	.	.	.	.
1/2-1/2*I	I	.	.	.	.
-1/6-1/2*I	1+I	-1	.	.	.
-1/2	3/2-1/2*I	-3/2+3/2*I	-I	.	.
-1/6+1/2*I	-2*I	1+3*I	-2-2*I	1	.
1/2+1/3*I	-5/2-5/6*I	5	-5+5/3*I	5/2-5/2*I	I

2.2.6. The matrix Gp as eigenmatrix of Pj

In the previous chapter we not only found one eigenvector (containing the bernoulli-numbers), but even a full set of eigenvectors. I called this matrix "$\mathbf{G_p}$".

This matrix has the following structure: (using $\beta_1 = +1/2$)

```
        | ß0   0    0    0    0    0    0    0 |
        | ß1   ß0   0    0    0    0    0    0 |
        | ß2   ß1   ß0   0    0    0    0    0 |
Gp =    | 0    ß2   ß1   ß0   0    0    0    0 | ⊗ P * Z(1)~
        | ß4   0    ß2   ß1   ß0   0    0    0 |
        | 0    ß4   0    ß2   ß1   ß0   0    0 |
        | ß6   0    ß4   0    ß2   ß1   ß0   0 |
        | 0    ß6   0    ß4   0    ß2   ß1   ß0 |
```

where "⊗" indicates the elementwise multiplication.

Thus each row of $\mathbf{G_p}$ is much similar to the coefficients of the Bernoulli-polynomials, except that there is a cofactor of $^1/_{k+1}$ for the entry of the *k'th* column. The coefficients of each row can then be interpreted as the coefficients of the *integrals of the bernoulli-functions*. The matrix of coefficients of the Bernoulli-functions is

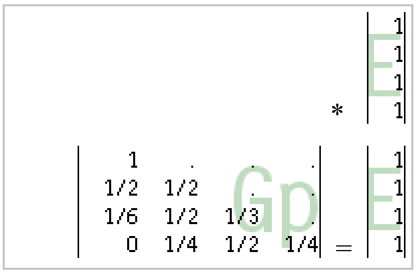
```
        | ß0   0    0    0    0    0    0    0 |
        | ß1   ß0   0    0    0    0    0    0 |
        | ß2   ß1   ß0   0    0    0    0    0 |
B   =   | 0    ß2   ß1   ß0   0    0    0    0 | ⊗ P
        | ß4   0    ß2   ß1   ß0   0    0    0 |
        | 0    ß4   0    ß2   ß1   ß0   0    0 |
        | ß6   0    ß4   0    ß2   ß1   ß0   0 |
        | 0    ß6   0    ß4   0    ß2   ß1   ß0 |
```

giving in each row r the coefficients for

$$B_r(x) = \Sigma_{c=0..r} B[r,c]*x^c$$

The rowsums in $\mathbf{G_p}$ equal 1, which means

$$Gp * E = E$$



so $\mathbf{E}$ is also an eigenvector of $\mathbf{G_p}$.

The alternating rowsums are

$$Gp * V(-1) = [1,0,0,...]\sim = I_0$$

Using the entries of a row r of $\mathbf{G_p}$ as coefficients of a polynomial in x:

$$G_{p\,r}(x)= \Sigma_{k>=0}\, G_p[r,k]*x^k$$

then

$$G_{p\,r}(0) = \beta_r$$
$$G_{p\,r}(1) = 1$$
$$G_{p\,r}(-1) = 0 \text{ for } r>0 \text{ and } 1 \text{ for } r=0$$

The functiongraphs show an interesting sinusoidal form in the range $-1<x<1$; where the multiple local maxima and minima seem to approximate to equal values for a certain $G_{p\,r}$-function, different from the respective Bernoulli-polynomials.

The summing-property of Gp

A matrixmultiplication with a vector **V***(n)* is an interesting summing procedure:

$$G_p*n*V(n) = V(1)+V(2)+...+V(n) = S(1,n)$$

which means, that $\mathbf{G}_p$ is the matrix, which solves the original problem of Jacob Bernoulli to find coefficients to simplify summation of natural numbers of equal powers:

$$G_{p,r}*n*V(n) = \Sigma_{k=0..n} k^r = 1^r + 2^r + 3^r + ... + n^r$$

and

$$G_p *(s*V(s) - (s-1)*V(s-1)) = V(s)$$

which is valid for all complex s.

The eigensystem of Gp

$\mathbf{G}_p$ has an interesting eigensystem itself: we meet the matrices of Stirling-numbers of second kind ($\mathbf{St}_2$) and of first kind ($\mathbf{St}_1$).

$$Gp = St_2 * {}_dZ(1) * St_2^{-1}$$

and even , $\mathbf{St}_1$ being the inverse of $\mathbf{St}_2$,

$$St_2^{-1} = St_1$$
$$Gp = St_2 * {}_dZ(1) * St_1$$

so describing $\mathbf{P}_j$ as

$$Pj = Gp * {}_dJ * Gp^{-1} = St_2 * {}_dZ(1)*St_1 \quad * \quad {}_dJ \quad * \quad St_2 * Z_d(-1)*St_1$$

Using ${}_d\mathbf{F}$ as diagonalmatrix containing the factorials beginning at *0!*, we can declare an interesting rescaling of these eigenmatrices; we can write:

$$D = St_2*{}_dF *{}_dJ$$
$$Gp = D * {}_dZ(1) * D^{-1}$$

referring to a matrix **D**, which shall be of importance in the summation-chapter.

Having all eigenvalues pairwise unequal, the eigenvectors of $\mathbf{G}_p$ are uniquely determined by the columns of $\mathbf{St}_2$, except of arbitrary column-scaling. These columns contain the **E**-vector (=**V***(1)*) as well as the vector (**V***(2)*-**V***(1)*), which both are thus invariant by leftmultiplication with $\mathbf{G}_p$. (Note, that the difference *(V(2)-V(1))* reflects numbers of the form 2^n-1, which are of special interest for prime *n*.)

Sums of columns of Gp

The columnwise sums are not easily determined, since the summing would imply to sum divergent (though alternating) series. The summation however can be done by Borel-summation (see chapter below) and show the following intimate relation with the zeta-function:

$E\sim * G_p = [\zeta(2), \quad \zeta(3), \quad \zeta(4), \quad \zeta(5),...]$
$E\sim * G_m = [\zeta(2)\text{-}1, \zeta(3)\text{-}1, \zeta(4)\text{-}1, \zeta(5)\text{-}1,...]$

Gp

1	.	.	.	.	.
1/2	1/2	.	.	.	.
1/6	1/2	1/3	.	.	.
0	1/4	1/2	1/4	.	.
-1/30	0	1/3	1/2	1/5	.
0	-1/12	0	5/12	1/2	1/6

E~

1	1	1	1	1	1
Z(2)	Z(3)	Z(4)	Z(5)	Z(6)	Z(7)

The knowledge about its Borel-sums allow to proceed with some interesting expressions for summing the zeta-values. This is be dealt with in the "summation"-chapter.

Some other identities, not completed:

$$\sum_{k=0}^{oo} \frac{x^k}{k!}\beta_k = \frac{e^x}{e^x - 1}$$

Gp

1	.	.	.	.	.
1/2	1/2	.	.	.	.
1/6	1/2	1/3	.	.	.
0	1/4	1/2	1/4	.	.
-1/30	0	1/3	1/2	1/5	.
0	-1/12	0	5/12	1/2	1/6

1	1	1/2	1/6	1/24	1/120
e/(e-1)	?	?	?	?	?

and second column appears to be: $d(e^x/(e^x\text{-}1))/dx = (e^x*(e^x\text{-}1)\text{-}e^x*e^x)/(e^x\text{-}1)^2 = - e^x/(e^x\text{-}1)^2$

Gp

1	.	.	.	.	.
1/2	1/2	.	.	.	.
1/6	1/2	1/3	.	.	.
0	1/4	1/2	1/4	.	.
-1/30	0	1/3	1/2	1/5	.
0	-1/12	0	5/12	1/2	1/6

0	1	1/2	1/3	1/4	1/5
gamma	?	?	?	?	?

2.2.7. The Matrix Eta as eigenmatrix of Pj

It is not obvious how to extend the eta-eigenvector-solution for $\mathbf{P}_j$ to a full system of eigenvectors. But inspired by the structure of $\mathbf{G}_p$ one can construct such an eigenmatrix just analoguously to $\mathbf{G}_p$.

This matrix has the following structure:

```
       | ε0   0    0    0    0    0    0    0 |
       | ε1   ε0   0    0    0    0    0    0 |
       | 0    ε1   ε0   0    0    0    0    0 |
Eta=   | ε3   0    ε1   ε0   0    0    0    0 | ⊗ P * Zd(1)
       | 0    ε3   0    ε1   ε0   0    0    0 |
       | ε5   0    ε3   0    ε1   ε0   0    0 |
       | 0    ε5   0    ε3   0    ε1   ε0   0 |
       | ε7   0    ε5   0    ε3   0    ε1   ε0 |
```

where "⊗" again indicates the elementwise multiplication. The scaling of the columns by Zd(1) is essentially arbitrary. Whether there is one version, which is to prefer over others, is not yet determined.

Eta performs alternating summing, similar to Gp, with the extend, that there is a constant summand which has to be reflected.

2.2.8. The Matrix Vh as eigenmatrix of Pj

It is again not obvious how to extend the **V***(1/2)*-eigenvector-solution for $\mathbf{P}_j$ to a full system of eigenvectors. But again using the structure of **G***p* one can construct such an eigenmatrix.

This matrix has the following structure: (using $v_k = 1/2^k$)

```
       | v0   0    0    0    0    0    0    0 |
       | v1   v0   0    0    0    0    0    0 |
       | v2   v1   v0   0    0    0    0    0 |
vh =   | v3   v2   v1   v0   0    0    0    0 | ⊗ P * Z(1)~
       | v4   v3   v2   v1   v0   0    0    0 |
       | v5   v4   v3   v2   v1   v0   0    0 |
       | v6   v5   v4   v3   v2   v1   v0   0 |
       | v7   v6   v5   v4   v3   v2   v1   v0 |
```

where "⊗" again indicates the elementwise multiplication.

- no discussion so far -

2.2.9. The Matrix Rec as eigenmatrix of Pj

The same has to be said for the matrix created by reciprocals *[1,1/2,1/3,...]=* **Z***(1)~*

This matrix has the following structure:

```
       | r0   0    0    0    0    0    0    0 |
       | r1   r0   0    0    0    0    0    0 |
       | r2   r1   r0   0    0    0    0    0 |
rec=   | r3   r2   r1   r0   0    0    0    0 | ⊗ P * Z(1)~
       | r4   r3   r2   r1   r0   0    0    0 |
       | r5   r4   r3   r2   r1   r0   0    0 |
       | r6   r5   r4   r3   r2   r1   r0   0 |
       | r7   r6   r5   r4   r3   r2   r1   r0 |
```

where "⊗" again indicates the elementwise multiplication.

- no discussion so far -

---

3. Summation procedures using matrices

One obvious problem with the summation of divergent series is the problem of unknown indexes. A simple example is usually shown as different parenthesing of the most simple divergent series:

$(1-1)+(1-1)+... =0 =?= 1 -(1-1) - (1-1) = 1$

Here the different parenthesing leads to different solutions and the naive approach is inconsistent and insufficient.

A powerful concept to overcome that obvious difficulties is to make use of formal powerseries. Here one assigns to each element of the series a power of a formal variable x, thus converting the series into an infinite polynomial in x, for which is then a limit for $x->1$ computed, if this is possible.

Different summation-methods were invented over the centuries, the most common ones named as "Cesaro-Summation", "Abel-summation", "Euler-summation" and Borel-summation". The approach, which is common to all these methods is, to find a transformation to the given series, which can be conventionally summed, and then find an operation, which re-assigns the value of the transformed series meaningfully to the original one. The assignment of a formal powerseries-parameter x can then be understood to keep the indexes of each element of the series consistent.

The matrix-notation implements this feature implicitely by its rigid row and column-structure.

So the divergent summation

$$\Sigma_{k=0..oo} (-1)^k = x$$

in matrix notation

$$E\sim * J = x$$

can be handled without extra notation overhead, provided that the condition for a formal powerseries summation are met, the most prominent: no implicte divergent sums or partial sums.

The technique to assign a finite value to such a divergent expression is then, to dominate the coefficients by another set of coefficients, then to compute a limit by a known formula or by approximating, and finally to relate that to the initial problem by an inverse operation to the multiplication by coefficients.

3.1. Introduction and examples

3.1.1. A simple averaging-concept: Cesaro-summation

Cesaro-Summation applies the simple idea, to use the limit of the mean of the convergents of the partial sums, if this exists. So, for a series like

$$s_{oo}= 1 - 1 + 1 - 1 .... =a_0 + a_1 + a_2 + a_3 + ...$$

having the finite partial sums

$$s_n = a_0 + a_1 + a_2+...+a_n$$

the Cesaro-sum is defined as:

$$s_{oo} = \lim_{n->oo} mean(s_0,s_1,s_2,...) = \lim_{n->oo} 1/(n+1)*\Sigma_{k=0..oo} s_k$$

For the current example this is

$$s_{oo} = \lim_{n->oo} 1/(n+1)*sum(1,0,1,0,....) =\lim_{n->oo} 1/(n+1)*n/2 =\lim_{n->oo} 1/(1+1/n)*1/2 = 1/2$$

In matrix notation this is, using **D** as partial-sum-operator, and **J** for the vector of 1 with alternating signs,

a) Construction of partial sums:

$$\underset{D}{\begin{vmatrix} 1 & . & . & . \\ 1 & 1 & . & . \\ 1 & 1 & 1 & . \\ 1 & 1 & 1 & 1 \end{vmatrix}} * \underset{J}{\begin{vmatrix} 1 \\ -1 \\ 1 \\ -1 \end{vmatrix}} = \underset{S}{\begin{vmatrix} 1 \\ 0 \\ 1 \\ 0 \end{vmatrix}}$$

b) Averaging is performed in two steps:

b.1.1) summing to sum of partial sums p

$$\underset{D}{\begin{vmatrix} 1 & . & . & . \\ 1 & 1 & . & . \\ 1 & 1 & 1 & . \\ 1 & 1 & 1 & 1 \end{vmatrix}} * \underset{S}{\begin{vmatrix} 1 \\ 0 \\ 1 \\ 0 \end{vmatrix}} = \underset{p}{\begin{vmatrix} 1 \\ 1 \\ 2 \\ 2 \end{vmatrix}}$$

b.1.2) dividing by number of element used for summing

$$\underset{Z1}{\begin{vmatrix} 1 & . & . & . \\ . & 1/2 & . & . \\ . & . & 1/3 & . \\ . & . & . & 1/4 \end{vmatrix}} * \underset{p}{\begin{vmatrix} 1 \\ 1 \\ 2 \\ 2 \end{vmatrix}} = \underset{s}{\begin{vmatrix} 1 \\ 1/2 \\ 2/3 \\ 1/2 \end{vmatrix}}$$

or shorter

b.2)

$$\underset{m}{\begin{vmatrix} 1 & . & . & . \\ 1/2 & 1/2 & . & . \\ 1/3 & 1/3 & 1/3 & . \\ 1/4 & 1/4 & 1/4 & 1/4 \end{vmatrix}} * \underset{S}{\begin{vmatrix} 1 \\ 0 \\ 1 \\ 0 \end{vmatrix}} = \underset{s}{\begin{vmatrix} 1 \\ 1/2 \\ 2/3 \\ 1/2 \end{vmatrix}}$$

$$\lim_{n\to oo} S_n = 1/2$$

The whole process in one matrix-formula is then

$$\lim_{n\to oo} = C * J$$
$$= \lim_{n\to oo} ({}_dZ(1) * D) * (D * J)\ [n]$$

where *[n]* indicates the element in the *n'th* row

$$\underset{C}{\begin{vmatrix} 1 & . & . & . & . & . \\ 1 & 1/2 & . & . & . & . \\ 1 & 2/3 & 1/3 & . & . & . \\ 1 & 3/4 & 1/2 & 1/4 & . & . \\ 1 & 4/5 & 3/5 & 2/5 & 1/5 & . \\ 1 & 5/6 & 2/3 & 1/2 & 1/3 & 1/6 \end{vmatrix}} * \underset{J}{\begin{vmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{vmatrix}} = \underset{e}{\begin{vmatrix} 1 \\ 1/2 \\ 2/3 \\ 1/2 \\ 3/5 \\ 1/2 \end{vmatrix}}$$

$$\lim_{n\to oo} S_n = \ldots 1/2$$

For the finite case it looks a bit mysterious, that the obvious different weighting of the elements of **J** for the computation of the means (highest (=*1*) for the first element, and decreasing with constant progression of *1/n*) does not affect the limit itself.

3.1.2. Principle of Euler-summation

The principle is the same as in Cesaro-summation: we search a reasonable mean for the partial-sums; only here the mean is using weights from the binomial-coefficients.

a) Construction of partial sums:

$$\underset{D}{\begin{vmatrix} 1 & . & . & . \\ 1 & 1 & . & . \\ 1 & 1 & 1 & . \\ 1 & 1 & 1 & 1 \end{vmatrix}} * \underset{J}{\begin{vmatrix} 1 \\ -1 \\ 1 \\ -1 \end{vmatrix}} = \underset{S}{\begin{vmatrix} 1 \\ 0 \\ 1 \\ 0 \end{vmatrix}}$$

b) Averaging is performed in two steps:

b.1.1) summing to binomially weighted sum of partial sums p

$$\underset{P}{\begin{vmatrix} 1 & . & . & . \\ 1 & 1 & . & . \\ 1 & 2 & 1 & . \\ 1 & 3 & 3 & 1 \end{vmatrix}} * \underset{S}{\begin{vmatrix} 1 \\ 0 \\ 1 \\ 0 \end{vmatrix}} = \underset{p}{\begin{vmatrix} 1 \\ 1 \\ 2 \\ 4 \end{vmatrix}}$$

b.1.2) dividing by row-sums of **P**

$$\underset{1/2}{\begin{vmatrix} 1 & . & . & . \\ . & 1/2 & . & . \\ . & . & 1/4 & . \\ . & . & . & 1/8 \end{vmatrix}} * \underset{p}{\begin{vmatrix} 1 \\ 1 \\ 2 \\ 4 \end{vmatrix}} = \underset{s}{\begin{vmatrix} 1 \\ 1/2 \\ 1/2 \\ 1/2 \end{vmatrix}}$$

or shorter

b.2) using *m = dV(1/2)*P* implements means with binomial-weighted values

$$\underset{m}{\begin{vmatrix} 1 & . & . & . \\ 1/2 & 1/2 & . & . \\ 1/4 & 1/2 & 1/4 & . \\ 1/8 & 3/8 & 3/8 & 1/8 \end{vmatrix}} * \underset{S}{\begin{vmatrix} 1 \\ 0 \\ 1 \\ 0 \end{vmatrix}} = \underset{S}{\begin{vmatrix} 1 \\ 1/2 \\ 1/2 \\ 1/2 \end{vmatrix}}$$

$$\lim_{n\to\infty} S_n = 1/2$$

The whole process in one matrix-formula is then

> $\lim_{n\to\infty} E\sim * J$
> $= \lim_{n\to\infty} ({}_dV(1/2) * P) * (D * J)[n]$
> where *[n]* indicates the element in the n'th row

$$\underset{E}{\begin{vmatrix} 1 & . & . & . \\ 1 & 1/2 & . & . \\ 1 & 3/4 & 1/4 & . \\ 1 & 7/8 & 1/2 & 1/8 \end{vmatrix}} * \underset{J}{\begin{vmatrix} 1 \\ -1 \\ 1 \\ -1 \end{vmatrix}} = \underset{S}{\begin{vmatrix} 1 \\ 1/2 \\ 1/2 \\ 1/2 \end{vmatrix}}$$

$$\lim_{n\to\infty} S_n = \ldots 1/2$$

The convergence is much clearer than with the Cesaro-method. This indicates a difference in the power of the both methods.

With the Euler-summation "more divergent" series can be summed compared to the Cesaro-method. Assume the vector **J** was just **V***(-1)*, and we could try to sum **V***(-2)*, which has stronger divergence than **V***(-1)*:

a) Construction of partial sums:

$$\underset{D}{\begin{vmatrix} 1 & . & . & . \\ 1 & 1 & . & . \\ 1 & 1 & 1 & . \\ 1 & 1 & 1 & 1 \end{vmatrix}} * \underset{2}{\begin{vmatrix} 1 \\ -2 \\ 4 \\ -8 \end{vmatrix}} = \underset{S}{\begin{vmatrix} 1 \\ -1 \\ 3 \\ -5 \end{vmatrix}}$$

b) Averaging is performed in two steps:

b.1.1) summing to binomially weighted sum of partial sums p

$$\underset{P}{\begin{vmatrix} 1 & . & . & . \\ 1 & 1 & . & . \\ 1 & 2 & 1 & . \\ 1 & 3 & 3 & 1 \end{vmatrix}} * \underset{S}{\begin{vmatrix} 1 \\ -1 \\ 3 \\ -5 \end{vmatrix}} = \underset{p}{\begin{vmatrix} 1 \\ 0 \\ 2 \\ 2 \end{vmatrix}}$$

b.1.2) dividing by row-sums of **P**

$$\underset{1/2}{\begin{vmatrix} 1 & . & . & . \\ . & 1/2 & . & . \\ . & . & 1/4 & . \\ . & . & . & 1/8 \end{vmatrix}} * \underset{p}{\begin{vmatrix} 1 \\ 0 \\ 2 \\ 2 \end{vmatrix}} = \underset{S}{\begin{vmatrix} 1 \\ 0 \\ 1/2 \\ 1/4 \end{vmatrix}}$$

$$\lim_{n\to\infty} S_n = 1/3$$

or shorter

b.2) using *m = dV(1/2)*P* implements means with binomial-weighted values

$$\underset{m}{\begin{vmatrix} 1 & . & . & . \\ 1/2 & 1/2 & . & . \\ 1/4 & 1/2 & 1/4 & . \\ 1/8 & 3/8 & 3/8 & 1/8 \end{vmatrix}} * \underset{S}{\begin{vmatrix} 1 \\ -1 \\ 3 \\ -5 \end{vmatrix}} = \underset{S}{\begin{vmatrix} 1 \\ 0 \\ 1/2 \\ 1/4 \end{vmatrix}}$$

$$\lim_{n\to\infty} S_n = 1/3$$

where the convergents of **S** from $\mathbf{S}_9$ to $\mathbf{S}_{16}$ are

… … 0.3359 0.3320 0.3340 0.3330 0.3335 0.3333 0.3334 0.3333 … …

If even stronger divergent series *(***V***(-n), where n>2)* shall be summed, then the Euler-summation can be repeatedly applied. It was shown, that the concept of repeated application and generalizations for fractional and complex parameters s in *V(-s)* the Eulersummation can be used for geometric series for *V(-s)* in the whole complex half-plane when *real(s)<1*.

3.2. A more direct approach using the binomialmatrix

3.2.1. The key idea

In the previous chapter I reminded the reader, that the limit of a divergent series is assigned by the limit of the mean of the partial sums of the series. This is a reasonable assumption and my discussion here may be a fall back compared to this concept, which proved to be useful, meaningful and is developed in much depth. Anyway, the binomial-matrix gives an idea, how the partial sums possibly can be left out and still a value can meaningfully be assigned to such a series. With my current study I considered mostly geometric series, and the results are fully compatible with the partial-sum-approach.

The key idea is, in a divergent summation of a series, expressed as vector **A**, (or currently in a powerseries **V***(x)*),

$$lim\ s = lim\ E\sim * A = lim\ E\sim * V(x)$$

to rewrite the summing operator **E~** by a product of two matrices **Q*P=E~** , of which **E~** is the result, and where the matrices **Q*P** are chosen in a way, that the reordering of summation-operations will imply only convergent summation. Say, to replace

$$E\sim * A = (Q\sim * P) * A$$

where **P*A =B** involves only convergent summation, and then also **Q* B** again involves only convergent summation, and then to assign the result **B = P*A**, *s* = **Q~*B** as value for the divergent expression *lim (***Q~ * P***)* *** A** = *lim* **E~ * A** = *lim s*. That this is possible for some families of series is the focus of the following chapter.

3.2.2. Using P with a powerseries

Let's recall the chapter discussing the matrix **P.**

It was stated, that

$$P*V(n) = V(n+1)$$

and also

$$1/2*V(1/2) \sim * P = E\sim$$

Now since $[1,-1,1,-1,...] = J$ and we want to sum this J this would be

$$E\sim * J = ??$$

But since also

$$E\sim = 1/2*V(1/2) * P$$

we have

$$\begin{aligned} E\sim \quad\quad\quad\quad & * J = ?? \\ (1/2*V(1/2) * P\ ) & * J = ?? \end{aligned}$$

Since the indexes for the summation are fixed by the matrix-notation, we can operate algebraically with the parentheses and rewrite

$$1/2*V(1/2) * (P * J) = ??$$

From the chapter above we know that (since **J** = **V***(-1)*)

$$P * J = P * V(-1) = V(0) = I_0$$

so we have

$$\begin{aligned} E\sim * J &= 1/2*V(1/2)\sim * (P * J) \\ &= 1/2*V(1/2)\sim * I_0 \\ &= [1/2,\ 1/4,\ 1/8,\ ...] * [1, 0, 0, 0,...]\sim = 1/2 \end{aligned}$$

and this looks much more straightforward than the Cesaro-or Euler-method, where the result is still a series-approximation even in the simple cases of pure powerseries.

The technique is, to find a multiplicative expression, which reduces the divergent series to a convergent series and to add another expression, which "inverts" the effect of that multiplication, while keeping the operands finite. If such a pair of matrix-multiplications can be found, then the divergent series can be summed.

The problem of the "dominating" nature of the applied matrix-multiplication is then related to the problem of "radius of convergence" of such a summation-method.

Let's see, how the alternating, and also absolutely divergent series $\Sigma_{k>=0} (-2)^k = ??$, in matrix-notation

$$\lim [1,1,1,1,\ldots]*[1,-2,4,-8,\ldots]\sim = ??$$
$$\lim \quad E\sim \quad * V(-2) \quad = ??$$

can be summed.

We set up the problem in matrix-notation as follows:

$$\lim E\sim * V(-2) = ??$$

Now we look for an expression for **E~** involving the matrix **P** like in the previous example. We know that

$$1/s*V(1/s)\sim * P^{s-1} = E\sim$$

valid for any $s>0$. On the other hand we know, that a power of **P** dominates a powerseries, represented in a **V**( )-vector, having

$$P^{-s} *V(s) = V(0)$$

or for the current example

$$P^2 *V(-2) = V(0)$$

We setup the equation and expect

$$\begin{aligned} E\sim * V(1-s) &= 1/s*V(1/s)\sim * P^{s-1} V(1-s) \\ &= 1/s*V(1/s)\sim * V(0) \\ &= 1/s \end{aligned}$$

provided, that the partial terms, as appropriately parenthezised, involve in at least one direction of the computation only convergent sums.

With the example of $1-s=-2$ we have $s=3$ and try with the equation:

$$\begin{aligned} E\sim * V(-2) &= (1/3*V(1/3)\sim P^2) * V(-2) \\ &= 1/3*V(1/3)\sim * (P^2 V(-2)) \\ &= 1/3*V(1/3)\sim * \quad V(0) \\ &= 1/3 \end{aligned}$$

which involves only convergent sums and gives a result, which is consistent with

$$\Sigma_{k=>0} s^k = 1/(1-s) = 1/(1-(-2)) = 1/3$$

Note, that in the leftmultiplication of **V**$(1/3)$ with $\mathbf{P}^2$ the convergent series $\Sigma_{k>=0} (2/3)^k = x$ occurs, which results to the value $x=3$.

3.2.3. Limit of the method in terms of range of convergence

As mentioned in the previous chapter, not all powerseries can be summed that way. This lack of power shares it with any method, be it Cesaro, Abel, Euler or Borel-summation. But it seems, that all these summations can be reformulated in the terms, I'm considering here. Konrad Knopp just starts an article with a comment like: *"for a summation method - just take any matrix, which suffices.."*

The crucial step in the matrix-formulation of the various method is, to find a pair of **V**$(1/s)\sim$ * $\mathbf{P}^{s-1}$, which is known to sum to **E**~ and on the other hand $\mathbf{P}^{s-1}$ dominates **V**$(1-s)$. If $s<0$ then the matrix-multiplication produces divergent expressions, and the outcome of a vector $E\sim$ is not provided by this method alone. Surely it is *known* that the summation of a geometric series is valid for all complex numbers except $s=1$. So if only an aproximation tool for some real-life-data is sought, one might apply this method. But looking at the method alone, one has to state, that the range of convergence is limited this way.

This range of convergence depends on $1/s$ in the formula $V(1/s)\sim * P^{s-1}$ and limits s to $s>0$. But since we ask for the summation of the powerseries $E\sim * V(1-s)$ in the rhs of the formula, we should replace $(1-s)$ by t and instead of

$$1/s\ V(1/s)\sim P^{s-1}\ V(1-s) = ??$$

we should write:

$$1/(1-t)\ V(1/(1-t))\sim P^{-t}\ V(t) = ?? \qquad // t<1$$

The range of convergence in terms of s is $s>0$, in terms of t this means: the range of convergence for this method is $t<1$, as fas as summation of $V(t)$ is intended.

An example, which is not summable:

If $1-s>=1/2$ this is not so. Then $s<=1/2$. Let's rewrite the above equation for $t=1-s$:

$$E\sim * V(1-s) = 1/s*V(1/s)\sim \quad * P^{s-1}\ V(1-s)$$
$$E\sim * V(t) \quad = 1/(1-t)*V(1/(1-t))\sim * P^{-t}\quad V(t)$$

Then, for this example we have $t=2$ and:

$$E\sim * V(2) \quad = (1/(-1)*V(1/(-1))\sim * P^{-2}\ )\ V(2)$$

which is for the parenthese, for instance for the first column on $\mathbf{P}^{-2}$:

$$-[1,-1,1,-1,1,...] * [1,-2,4,-8,16,-32,...]\sim = -E\sim*[1,2,4,8,16,....]\sim = -\Sigma_{k>=0} 2^k$$

which is the same divergent series, whose value we just want to compute.

But now we can look to the problem from the opposite site.

$$2*V(2)\sim*P^{1/2-1}*V(1-1/2) = (2*V(2)\sim * P^{-1/2}\ )* V(1+1/2)$$

and the first column of the matrix generated by the left parenthese gives the summation-problem

$$2*[1,2,4,8,16,...]*[1,-1/2,1/4,-1/8,...]\sim = 2[1,1,1,1,1,..]*[1,-1,1,-1,...]\sim = x$$

which we already solved with $x=2*1/2 = 1$. For all entries we get the same result, so that we know, that for the whole parenthese we have the value

$$(2*V(2)\sim * P^{-1/2}\ ) = E\sim$$

and the right multiplicator is **V**$(3/2)$, which leads then to the problem of summing **V**$(3/2)$.

3.2.4. Summing zeta-type series with the binomialmatrix P resp Pj

Things are not so obvious, if series of the zeta-type shall be summed with that technique. So the question may be asked: can something summed like

*lim E~ * Z(s) = ??*

using

$(x\ V(x) * P^y) * Z(s) = ??$

as before?

Helmut Hasse gave an impressing example, how this can also in the divergent case be summed to known number-theoretical values.

First he states, that arithmetic progressions like in **Z**(-1) or of higher degree **Z**(-2),**Z**(-3),... are limited by application of the binomial-theorem with an appropriate degree. To make things short, this means, that for

*Pj * Z(-1), Pj * Z(-2), ...*

only a finite number of relevant terms *(<>0)* occur.

Translated this to the matrix scheme one may construct the matrix **ZV** by concatenating all *Z(-n)*-vectors of *n>=0*.

ZV := Z(0) || Z(-1) || Z(-2) || ... = {V(0)~ ,V(1)~, V(2)~ ,..}

ZV

1	1	1	1	1	...
1	2	4	8	16	...
1	3	9	27	81	...
1	4	16	64	256	...
...					...

Then the matrixmultiplication would be set up:

Z(1)~ Pj * ZV = result*

Pj

1	0	0	0	0	...
1	-1	0	0	0	...
1	-2	1	0	0	...
1	-3	3	-1	0	...
1	-4	6	-4	1	...
1	-5	10	-10	5	...

Pj * ZV

1	1	1	1	1	...
0	-1	-3	-7	-15	...
0	0	2	12	50	...
0	0	0	-6	60	...
0	0	0	0	24	...
0	0	0	0	0	...

Z(1)	[1, 1/2, 1/3, 1/4 1/5...]	=	1 1/2 1/6 0 -1/30 ...	*result = B~*

where the resulting vector is just the row-vector of Bernoulli-numbers, call it **B~**.

In his article he proceeds then to prove, that not only zeta-series with negative integer exponents lead to the bernoulli-numbers, but that even using any complex exponent the appropriate zeta-value occurs, thus (re-)proving the zeta/bernoulli-relation.

$Z(1)\sim * Pj * Z(-n) = \beta_n$ *// for natural numbers n*

*Z(1)~ * Pj * Z(-s) = s*zeta(1+s) = s* E~ *Z(1-s)* *// for any complex number*

proving the identity:

$-n*\zeta(1-n) = \beta_n$

From the view of the matrix-summation this is an interesting result, since the summation of **Z**(1)~ ***P**$_j$ involves divergent series; to compute **Z**(1)~ with the first column of **P**j this would mean to sum $\zeta(1)$.

In a comment to a preprint of the article, Konrad Knopp had then mentioned (according to the footnote of H.Hasse in his printed article), that he already had a very similar result in using *V(1/2)~* instead of *Z(1)~*:

$1/2\ V(1/2)\sim * Pj * Z(-n) = \eta(-n) = \varepsilon_n$

where $\eta(n)$ is the alternating zeta-series.

For the arguments, which Helmut Hasse used, I call the above upper-triangular matrix **P***j****ZV** "Delta"-matrix **D**; and with this delta-matrix this specific approach can even be a bit continued.

3.2.5. A byproduct of the Hasse-summation

The continuation goes in two aspects.

1) What else is known about the matrix **D**?

2) could the above matrix-multiplication-scheme be continued to more than one single row,

say, for instance at the rhs in

$Z(1)\sim * Pj * ZV = B\sim$

to, say, to a system involving **G***p~*, since **B**~ is just the first row of **G***p*~ ?

The latter is easily confirmed: just set ($Z_d(1)$ indicating the diagonalmatrix containing elements of $Z(1)$):

$Pj * ZV = D\sim$
$X\sim * Z_d(1) * D\sim = Gp\sim$
$X = Z_d(-1) * D^{-1} * Gp$

Now aspect 1) becomes interesting: what else is known about matrix **D**?

D is obviously a scaled and signed version of the *Stirling* matrix of the *second kind (*$\mathbf{St}_2$*)*, such that with a diagonalmatrix $\mathbf{F}_d$ containing the factorials *[0!,1!,2!,...]* in the diagonal, we have

$D = St_2 * F_d * J_d$

Since this is only a column-scaling and signing of $\mathbf{St}_2$, matrix **D** is thus also an eigenmatrix of $\mathbf{G}_p$, such that

$Gp = D * Z_d(1) * D^{-1}$

and the previous equation can be simplified:

$X = Z_d(-1) * D^{-1} * Gp$
$= Z_d(-1) * D^{-1} * D * Zd(1) * D^{-1}$
$= D^{-1}$

(where also $\mathbf{D}^{-1}$ is just a scaled version of the matrix $\mathbf{St}_1$, the triangular matrix containing the *Stirling* numbers of the *first kind*, which is just the inverse of $\mathbf{St}_2$).

So the scheme, which Helmut Hasse employs to lead to the zeta-summation and the connection of zeta with bernoulli-numbers, can be understand as just the first row of a matrix, which exhibits another relation between well known number-theoretic matrices: here with the eigensystem of the **G**p-matrix.

The Hasse-model

$Z(1)\sim Pj\ ZV = B\sim$

can thus consistently be extended to

$D^{-1}\sim * Z_d(1) * D\sim = Gp\sim$
or, transposed:
$D * Z_d(1) * D^{-1} = Gp$

and decomposed in its stirling-components:

$St_2 * F_d * J_d \quad * Z_d(1) * J_d * F_d^{-1} * St_1 = Gp$

or

$ZV\sim * Pj\sim \quad * Z_d(1) * J_d * F_d^{-1} * St_1 = Gp$

$ZV\sim * Pj\sim \quad * Z_d(1) * \quad D^{-1} = Gp$

In the last form inserting a rowvector **Z***(s)~* instead of the complete **ZV**~ matrix resembles Hasses result:

$Z(s)\sim * Pj\sim \quad * Z_d(1) * \quad D^{-1} = X\sim \ = [\ s*\zeta(s+1),\ \ ??,\ ??,\ \ ??,\ \ldots]$

where **X**~ is a row vector whose first entry is $s*\zeta(s+1)$. and possibly can interpreted as a continuous interpolation of the (discrete indexed) rows of **G***p*-matrix.

3.3. Excurs: a P-similar matrix of higher order

3.3.1. Matrix Q ("Laguerre"-Matrix)

Definition:

Remember that **P** could be defined as matrix-exponential of a subdiagonal-matrix containing natural numbers:

P := exp(sd(-1, x)) *// filling consecutive numbers x into the first subdiagonal, starting at x=1*

Now let's define a matrix **Q** using x^2 instead of x:

Q := exp(sd(-1, x^2)) *// filling x^2 into the first subdiagonal, starting at x=1*

1	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0
2	4	1	0	0	0	0	0
6	18	9	1	0	0	0	0
24	96	72	16	1	0	0	0
120	600	600	200	25	1	0	0
720	4320	5400	2400	450	36	1	0
5040	35280	52920	29400	7350	882	49	1

The structure of **Q** is simply a scaling of **P** by factorials and inverse of factorials:

$$Q = {}_dF * P * {}_dF^{-1}$$

The row-scaled and columns-signed version

$$Laguerre = {}_dF^{-1} * Q * {}_dJ = Pj * {}_dF^{-1}$$

is also known as "lower triangular matrix of the laguerre-polynomials", (see mathworld and OEIS) and involved in procedures for summation of divergent series occuring in quantum-physics.

The inverse

The inverse is, since **Q** itself is only a similarity-scaling of the **P**-matrix, the row and column-signed copy of **Q** itself:

$$Q^{-1} = {}_dJ*Q*{}_dJ$$

Eigensystem of Qj:

The Eigenvalues of **Q***j* $(:= Q * {}_dJ)$ are the entries of the diagonal, they are all alternating 1 or -1:

The Eigenmatrix: $\mathbf{Q}j_{ew}$ (one arbitrary version, since all eigenvalues are ±1):

1	0	0	0	0	0	0	0
1/2	1/4	0	0	0	0	0	0
1/3	1/2	1/9	0	0	0	0	0
0	3/4	1/2	1/16	0	0	0	0
-4/5	0	4/3	1/2	1/25	0	0	0
0	-5	0	25/12	1/2	1/36	0	0
120/7	0	-20	0	3	1/2	1/49	0
0	210	0	-245/4	0	49/12	1/2	1/64

is such that

$$Qj \;*\; Qj_{ew} = Qj_{ew} * {}_dJ$$

It occurs, that the eigenmatrix has fractional entries in the first column for indexes *2k*, *iff 2k+1 is prime* and the denominator is just that prime *2k+1*. This was recently proved in a thread in the news-group sci.math using the Clausen-von Staudt theorem and the Wilson-prime-theorem.

The overall structure is a scaling of **G***p* in the same manner as **Q** is a scaling of **P**:

$$Qj_{ew} = {}_dF * Gp * {}_dF^{-1}$$

since

$$Qj \quad * \quad Qj_{ew} = Qj_{ew} \quad * J$$

and

$${}_dF * P * {}_dF^{-1} \quad * {}_dF\, Gp * {}_dF^{-1} = {}_dF\, Gp * {}_dF^{-1} * J$$
$${}_dF * P \quad * \quad Gp * {}_dF^{-1} = {}_dF\, Gp * {}_dF^{-1} * J$$
$$P \quad * \quad Gp * {}_dF^{-1} = \quad Gp * {}_dF^{-1} * J$$
$$P \quad * \quad Gp \quad = \quad Gp \quad * J$$

Summation properties of Q and Qj :

Sinc it is

$${}_dF^{-1} * Q * {}_dF = P$$

it follows, that

$${}^1/_2 * V({}^1/_2)\sim * {}_dF * Q * {}_dF = E\sim$$

and the expression

$${}^1/_2 * V({}^1/_2)\sim * {}_dF * Q * X = sum$$

performs summing of a vector **X**, such that in fact $x_k/k!$ is summed:

$$F^{-1} * X = sum$$

For instance

$$({}^1/_2 * V({}^1/_2) * {}_dF^{-1} * Q) \quad * \quad V(1) = sum\ ???$$
$$= ({}^1/_2 * V({}^1/_2) * {}_dF^{-1} * Q * {}_dF^{1}) * ({}_dF^{-1} * V(1))$$
$$= \quad E\sim \quad * {}_dF^{-1} * V(1)$$
$$= F^{-1} * V(1)$$
$$= \Sigma_{k>=0}\, {}^1/_{k!}$$
$$= e$$

and generally

$$({}^1/_2 * V({}^1/_2) * {}_dF^{-1} * Q) \quad * \quad V(s)$$
$$= F^{-1} * V(s)$$
$$= e^s$$

Summing the bernoulli-numbers, for instance in $\mathbf{B}_m = \mathbf{G}_m[*,0]$ gives:

$$({}^1/_2 * V({}^1/_2) * {}_dF^{-1} * Q) \quad * \quad B_m$$
$$= E\sim * {}_dF^{-1} * B_m$$
$$= (e-1)^{-1}$$

3.3.2. Generalizing the P, Q-concept

The matrix **Q** was just an ad-hoc-invention, which was later found to be exactly the already known Laguerre-matrix. But in the logarithmmatrix $\mathbf{L}_p$, whose exponential is **P**, to set squares of natural numbers instead of the numbers themselves is an eye-opener: why not examine the behaviour of such matrices with general powers of n?

The first candidate is then the matrix, where the zero'th power is inserted, thus a vector of all ones *[1,1,1,1,...]* is set into the first subdiagonal.

Well- another generalization could be to use the second subdiagonal and insert the second column of the Pascal-triangle *[1,3,6,10,...]*. One finds then coefficients, which occur, when the Gaussian error-function is analyzed and the derivatives are computed. I won't discuss this here, but that is a much interesting connection, estabilishing a very interesting family of basic number-theoretic triangles and coefficients...

The entries of the matrix $\mathbf{P}_0$

Using the zero'th powers of N, say **Z***(0)* in the first subdiagonal of $\mathbf{L}_p$, and exponentiate this, lets us express the pascal matrix as $\mathbf{P}_1$ and the new matrix as $\mathbf{P}_0$. The Laguerre-matrix is then $\mathbf{P}_2$ and so on (I didn't check for occurences of the higher order $\mathbf{P}_q$ matrices for existence in articles yet).

Then, while computing the matrixdiagonal all nominators are *1*, and the denominators are the reciprocals of the factorials, in GP/Pari-code, for a finite dimension n

```
P0 = matrix(n,n, x, y, if(x<=y,1/(x-y)!,0))
```

and this is just the result, if $\mathbf{P}_1$ is post- and premultiplied by the diagonalmatrix of factorial- resp reciprocals of the factorials:

$$P_0 = {}_dF^{-1} * P * {}_dF$$

Now the first column of $\mathbf{P}_0$ contains the vector *[1, 1/1!, 1/2!, 1/3!,...]~ ,* the diagonal contains the identity-matrix, and the other columns contain some compositions of these reciprocal factorials. Precisely it is

$$\mathbf{P}_q\,[row,column] = \mathbf{P}_q[r,c] = \frac{1}{r-c} * \left(\frac{r!}{c!}\right)^q$$

where an important property is:

$\mathbf{P}_q[r,c] = 0 \quad for\ (r\text{-}c)<0$
$\mathbf{P}_q[r,c]= 1 \quad for\ (r\text{-}c)=0$

which is given by the construction of the triangular form of the matrices.

The inverse

This factorial scaling is just a similarity-transform, thus it's inverse is, like the inverse of **P** just pre- and postmultiplication by ${}_d\mathbf{J}$.

$$P_0^{-1} = {}_dJ * P_0 * {}_dJ$$

Summation-properties by leftmultiplication with a vector

We have that

$$\mathbf{P}_q\,[row,column] = \mathbf{P}_q[r,c] = \frac{1}{r-c} * \left(\frac{r!}{c!}\right)^q$$

inserting *0* and *1* when *r-c<0* or *r-c=0*

Example: For *q=1* this gives the usual Pascal-matrix **P** (= $\mathbf{P}_1$), since these are just the binomial-coefficients, filling the matrix in a triangular shape

For $q=0$, giving $\mathbf{P}_0$ this is for each column the reciprocal of the factorial, shifted by one row for each subsequent column, again with each entry being zero, if $c>r$.

Now the columnwise-summing property can be derived explicitely. Let, for instance, consider the vector V(x)~ as leftmultiplicator in this paragraph, and thus define a function $f(x)$ as

$$f(x) = V(x)\sim * E = \Sigma_{k=0..oo}\, x^k = 1 + x + x^2 + x^3 + \dots.$$

and the question is, for any q and x:

$$V(x)\sim * P_q = ??$$

In the above matrixmultiplication the index k of summation is the current index for the row of the righthand matrix $\mathbf{P}_q$, so we replace r by k in the formula:

$$V(x)\sim * P_q \quad = \left[\sum_{k=0}^{oo} \frac{x^k}{k!}\left(\frac{k!}{0!}\right)^q \quad \sum_{k=0}^{oo} \frac{x^k}{(k-1)!}\left(\frac{k!}{1!}\right)^q \quad \sum_{k=0}^{oo} \frac{x^k}{(k-2)!}\left(\frac{k!}{2!}\right)^q \quad \dots \quad \sum_{k=0}^{oo} \frac{x^k}{(k-c)!}\left(\frac{k!}{c!}\right)^q \quad \dots \right]$$

Using P0

For $q=0$, adressing $\mathbf{P}_0$, the cofactors with a power of q disappear and we have:

$$= \left[\sum_{k=0}^{oo} \frac{x^k}{k!} \quad \sum_{k=0}^{oo} \frac{x^k}{(k-1)!} \quad \sum_{k=0}^{oo} \frac{x^k}{(k-2)!} \quad \dots \quad \sum_{k=0}^{oo} \frac{x^k}{(k-c)!} \quad \dots \right]$$

Adapting the fractions by extraction of powers of x:

$$= \left[\sum_{k=0}^{oo} \frac{x^k}{k!} \quad x\sum_{k=0}^{oo} \frac{x^{k-1}}{(k-1)!} \quad x^2\sum_{k=0}^{oo} \frac{x^{k-2}}{(k-2)!} \quad \dots \quad x^c\sum_{k=0}^{oo} \frac{x^{k-c}}{(k-c)!} \quad \dots \right]$$

and since for $k<c$ the elements of the sum are zero, all these sums are just the same as that of column 0, since we can replace k by $k+c$ and get

$$= \left[\sum_{k=0}^{oo} \frac{x^k}{k!} \quad x\sum_{k=-1}^{oo} \frac{x^k}{k!} \quad x^2\sum_{k=-2}^{oo} \frac{x^k}{k!} \quad \dots \quad x^c\sum_{k=-c}^{oo} \frac{x^k}{k!} \quad \dots \right]$$

$$= \left[exp(x) \quad x\, exp(x) \quad x^2 exp(x) \quad \dots \quad x^c exp(x) \quad \dots \right]$$

$$= exp(x) * \left[1 \quad x \quad x^2 \quad \dots \quad x^c \quad \dots \right]$$

So we have the simple result

$$V(x)\sim * P_0 \quad = V(x)\sim * exp(x)$$

which means, that any vector V(x) (representing a powerseries) is a left eigenvector of $\mathbf{P}_0$. Since exp(s) converges for any complex s, this relation holds for the whole complex plane.

Using P_1

For $\mathbf{P} = \mathbf{P}_1$, where $q=1$, and again with $f(x) = \mathbf{V}(x)\sim * \mathbf{E} = 1 + x + x^2 + x^3 + \dots$ we get

$$V(x)\sim * P_1 \quad = \left[\sum_{k=0}^{oo} \frac{x^k}{k!}\frac{k!}{0!} \quad \sum_{k=0}^{oo} \frac{x^k}{(k-1)!}\frac{k!}{1!} \quad \sum_{k=0}^{oo} \frac{x^k}{(k-2)!}\frac{k!}{2!} \quad \dots \quad \sum_{k=0}^{oo} \frac{x^k}{(k-c)!}\frac{k!}{c!} \quad \dots \right]$$

The factorials cancel out

$$= \left[\sum_{k=0}^{oo} x^k \quad \frac{x}{1!}\sum_{k=0}^{oo} k\, x^{k-1} \quad \frac{x^2}{2!}\sum_{k=0}^{oo} k(k-1)x^{k-2} \quad \dots \quad \frac{x^c}{c!}\sum_{k=0}^{oo} k!\, x^{k-c} \quad \dots \right]$$

which makes it more obvious now, that an important feature is, that each column contains the derivative of the previous one as a cofactor, which was not obvious from the $\mathbf{P}_0$-case.

We can now write:

$$= \left[f(x) \quad \frac{x}{1!} f'(x) \quad \frac{x^2}{2!} f''(x) \quad \dots \quad \frac{x^c}{c!} f^{(c)}(x) \quad \dots \quad \right]$$

and if we know the value of the derivatives of *f(x)*, we have known results in all columns.

For the convergent case ($|x|<1$) we have

$$f(x) \quad = \quad (1-x)^{-1}$$

and the *c'th* derivative of *f(x)* at point *x* is:

$$f^{(c)}(x) \quad = \frac{c!}{(1-x)^{c+1}}$$

This gives:

$$V(x)\sim * P_1 \quad = \quad \left[\frac{1}{1-x} \quad \frac{x}{1!} \frac{1!}{(1-x)^2} \quad \frac{x^2}{2!} \frac{2!}{(1-x)^3} \quad \dots \quad \frac{x^c}{c!} \frac{c!}{(1-x)^{c+1}} \quad \dots \right]$$

or

$$= \frac{1}{1-x} \left[\quad 1 \quad \frac{x}{(1-x)} \quad \frac{x^2}{(1-x)^2} \quad \dots \quad \frac{x^c}{(1-x)^c} \quad \dots \right]$$

and finally, as expression in matrix-notation:

$$V(x)\sim * P_1 \quad = 1/(1-x) * V(x/(1-x))$$

For powers of **P,** say the *r'th* power $\mathbf{P}^r$ we get, (as far as convergence can be assumed):

$$s \cdot \mathbf{V}(s) \sim * \mathbf{P}_1^r = \frac{s}{1-sr} \cdot \mathbf{V}(\frac{s}{1-sr}) \sim$$

or differently written, replacing *s* by *1/s*:

$$\frac{1}{s} \cdot \mathbf{V}(\frac{1}{s}) \sim * \mathbf{P}_1^r = \frac{1}{s-r} \cdot \mathbf{V}(\frac{1}{s-r}) \sim$$

Selecting $r = s-1$ gives then the summation-vector $\mathbf{V}(1)\sim = E\sim$.

(See also G.H. Hardy, Pg 248 ff, chapter about Hausdorff-means)

Applying the observation about the derivatives, we may also rewrite the $\mathbf{P}_0$-operation using derivatives in the same manner as with $\mathbf{P}_1$, noting that $\mathbf{P}_0 = \mathbf{F}^{-1} * \mathbf{P}_1 * \mathbf{F}$:

$$= \left[f(x) \quad 1! \frac{1}{1!} x \frac{f'(x)}{1!} \quad 2! \frac{1}{2!} x^2 \frac{f''(x)}{2!} \quad \dots \quad c! \frac{x^c}{c!} \frac{f^{(c)}(x)}{c!} \quad \dots \quad \right]$$

Generally, for any matrix $\mathbf{P}_q$, using again *f(x)* as sum of the adressed series

$$f(x)=V(x)\sim * P_q[0] \qquad \text{// adressing the first column of } P_q$$

we have:

$$V(x)\sim * P_q \quad = \left[f(x) \quad \frac{x}{1!^{q-1}} f'(x) \quad \frac{x^2}{2!^{q-1}} f''(x) \quad \frac{x^3}{3!^{q-1}} f^{(3)}(x) \quad \dots \quad \frac{x^k}{k!^{q-1}} f^{(k)}(x) \quad \dots \right]$$

and for the first three q=0,1,2 it is:

$$\begin{bmatrix} V(x)*P_0 \\ V(x)*P_1 \\ V(x)*P_2 \end{bmatrix} = \begin{bmatrix} exp(x)[1 & x & x^2 & x^3 & \cdots & x^k & \cdots] \\ \frac{x}{1-x}*[1 & \left(\frac{x}{1-x}\right)^1 & \left(\frac{x}{1-x}\right)^2 & \left(\frac{x}{1-x}\right)^3 & \cdots & \left(\frac{x}{1-x}\right)^k & \cdots] \\ ??=f(x) & \frac{x}{1!}f'(x) & \frac{x^2}{2!}f''(x) & \frac{x^3}{3!}f^{(3)}(x) & \cdots & \frac{x^k}{k!}f^{(k)}(x) & \cdots] \end{bmatrix}$$

where for higher q closed-form expressions for f(x) and the derivatives must be found to proceed with the here proposed method of summation.

4. Snippets/ Citations

4.1. Relation between Zeta and Eta-function

$$\frac{\zeta(s)}{2^s}=\frac{\eta(s)}{2^s-2}$$

$$(1-\frac{2}{2^s})\zeta(s)=\eta(s)$$

The same relation occurs between the bernoulli-numbers β_n and "eta"-numbers ε_n

4.2. Leonhard Euler: divergent summation

quarum ſummae ſint finitae, atque adeo negatiuae, ſeu nihilo minores. Cum enim fractio $\frac{1}{1-a}$ per diuiſionem in ſeriem euoluta det: $1+a+a^2+a^3+a^4+$ etc. deberet eſſe:

$-1=1+2+4+8+16+$ etc.

$-\frac{1}{2}=1+3+9+27+81+$ etc.

quod aduerſariis non immerito abſurdiſſimum videtur cum per additionem numerorum affirmatiuorum nunquam

"On divergent series" (http://math.dartmouth.edu/~euler/docs/originals/E247.pdf)

4.3. Konrad Knopp über das Eulersche Summierungsverfahren

4. Das *E*-Verfahren. Es knüpft an die sog. *Eulersche Reihentransformation*[7]) an, auf die weiter unten noch näher eingegangen werden soll, und benutzt demgemäß an Stelle des nicht vorhandenen lim s_n den Grenzwert der Folge

$$s_n'=\frac{\binom{n}{0}s_0+\binom{n}{1}s_1+\cdots+\binom{n}{n}s_n}{2^n}$$

zur Limitierung, falls dieser vorhanden ist. Er soll dann als

$$E\text{-}\lim s_n \quad \text{bzw.} \quad E\text{-}\sum a_n$$

bezeichnet werden.

4.4. Borel-summation for columns of Gm :

4.4.1. Konrad Knopp, P 49

2. Das *B*-Verfahren. Von E. Borel[5]), der zuerst diese Dinge unter einem allgemeineren Gesichtspunkte betrachtet hat, rührt das Verfahren her, das an Stelle von $\lim s_n$ den Grenzwert

$$\lim_{x \to +\infty} e^{-x} \cdot \sum_{n=0}^{\infty} s_n \frac{x^n}{n!}$$

benutzt, falls dieser vorhanden ist. Von ihm rührt auch die Bezeichnung *limite généralisée* für einen solcherart gewonnenen Wert her. Wir bezeichnen ihn als

$$B\text{-}\lim s_n \qquad \text{bzw.} \qquad B\text{-}\sum a_n$$

und nennen die Folge demgemäß *B-limitierbar*, die Reihe *B-summierbar* zum Werte s.

4.4.2. Sci.math: what is the sum of

$$\sum_{k=0}^{oo} c(k,n) * \beta_{k-n} = ?$$

Robert Israel in sci.math:

first example; n=1. Use

$$F(z) = \sum_{k=0}^{oo} \beta_k * z^k = \Psi\left(1,1+\frac{1}{z}\right) * \frac{1}{z}$$

then

$$(z F(z))' = \sum_{k=0}^{oo} \beta_k * (k+1) * z^k = -\Psi\left(2,1+\frac{1}{z}\right) * \frac{1}{z^2}$$

$$S(1) = 2(\zeta(3)-1)$$

(In the current notation this means :) $E' * G_m[*,1] = S(1)/2 = \zeta(3)-1$

Generally for any n

$$\frac{d^n}{dz}\left(z^n F(z)\right) = \sum_{k=0}^{oo} \beta_k * (k+n)\dots(k+1) * z^k = (-1)^n \Psi\left(n+1,1+\frac{1}{z}\right) * \frac{1}{z^{n+1}}$$

$$S(n) = (-1)^n \frac{\Psi(n+1,2)}{n!} = (n+1)(\zeta(n+2)-1)$$

(In the current notation this means :)

$E' * G_p[*,n] = S(n)/(n+1) - \beta_1^- + \beta_1^+ = \zeta(n+2)$ // β_1^- taken as $-1/2$, $\beta_1^+ = +1/2$

A single short note:

$\Sigma(\beta_r/r!) = 1/(e-1) = \sim 0.581$

Another Borel sum using bernoulli-numbers:
Am 21.08.2006 00:30 schrieb Gene Ward Smith:
>
> Here's another interesting Borel sum:
>
> `zeta(0) + zeta(-1) + zeta(-2)+... = -1/2 - sum Bn/n`
>
> where zeta(s) is the Riemann zeta function. Then we get
>
>

$$\sum_{k=0}^{oo} \zeta(-k)\frac{x^k}{k!} = \frac{(x-1)e^x+1}{x(e^x-1)} = -\gamma$$

(restated in current notation:) $= -\Sigma_{k>=0}\, \beta_{k+1} /(k+1)! * x^k \; -1/2$

> The Borel sum is

$$\int_0^{oo} \frac{1}{xe^x}\frac{(x-1)e^x+1}{e^x-1}dx = \int_0^{oo}\frac{1}{xe^x}\frac{xe^x-(e^x-1)}{e^x-1}dx$$

$$= \int_0^{oo}\frac{1}{xe^x}\frac{xe^x-(e^x-1)}{e^x-1}dx$$

> $$= \int_0^{oo}\frac{1}{xe^x}\left(\frac{xe^x}{e^x-1}-1\right)dx$$

$$= \int_0^{oo}\left(\frac{1}{e^x-1}-\frac{1}{xe^x}\right)dx$$

$$= -\gamma \qquad \text{//where } \gamma \text{ is the Euler-constant}$$

>

5. Online-Ressources:

This shows how the Pascal-matrix can be constructed with the matrixexponential:
[Helms1] Gottfried Helms (Kassel)
http://go.helms-net.de/math/pdf/PascalDreieckMatrixLog.pdf

The recursion-formula from the definition of the Bernoulli-numbers (Eq. 30 and 31):
[mathworld] Eric Weissstein et al.
http://mathworld/wolfram.com/Bernoullinumbers

The original article of Jakob Bernoulli: (at university of Michigan not always available)
[Mich] *http://www.hti.umich.edu/u/umhistmath/*Bernoulli, Jakob: Wahrscheinlichkeits-rechnung (Ars conjectandi) von Jakob Bernoulli (1713) Uebers. und hrsg. von R. Haussner. (digitalisiert bei:
http://www.hti.umich.edu/cgi/t/text/text-idx?c=umhistmath;idno=ABZ9501

An article about the bernoulli-numbers in the view of a continuous definition of that numbers
[Luschny] : Peter Luschny (Straßburg)/ Hermann Kremer(Darmstadt)
http://www.dsmath.de/archiv/zahlen/BernoulliEuler.pdf

An article about the pascal-matrix:
[Edelman] Alan Edelman & Gilbert Strang, MIT
http://web.mit.edu/18.06/www/pascal-work.pdf

[OEIS_A002425] N.J.A.S. Sloane, "Online encyclopedia of integer sequences"
(*http://www.research.att.com/~njas/sequences/A002425*)

Helmut Hasse:
ARTICLE - PAGE 456
Ein Summierungsverfahren für die Riemannsche ζ-Reihe
Hasse,H.,
In PERIODICAL VOLUME 32
PURL: *http://www-gdz.sub.uni-goettingen.de/cgi-bin/digbib.cgi?PPN266833020_0032*
Mathematische Zeitschrift

Konrad Knopp:
CHAPTER - PAGE 474
§ 59. Allgemeine Bemerkungen über divergente Zahlenfolgen und die Verfahren zu ihrer Limitierung.
In MONOGRAPH
PURL: *http://www-gdz.sub.uni-goettingen.de/cgi-bin/digbib.cgi?PPN378970429*
Theorie und Anwendung der unendlichen Reihen
Knopp,Konrad

G.H Hardy, Divergent Series, Oxford University Press, 1973

A extensive compilation of facts on binomial-identities:
http://binomial.csuhayward.edu/index.html
