Mathematical Miniatures

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Cycles in the Collatz-problem

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Abstract: The possibility of occurence of cycles in the Collatz-problem is discussed. Here I use my approach to the collatz-problem with the means of an exponential diophantine expression. Although I don't arrive at a proof or disproof of cycles I find some strong arguments on a path of rational approximation, which also shows a connection to an unsolved aspect in the Waring-problem. My discussion is based on the consideration of a compressed version of the Collatz-transformation, which reduces to the consideration of odd numbers only.

The approach here can easily be extended to connected cycles, analoguously to that of m-cycles in [Steiner] and [de Weger], though I didn't append that formulas yet. This will be continued in the next days/weeks.

Contents:

1. Notation/Basic Definitions

1.1. The forward-transformation T()

In the following I rewrite the Collatz-statement in a compressed form. Instead of

1.1.1. $$Collatz(a) \to b \quad := \quad b = \begin{cases} 3a+1 & \text{if } a \text{ is odd} \\ \frac{a}{2} & \text{if } a \text{ is even} \end{cases}$$

I write in all the following:

1.1.2. **forward-transformation:**

$$T(a;A) \to b \quad := \quad b = \frac{3a+1}{2^A} \qquad a,b>0, a \text{ and } b \text{ are odd}$$ *, A>0 is integer*

such, that *A* is the highest exponent of *2*, keeping the condition *a* and *b* are odd integers.

The use of the parameter *A* may seem to be of no use, since it is completely determined by the value of *a*. But this notation allows then to discuss a structure of an iterated transformation using *a* as an unknown variable by means of its exponents only.

1.1.3. *In any equation for members of an iterated transformation only positive odd integers are assumed.*

An **iterated forward transformation** is then written as:

1.1.4. $$T(a;A,B) := T(T(a;A);B) = \frac{\frac{3a+1}{2^A}*3+1}{2^B}$$ *of any finite number of parameters A,B,...*

1.2. The backward-transformation C()

The use of exponential parameters allows then to discuss the reverse operation with the same notational scheme:

1.2.1. $$C(b;A) \to a \quad := \quad a = \frac{b*2^A-1}{3} \qquad a,b>0, a \text{ and } b \text{ are odd}$$

and

1.2.2. **backward-transformation:**

$$C(b;B,A) \to a \quad := \quad a = \frac{\frac{b*2^A-1}{3}*2^B-1}{3} \qquad a,b>0, a \text{ and } b \text{ are odd}$$

where now the exponential parameters *A* and *B* are free parameters even if *b* is given (though with some modular restrictions).

1.3. Restrictions on the parameters of T(a;A,B,...,H)

The exponents *A,B,...* are restricted by the Collatz-definition and its domain to

1.3.1. *1<=A,B,C,...*

(but note, that this restriction can easily be omitted, when generalizing the problem into different characteritics like allowing negative numbers or different iteration formulae)

and *T()* has then the basic characteristic:

1.3.2. *for an exponent* A=1 *the T()-transformation is ascendent*

1.3.3. *for all exponents* A>1 *the T() - transformation is descendent,*
except if a=1 *and* A=2*, where it cycles*[1]*.*

1.4. Reformulation of the Collatz-conjecture in terms of T()

With this notation the Collatz-conjecture for *T()* is

1.4.1. *For all positive odd integers a there exists a finite set of exponents A,B,C,...Z such that*
$$T(a;A,B,C,...Z)=1$$

and conversely for the backward *C()*-transformation :

1.4.2. *all a of the domain can be computed by the inverse transformation C() with a finite set of exponents A,B,...Z starting from a=1*
$$C(1;A,B,C,...Z)=a$$

meaning: each odd integer is constructable by an appropriate set of exponents starting from *1* using the iterated *C()*-transformation.

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[1] Note, that extending the domain to negative integers, we also have a=-1 and A=1 as another cycle and few other known cycles using negative a.

1.5. A "canonical" form

An iterated transformation

b=T(a;A,B,C,D...H)

with *N* terms and the sum

S=A+B+C+...+H

can explicitely be written as:

1.5.1. $$T(a;A,B,C,D,..,H)=\frac{3^N}{2^S}a+\frac{3^{N-1}+3^{N-2}2^A+3^{N-3}2^{A+B}+\ldots+3^1 2^{A+B+\ldots+F}+2^{A+B+\ldots+G}}{2^S}$$

or

1.5.2. $$T(a;A_1,A_2,A_3,A_4,..,A_N)=\frac{3^N}{2^S}a+\frac{\sum_{k=1}^{N}3^{N-k}*2^{\sum_{j=0}^{k-1}A_j}}{2^S} \qquad \text{setting } A_0=0$$

We could call this a "canonical" form, where the most right term is indepedent of the variable *a* is then

1.5.3. ***canonical form*** *of T()*

$$T(a;A,B,C,D,..,H)=\frac{3^N}{2^S}a+T(0;A,B,C,D,...,H)$$

and of the reverse transformation

1.5.4. ***canonical form*** *of C()*

$$C(a;H,.....,D,C,B,A)=\frac{2^S}{3^N}a+C(0;H,...,D,C,B,A)$$

and in *(1.8)* we see, that given a fixed set of exponents, infinitely many *a*'s can be transformed by this characteristic transformation *T(a; ...)* as far as they have the same residue *(mod 2^S)*.

1.6. A view into transformation *T()* as a bitstring-operation

The transformation T() can be expressed by a very intriguing bitstring-notation. Since the members of a transformation we have for some a the bistring, say "11011001". Then the arithmetic goes

```
a*2        "  110110010"
+a        +"   11011001"
+1        +"          1"
=3a+1     =".1010001100"
/2^A      =  ".10100011"       // just shift right by deleting all trailing zeros
```

and the question of arriving at 1 is, whether this mixing of digits and cutting of zeros leads to the bit-string *"1"*.

2. Looking at exponents, elements a,b kept indeterminate: simple observations

2.1. multitude of solutions according to one residue class (mod 2^S)

Eq. *(1.8)* says, that given a certain transformation, say:

2.1.1. $b=T(a; A,B,C,D)$ where $N=4$, $S=A+B+C+D$

we can find a minimal solution (a_1,b_1) in terms of a modular class of 2^S :

2.1.2. $$b_1 = \frac{3^N}{2^S} a_1 + T(0; A,B,C,D)$$

where the canonical transformation $T(0;A,B,C,D)$ is independent of a and defines a <u>unique residue class</u> *modulo* 2^S.

The next possible solution a_2,b_2 in the domain is then the same residue-class:

2.1.3. $$\begin{aligned} b_1 &= \frac{3^N}{2^S} a_1 + T(0; A,B,C,D) \\ b_2 &= \frac{3^N}{2^S} a_2 + T(0; A,B,C,D) \quad \text{where } a_2 = a_1 + 2 * 2^S \end{aligned}$$

and then:

2.1.4.	$b_2 = b_1 + 2*3^N$	$a_2 = a_1 + 2*2^S$
	$b_k = b_1 + 2*(k-1)*3^N$	$a_k = a_1 + 2*(k-1)*2^S$

Example[1]:

For instance, for the transformation $b=T(a;1,2,3,4)$ we find after a first solution (a_1,b_1) infinitely many variants (a_k,b_k) as

2.1.5. $$b = T(a;1,2,3,4) \quad \Rightarrow \quad \begin{aligned} &(a_0 \to b_0) = (11 && \to 1) \\ &(a_1 \to b_1) = (2059 && \to 163) \\ &\quad \dots \\ &(a_k \to b_k) = (11 + 2^{10} * 2k && \to 1 + 3^4 * 2k) \\ &\qquad k = 0..inf \end{aligned}$$

The term k is needed with cofactor 2, using $2*k$, since the result b_k must be odd to fall into the domain of $T()$.

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[1]
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\\ An example of Pari/GP - code
\\ =====================================================================
\\ finds (rational) T0 from canonical a1 = T(0,[A,B,C,...])
T0 (m=[1])= local(lae,a,v); \
            lae = matsize(m)[2]; \
            a = 0 ; for(k=1,lae, a =(3*a+1)/2^m[k]);
   return (a);

\\ finds values a->b for b=T(a;[A,B,C,...])
\\ if k==1, return this values,  if k>1 return k'th next value
TFind_ab(v=[1],k=1)=local(t0,d,e,lae,w,a,b); \
      t0  = T0(v); d=denominator(t0); e=numerator(t0); lae=length(v); \
       w = - e / 3^lae % d; \
       a = w; if( T(a,v ) % 2 == 0 , a += d); \
       b =(( a + 2*(k-1)* d) * 3^lae + e)/d;  \
 return([a,b])
```

2.2. the construction of a "glide" (Oliveira,Lagarias) of arbitrary length

With that tool we can construct transformations of arbitrary length, where all intermediate members of the transform are $a_1,a_2,a_3,...a_n > a_0$, (called "*glides*" for instance in *Lagarias)*: just select appropriate exponents.

A sequence of exponents e_k containing only *1* and *2*, which follows the rule, that in the product

$$\frac{3}{2}*\frac{3}{4}*\frac{3}{2}*\frac{3}{4}*\frac{3}{2}*\frac{3}{4}*\frac{3}{4}*\frac{3}{2}*\frac{3}{4}*\frac{3}{2}*\frac{3}{4}*\frac{3}{4}*\ldots=\prod_{k=1}^{N}\frac{3}{2^{e_k}}=p_N$$

all partial products $p_k>1$ can serve as glide-generator.

For each denominator *2* we choose an exponent *1* and for each denominator *4* an exponent *2*:

$$a_n = T(a_0, 1,2,\quad 1,2,\quad 1,2,2,\quad 1,2,\quad 1,2,2,\quad 1,2,\quad 1,2,\quad 1,2,2,\ldots\ ) \qquad \Rightarrow a_k > a_0$$

for all k<=n

which reflects also the convergents of the continued fraction of *log(3)/log(2)*[1].

Thus any length of a *glide* can be constructed simply by setting exponents, and the smallest pair *(a,b)* satisfying such a glide of a specific length can then be determined by solving the modulus-conditions. Note however, that allowing **any** exponent this does not say, whether there are glides of the same length with smaller pair of *(a,b)*, so this method alone does not construct glide-*records*. I did not investigate this yet.

2.3. An exhaustive separation of the integers into infinitely many classes

Another curious observation is the following:

Using a segmentation of the set of positive integers into classes of the following form shows, that for each initial value of *a* a specific first exponent in the transformation is required and the result belongs to one of two residues classes (*mod 6*), though further analyses did not provide useful results. Note that the constant terms in *structure* reflect the two sequences

S_3: *(3,13,53,...(10*4* ^{k}*-1)/3)* *and*
S_1: *(1,5,21,...(4* ^{k}*-1)/3, ...)*

whose first *T()*-transforms are all *5* resp. all *1*:

classnr	structure of a	exponent A	result	
0	$2i$	–	–	*not allowed*
1	$4i+3$	1	$6i+5$	*ascending*
2	$8i+1$	2	$6i+1$	*descending*
3	$16i+13$	3	$6i+5$	*descending*
4	$32i+5$	4	$6i+1$	*descending*
5	$64i+53$	5	$6i+5$	*descending*
6	$128i+21$	6	$6i+1$	*descending*
...	...	...	...	

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[1] The relation becomes obvious, if we recursively denote a partial sequence *1,2* as a_0, *1,2,2* as b_0, then the occuring two types of sequences of a_0,b_0 as a_1,b_1 and so on. The lengthes of $a_0,a_1,a_2,...$ reflect then the coefficients of the continued fraction of *log(3)/log(2)*.

That the given segmentation into classes covers the whole set of positive integers can be shown by induction.

A table-view

<table>
<tr><td rowspan="8">classes 0..inf covering all positive numbers

split (mod 2) into:</td><td>class 0 ▼
covering
0 (mod 2)</td><td colspan="6">note: class 0 need not be considered in the compressed transformation</td></tr>
<tr><td rowspan="7">classes 1..inf covering
1 (mod 2)

split (mod 4) into:</td><td rowspan="6">classes 2..inf ▼
covering
1 (mod 4)

split (mod 8) into:

▼in all split-groups T() is descending)</td><td>class 2 ▼♂
covering
1 (mod 8)</td><td colspan="4">note: class 2 contains the trivial cycle, T(1;2,2...) is neither ▼nor ▲</td></tr>
<tr><td rowspan="5">classes 3..inf covering
5 (mod 8)

split (mod 16) into:</td><td rowspan="4">classes 4..inf covering
5 (mod 16)

split (mod 32) into:</td><td>class 4 ▼♂
5 (mod 32)</td><td colspan="2"></td></tr>
<tr><td rowspan="3">cl 5..inf
21 (mod 32)</td><td rowspan="2"></td><td>cl 6
▼♂</td></tr>
<tr><td>...</td></tr>
<tr><td>cl 5
▼♀</td><td></td></tr>
<tr><td>class 3 ▼♀
covering
13 (mod 16)</td><td colspan="3"></td></tr>
<tr><td>class 1 ▲♀
covering
3 (mod 4)</td><td colspan="5">note: class 1 is the sole ascending transformation T()</td></tr>
</table>

▼*T() is descending* ♂***T()≡1*** *(mod 6)*
▲*T() is ascending* ♀***T()≡5*** *(mod 6)*

The benefit of this table occurs, if we consider a certain number *a*, which may be described by a certain class:

Say, $a = 32\ i + 5$, then $b = 6\ i + 1$, which is smaller than *a* because of the common parameter *i*, and on the other hand, *b* must again have the structure of one of the classes. If it is, for instance, of class *1*, so $b = 4\ j + 3$, then *j*>*i*, and *c* will be $c = 6\ j + 5$ which is obviously *c*>*b* . If then *c* is of the class $128\ k+21$ then *d* is $d=6\ k+1$ and obviously smaller than *c*.

I evaluated such modular transformation-tables for more than one step, so involving *(mod 18)* instead of *(mod 6)* (considering *2*-step-transformations) and *(mod 54)* (considering *3*-step-transformations), but with not much new insight: for all possible combinations of transformations one seems to find possible candidates, and the Fermat-method of *infinite descent*, constructing a contradiction seems to not work on any level of complexity.

3. The cycle-problem: the general cycle

3.1. Overview; sufficient modular argument not yet found

For a general transformation, where "general" means here: no special restrictions for the exponents, we may write:

3.1.1. $$b = T(a; A, B, C, D, ..., H)$$ *with N exponents and S=sum of exponents*

For a general cycle *b* must equal *a*, this means

3.1.2. $$a = T(a; A, B, C, D, ..., H)$$

Using *(1.8)* this is

3.1.3.
$$a = \frac{3^N}{2^S} a + T(0; A, B, C, D, ..., H) = \frac{3^N}{2^S} a + \frac{Q(0; A, B, C, D, ..., H)}{2^S}$$
$$a = \frac{Q(0; A, B, C, D, ..., H)}{2^S - 3^N} = \frac{3^{N-1} + 3^{N-2}2^A + 3^{N-3}2^{A+B} + ... + 3^1 2^{A+B+...+F} + 2^{A+B+...+G}}{2^S - 3^N}$$

From this for a given set of exponents we'll find exactly one solution, which may or may not be in the allowed domain of positive odd integers (*a* might be negative and/or – in most cases – rational).

Note, that *A=B=C=...=2* gives *a=1* and we have the "trivial" cycle:

$$1 = T(1; 2, 2, 2, ..., 2)$$ *for arbitrary many exponents*

and allowing negative integers for *a* we have besides two known others:

$$-1 = T(-1; 1, 1, 1, ..., 1)$$ *for arbitrary many exponents*

For all greater A with N occurences (N steps) we get rational soultions due to the formula

a=T(a;A,A,A,...A) = $$a\frac{3^N}{2^{AN}} + \frac{\frac{3^N - 2^{AN}}{3 - 2^A}}{2^{AN}}$$

$$a\left(\frac{2^{AN} - 3^N}{2^{AN}}\right) = \frac{3^N - 2^{AN}}{(3 - 2^A)2^{AN}}$$

Finally

$$a = \frac{1}{2^A - 3} \le 1$$

Modular arguments against a general cycle based on this expression were not found yet, and variants of the collatz-problem, say the *5x+1* variant, having the same structure in their canonical expression, actually do have cycles in the allowed domain.

3.2. Approximation arguments

But the above formula can be investigated in terms of *approximation*, and some results, which exclude several small lengthes of general cycles are achievable already with this tool.

Let's write all intermediate steps of a hypothetical cycle *a = T(a;A,B,C,D)* as

3.2.1. *b=T(a;A), c=T(b;B), d=T(c;C), a = T(d;D)*

Recall their meaning as one-step-transformations:

3.2.2. $$b = \frac{3a+1}{2^A} \quad c = \frac{3b+1}{2^B} \quad d = \frac{3c+1}{2^C} \quad a = \frac{3d+1}{2^D}$$

Multiply to build the trivial product:

3.2.3. $$bcda = \frac{3a+1}{2^A} * \frac{3b+1}{2^B} * \frac{3c+1}{2^C} * \frac{3d+1}{2^D}$$

Rearrange the lhs and the denominators

3.2.4. $$2^{A+B+C+D} = \frac{3a+1}{a} * \frac{3b+1}{b} * \frac{3c+1}{c} * \frac{3d+1}{d}$$

write *S = A+B+C+D* and cancel:

3.2.5. $$2^S = \left(3+\frac{1}{a}\right) * \left(3+\frac{1}{b}\right) * \left(3+\frac{1}{c}\right) * \left(3+\frac{1}{d}\right)$$

is required to allow a cycle. This formula can easily be extended to any length *N*.

3.3. Example: disproof of the general-cycle of 2 steps length

Theorem:

3.3.1.	*A 2-step-cycle cannot exist.*

Assume the contrary. Then, with *b = T(a;A); a=T(b;B), S=A+B* we must have

3.3.2. $$2^S = \left(3+\frac{1}{a}\right) * \left(3+\frac{1}{b}\right)$$

But the range of results of the rhs are for *a,b* between their maximum *infinity* and minimum *1*

3.3.3. $$3^2 = 9 = \left(3+\frac{1}{\inf}\right) * \left(3+\frac{1}{\inf}\right) < \left(3+\frac{1}{a}\right) * \left(3+\frac{1}{b}\right) < \left(3+\frac{1}{1}\right) * \left(3+\frac{1}{1}\right) = 4^2$$

So, for the smallest *a,b=1* we had the rhs*=16*, and for increasing *a,b* the rhs converges to *9*, so the possible range for the rhs is *9<rhs<=16*.

The only perfect power of *2*, $2^S>9$ is $16=2^4$, so for any *a>1* the rhs is nearer to 3^2 than the lhs (and no other solution is possible, so *a=1*, *A=2*, and the cycle is already *1=T(1;2)*.

Let's call

3.3.4.	$PC_2(3^N) := 2^S$ the smallest S'th perfect power of 2 greater than 3^N

for the following.

The focus is here the goodness of approximation of rhs to 3^N, which is empirically much better for several N than that of the lhs (the $PC_2()$-expression) , especially with increasing a.

So the characteristics of the lhs and rhs may contradict and thus may make the inequality (even more the final attempted equality) impossible - at least in certain ranges.

3.4. Example: disproof of the general cycle of 3 steps length

Example with $N=3$: let $b=T(a;A)$, $c=T(b;B)$, $a=T(c;C)$, $S=A+B+C$. The following equation must be satisfied, if a 3-step-cycle exists:

3.4.1. $$2^S = \left(3+\frac{1}{a}\right)*\left(3+\frac{1}{b}\right)*\left(3+\frac{1}{c}\right)$$

Now for the rhs we have the bounds

3.4.2. $$\left(3+\frac{1}{\inf}\right)*\left(3+\frac{1}{\inf}\right)*\left(3+\frac{1}{\inf}\right) = 27..64 = \left(3+\frac{1}{1}\right)*\left(3+\frac{1}{1}\right)*\left(3+\frac{1}{1}\right)$$

or, rewritten, we have some range for the equation 3.4.1:

$(3 + 0)(3 + 0)(3+ 0) = 27$ //lower bound

....

$2^S = (3+1/a)(3+1/b)(3+1/c) = 32$

....

$(3 + 1)(3 + 1)(3+ 1) = 64$ // upper bound, trivial cycle

and we had two possible solutions, $3^3<2^S=2^5$ and $2^S=64=2^6$. where the latter would again define the trivial cycle, which we won't discuss here.

Here we see, that for $N=3$ the powerceil-function $PC_2(3^3)=32$ on the lhs has a good approximation to $3^N=27$, so good, that we may find a solution for a,b,c on the rhs, which are possibly in a reasonable range.

So we search for a solution of the only admissible equation

3.4.3. $$\left(3+\frac{1}{a}\right)*\left(3+\frac{1}{b}\right)*\left(3+\frac{1}{c}\right) = 32$$

For a first estimate assume first $a=b=c$ then we had

$(3+1/b)^3 = 32 = 2^5$

$(3+1/b) = 2^{5/3}$

$1/b = 2^{5/3} - 3$

3.4.4. $b = 1/(2^{5/3} - 3) \sim 5.72$

and a should be smaller and c should be greater than b.

Although for higher N the following consideration is not of much value, it shows another general reasoning about bounds, so I introduce it here:

Since we want a cycle of different elements, all elements must be odd and cannot be divisible by *3*, we could choose the smallest possible numbers to insert on the rhs, recalling that this makes the **worst** approximation to 3^N so that the $PC_2()$-expression could be a **better** approximation, which is required from the above formula for the existence of a cycle. The smallest possible set of *(a,b,c)* would be

$$(a,b,c) = (5,7,11)$$

and we insert that in the previous formula:

3.4.5. $$\left(3+\frac{1}{5}\right)*\left(3+\frac{1}{7}\right)*\left(3+\frac{1}{11}\right) \sim 31.086$$

which is already a better approximation to $3^N=27$ than the lhs with the $PC_2(3^N)$-term which were $PC_2(3^3)=32$.

So for even the smallest possible selection for *a,b,c* the inequality

$$\left[PC_2(3^3) \le\right] \quad 2^S = \left(3+\frac{1}{a}\right)*\left(3+\frac{1}{b}\right)*\left(3+\frac{1}{c}\right)$$

cannot be satisfied since actually

3.4.6. $$\left[PC_2(3^3) \le\right] \quad 2^S > \left(3+\frac{1}{a}\right)*\left(3+\frac{1}{b}\right)*\left(3+\frac{1}{c}\right) \text{ for all} (a \ge 5, b \ge 7, c \ge 11)$$

This way of arguing is the core of the following:

> the (generally) bad approximation of 2^S (or more precisely $PC_2(3^N)$) to 3^N compared to the product in the rhs, especially, if the values of the elements of the projected cycle are known to be high for other reasons (for instance $a,b,c>2^{58}$ because of computational checks) prevents the existence of many projected cycles.

From here we can derive some general conditions, which the exponents and/or the members of a cycle must allow. First observe

1) the product of the parentheses at rhs P(N) will not exceed 4^N *,*
it equals 4^N *if all members a,b,c,...=1*
2) the product P(N) will be greater than 3^N*.*

then the following conditions must be satisfied by selection of exponents or members:

3) the product P(N) must be a perfect power of 2 >= $PC_2(3^N)$
4) the sum S of exponents cannot be arbitrarily constructed
but is bounded by log(3)/log(2)<S/N<2

to allow a cycle in the collatz-problem.

Since $P(N)$ must be a perfect power of *2* and must be greater than 3^N, its minimal value must be equal or greater than $PC_2(3^N)$ (= powerceil2(3^N)), which means the next perfect power 2^S greater than 3^N)

3.5. Generalization, the critical inequality for general cycles

We can restate the formula using this bounds, where the obvious generalization from the *3*-step-example to the *N*-step is made:

Critical inequality for the general cycle (complete form):

3.5.1. $$2^N < PC_2(3^N) \le 2^S = \prod_{k=1}^{N}\left(3+\frac{1}{a_k}\right) < 4^N$$

where the "critical condition" in the view of approximation is the less or equal -relation with the product only:

Critical inequality for the general cycle (short form):

3.5.2. $$PC_2(3^N) \le 2^S = \prod_{k=1}^{N}\left(3+\frac{1}{a_k}\right)$$

which -if it cannot be satified using a certain length *N*- excludes the possibility of a cycle of this length.

See for instance the excerpt of a table for *N=1 to 200* which is fully documented in Appendix 1. For ease of documentation the formula is normed by the 3^N-term so we have

3.5.3. $$\frac{PC_2(3^N)}{3^N} \le \frac{2^S}{3^N} = \prod_{k=1}^{N}\left(1+\frac{1}{3a_k}\right)$$

and restated

3.5.4. $$\text{ratio} = \frac{PC_2(3^N)}{3^N\prod_{k=1}^{N}\left(1+\frac{1}{3a_k}\right)} \le 1$$ *to make a cycle possible from approximation arguments*

$$\text{ratio} = \frac{2^h * PC_2(3^N)}{3^N\prod_{k=1}^{N}\left(1+\frac{1}{3a_k}\right)} = 1$$ *to actually establish a cycle with some integer h>=0*

a_1 is taken as smallest element $a_1=5$
all following elements a_k are taken as the next possible odd integer, 6i+1 or 6i-1

The following table lists results for the tests for general cycles for lengthes up to N=200 .

N:= cyclelength
ug:= $PC_2(3^N)/3^N$
prod:= $(3+1/a_1)(3+1/a_2)\ldots/3^N$
ratio:= ug/prod must be <=1 to make this cycle possible by satisfying the critical equation

n	$PC_2(3^N)/3^N$	prod	ratio	ug < prod: cycle can exist
1	1.333333	1.066667	1.250000	-false-
2	1.777778	1.117460	1.590909	-false-

3	1.185185	1.151323	1.029412	-false-
4	1.580247	1.180844	1.338235	-false-
5	1.053498	1.203998	0.875000	
6	1.404664	1.225120	1.146552	-false-
7	1.872885	1.242876	1.506897	-false-
8	1.248590	1.259447	0.991379	
9	1.664787	1.273924	1.306818	-false-
(...)				
195	1.908332	1.772149	1.076846	-false-
196	1.272221	1.773152	0.717492	
197	1.696295	1.774149	0.956118	
198	1.130864	1.775143	0.637055	
199	1.507818	1.776130	0.848934	
200	1.005212	1.777116	0.565642	

The twofold value of this formula is,

for small N: empirically the product is far nearer to the N'th power of 3 than the appropriate powerceil2-function and this condition for a general cycle can easily be verified for a sample of small lengthes N.

for high values of a: the approximation of the product P(N) to 3^N *is extremely good, far too good to be worse than the* $PC_2()$*-approximation for small N. Knowing from empirical research that no* $a<2^{58}$ *is actually a member of a cycle, we can estimate, up to which N no general cycle can exist.*

3.6. Lower bounds for N, given a minimal member a of a cycle.

Assume the (unrealistic) assumption for the menbers of the hypothesized cycle, that they all are in the range *a, a+2,a+k*2,...a+2*n+d* where also the numbers *a+k*2 == 0 (mod 3)* are excluded and *d* reflects the overhead given by this additional restriction, then we have according to

3.6.1. $$PC_2(3^N) \le 2^S = \prod_{k=0}^{n-1}\left(3+\frac{1}{a_k}\right)$$

and to have a rough generous limit let's set all $a_k=a=2^{58}$, (where our additional assumption gives then bounds for an unrealistically short cycle), then we have (or: is required)

$$PC_2(3^N) \le \left(3+\frac{1}{2^{58}}\right)^N$$

$$\frac{PC_2(3^N)}{3^N} \le \left(1+\frac{1}{3*2^{58}}\right)^N$$

$$lg_3\left(\frac{PC_2(3^N)}{3^N}\right) \le N\, lg_3\left(1+\frac{1}{3*2^{58}}\right)$$

$$lg_3\left(\frac{PC_2(3^N)}{3^N}\right) \le \frac{N}{ln(3)}\left(\frac{1}{3*2^{58}}-\frac{1}{2*9*2^{116}}+\frac{1}{27*2^{174}}-\dots\right)$$

ln(3) is about *1.098* and approximating it with *1* increases the bound at the rhs, thus makes it again easier for a cycle to exist. So we have to satisfy, with a certain *N* at least the bound:

$$lg_3\left(\frac{PC_2(3^N)}{3^N}\right) \le N\left(\frac{1}{3*2^{58}}-\frac{1}{2*9*2^{116}}+\frac{1}{81*2^{174}}-\dots\right)$$

From empirical evaluation of the continued fraction of *log(3)/log(2)* we find that even the best approximations in the lhs of the above equation actually are poor. Empirically I got (best approximation depending on N), using the convergents of the continued fractions for N up to $\sim 10^{19}$, best approximations of about $1/N$ to 1 by $PC_2(3^N)/3^N$:

```
24  1.000000000000000010635580339  0.999999999999999999477187338
    2^9115015689657667             2^9881527843552324
    ------------------             ------------------
    3^5750934602875680             3^6234549927241963

25  1.00000000000000000179327108   0.99999999999999999656514447
    2^20674557256070414 7          2^21662710040425647 1
    --------------------           --------------------
    3^13044193314771494 0          3^13667648307495690 3

26  1.00000000000000000015168663   0.99999999999999999835841555
    2^63011824552566476 5          2^42337267296496061 8
    --------------------           --------------------
    3^39756034937038678 3          3^26711841622267184 3

27  1.00000000000000000002696849   0.99999999999999999987528186
    2^735467337374727303 3         2^672455512822160826 8
    ---------------------          ---------------------
    3^464028225929692645 6         3^424272190992653967 3

28  1.00000000000000000001012431   0.99999999999999999998315582
    2^4349792199695797343 3        2^3614324862321070040 0
    ----------------------         ----------------------
    3^2744413320641117195 3        3^2280385094711424549 7
29  1.00000000000000000000340444   0.99999999999999999999328013
    2^12313909261712664726 6       2^7964117062016867383 3
    -----------------------        ----------------------
    3^7769211735993658940 3        3^5024798415352541745 0
```

A guess, which is a lower bound for the remaining fraction as function of N, based on plots of this approximations is the following:

3.6.2. $$1+\frac{1}{9N\,lg_3(N)}<\frac{PC_2(3^N)}{3^N}$$

taking logarithms this is

3.6.3. $$log(3)\left(\frac{1}{9N\,lg_3(N)}-\frac{1}{2}\frac{1}{(9N\,lg_3(N))^2}+\frac{1}{3}\frac{1}{(9N\,lg_3(N))^3}\ldots\right)<lg_3\left(\frac{PC_2(3^N)}{3^N}\right)$$

so the lhs in the above equation must be smaller than the rhs in the equation before and we get for N a lower bound:

3.6.4. $$\frac{1}{9N\,lg_3(N)}<lg_3\frac{PC_2(3^N)}{3^N}\le N\left(\frac{1}{3*2^{58}}-\frac{1}{2*9*2^{116}}+\frac{1}{27*2^{174}}-\ldots\right)$$

Taking the lhs and rhs only we get a rough estimate for N:

3.6.5. $$\frac{1}{9N^2\,lg_3(N)}\le\frac{1}{3*2^{58}}-eps$$

Numerical computation says then that by

$$58\ln(2)-\ln(3)+\ln(\ln(3))\le 2\ln(N)+\ln(\ln(N))$$
$$N\ge 76260075$$

the length of a cycle must be at least *N=76 260 075*, or *N> 1.14*2^26* or *N> 1.77*3^16* or *N> 7.63*10^7* to allow a general cycle. The minimal length of a general cycle given at eq *2.26* in [lagarias] is $N>275000=2.75 * 10^5$ based on the assumption of $a>=2^{40}$; my formula gives *N>182317* for this.

3.7. A loose end here

For the derivation of this formula I assumed the unrealistic structure of the cycle, that all members a_k equal a_0 ; actually they must all be greater and also at least increasing by *2* or *4*, depending on the forbidden numbers divisible by *3*. So for an assumed cycle of length $N=10^8$ the last member a_{n-1} is in fact at least about $a_n \sim a_0+3* 10^8$ and the rhs of the critical equation decreases again by something according to the new estimated formula

3.7.1.
$$PC_2(3^N) \le 2^S = \prod_{k=0}^{n-1}\left(3+\frac{1}{a_0+3k}\right) = 3^N \prod_{k=0}^{n-1}\left(1+\frac{1}{3a_0+9k}\right)$$

and taking logarithms this is

3.7.2.
$$\frac{1}{9N\,lg_3(N)} < lg_3\left(\frac{PC_2(3^N)}{3^N}\right) \le \sum_{k=0}^{N-1} lg_3\left(1+\frac{1}{3a_0+9k}\right)$$
$$\frac{1}{9N\,lg_3(N)} < \sum_{k=0}^{N-1} lg_3\left(1+\frac{1}{3a_0+9k}\right)$$

Generally the given fomula has its value in allowing the following formulation:

without loss of generality a can be assumed to be the smallest element of the cycle.
If the critical condition cannot be satisfied for a certain a, then no higher a can satisfy the critical condition.

This theorem allows to exclude all search for greater *a* once for a certain *a* this hypothetic cycle was disproven - so we don't need to look at *a+2*, if the condition is already not satisfied for *a*.

4. The primitive cycle ("1-cycle" [Simons/deWeger], "circuit" [Steiner])

4.1. Definitions

With the considerations of the previous chapter it was not possible to exclude the possibility of a general cycle finally, but at least one finds an estimate for a lower bound for the length, depending on the value of its smallest number a_0.

To simplify things one could next look at cycles of special forms; the most primitive the one, which has only ascending steps and then one single descending step. One may call such a cycle a *"1-peak-cycle"*. Such a cycle has the form:

4.1.1. $a = T(a;1,1,1,..1,A)$ *with length N, N-1 ones and S=N-1+A*

Before studying the 1-peak-cycle let's introduce some more convenient notations.

Write a *"1-peak-transformation"*, not necessarily forming a cycle:

4.1.2. $PT(a;N:A) \quad := \quad b = T(a;1,1,1,..1,A)$

with length N, the first N-1exponents being 1 and S=N-1+A

Write concatenations of several such *"1-peak-transformations"* as *"m-peak-transformation"*:

4.1.3. $b = PT(a;N_1:A_1,N_2:A_2,...N_m:A_m,)$

with overall length $N=N_1+N_2+...+N_m$, $S=(N-m)+A_1+A_2+...A_m$

and a *"m-peak-cycle"* then equalling $b=a$:

4.1.4. $a = PT(a;N_1:A_1,N_2:A_2,...N_m:A_m)$

It is obvious, that any general cycle can be understood as a "*m-peak-cycle*" where possibly some of the partial "*1-peak-transformations*" are allowed to be degenerate, meaning they have the length *1* and only one exponent $A>1$.

This type of cycle was also studied by several researchers; using the notation "*1-cycle*" and "*m-cycle*" and indeed for this type of cycles definitive results could be proven:

* *There is no 1-cycle of any length except the trivial one (Ray Steiner, 1978)*
* *There is no 2-cycle of any length (John Simons, 1996)*
* *There are no 3..68-cycles (Benne de Weger/John Simons, 2002)*
* *several m>68 m-cycles are also excluded (by similar approximation arguments to mine) (deWeger/Simons, 2002)*

We shall see, that the "critical equation", as stated in the chapter about the general cycle provides sharp bounds, which cannot be satisfied by the quality of approximation of $PC_2(3^N)/3^N$.(and an interesting relation to a still open detail in the problem of sums of like powers by E.Waring [1] occurs).

The canonical form of a *1*-peak-cycle/*1-cycle* is:

4.1.5.
$$\begin{aligned} a = T(a;1,1,1,1,...,1,A) &= PT(a;N:A) \\ &= \frac{3^N}{2^S}a + \frac{3^{N-1}+3^{N-2}2^1+...+3^1 2^{N-2}+2^{N-1}}{2^S} \\ &= \frac{3^N}{2^S}a + \frac{3^N-2^N}{2^S} \end{aligned}$$

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[1] (see mathworld, mentioned in the entry powerfraction)

This transformation can be separated into two steps: an only-ascending step, involving only the *1*-exponents, and the final step involving the *A*-exponent.

Rewritten using *L=N-1* this is:

4.1.6.
$$b = T(a;1,1,1,1,...,1) \quad = \frac{3^L}{2^L}a + \frac{3^{L-1}+3^{L-2}2^1+...+3^1 2^{L-2}+2^{L-1}}{2^L} = \frac{3^L}{2^L}a + \frac{3^L-2^L}{2^L}$$
$$a = T(b;A) \quad = \frac{3b+1}{2^A}$$

The structure of *b* and *a* from the first of these equations can be written in terms of a common free parameter *k*. First rearrange:

4.1.7.
$$b = \frac{3^L}{2^L}a + \frac{3^L-2^L}{2^L} = \frac{3^L(a+1)}{2^L} - 1$$
$$b+1 = \frac{3^L(a+1)}{2^L}$$
$$\frac{b+1}{3^L} = \frac{a+1}{2^L}$$

and from this follows, that the numerators must be equal multiples, say *k*'th multiples, of their denominators and must be described by:

4.1.8. $$b = k*3^L - 1 \qquad\qquad a = k*2^L - 1$$

check that this gives the required identity:

$$\frac{(k*3^L-1)+1}{3^L} = \frac{(k*2^L-1)+1}{2^L}$$
$$k \quad = \quad k$$

Moreover, since *b* must be odd, *k* must be even and the first theorem for *1-peak*-transformations is

> 4.1.9. *Given a 1-peak-transformation b=PT(a;L:1) then it follows for a and b, that*
> $a = k*2*2^L - 1$ *and*
> $b = k*2*3^L - 1$

This also means that the intermediate members of the purely ascending part of a *1-peak*-transformation are

> 4.1.10. $(a_0, a_1, a_2, ...a_L) = 2*k*(2^L,\ 2^{L-1}*3^1,\ 2^{L-2}*3^2,\ ...,\ 3^L) - 1$

So for a three-step-transformation *b=T(a;1,1,1)* we have *L=3* and the first three solutions

(k=0; $(a_0, a_1, a_2, a_3) = 0\ (8, 12, 18, 27.) - 1 = (-1,-1,-1,-1)$ --- not in the domain)
k=1; $(a_0, a_1, a_2, a_3) = 2\ (8, 12, 18, 27.) - 1 = (15, 23, 35, 53)$
k=2; $(a_0, a_1, a_2, a_3) = 4\ (8, 12, 18, 27.) - 1 = (31, 47, 71, 107)$

To form a *1-peak*-cycle, a single descending transformation with exponent *A>1* must be appended.

$$a = T(b;A)$$

since $b=2k*3^L - 1$ the structure of *a* must then also be:

4.1.11. $$a = \frac{3(2k*3^L-1)+1}{2^A} = \frac{2k*3^N-2}{2^A} = \frac{k*3^N-1}{2^{A-1}}$$

Here again occurs, that k must be odd such that the numerator is divisible by the denominator and we can complete the description of the structure of a:

4.1.12. $$a = k*2^N - 1$$ $$a = \frac{k*3^N - 1}{2^{A-1}}$$ simultaneously, k odd, >0

From here the critical equation for the 1-peak-cycle can be derived:

by equalling both structure-descriptions of a:

4.1.13. $$k*2^N - 1 = \frac{k*3^N - 1}{2^{A-1}}$$

and finally to allow a *1-peak-cycle*, we need a length N of the cycle, such with a free odd positive parameter k and $A>1$ the following equality holds:

4.1.14. $$2^{A-1} = \frac{k*3^N - 1}{k*2^N - 1}$$ $A>1$, k odd >0

It is interesting, that k cannot even be 1 in this formula.

> Proof: The term 3^N-1 contains powers of 2 in a systematic form; if N is odd, then always $A=2$; but A must also be related to N by $A\sim N*(log(3)/log(2)-1)$, so for odd N there is no further solution except one ($A=2$, $N=1$, $a=1$), which describes the "trivial cycle".
>
> So, for a second solution, N must be even. But then in the denominator the form 2^N-1 contains the primefactor 3, but the numerator does systematically not, and we get a noninteger result for all $N=/=1$

To relate this result to the critical inequality for general cycles note, that for a *1-peak-cycle* the transformation is $a=PT(a;N:A)$, thus $S=N-1+A = N*(A-1)$ and multiplying with 2^N gives:

4.1.15. $$2^S = 2^N \frac{k*3^N - 1}{k*2^N - 1}$$

and finally it must be solvable for N,S and a free parameter $k>0$, odd,

4.1.16. $$\frac{PC_2(3^N)}{3^N} \le \frac{2^S}{3^N} = \frac{2^N}{3^N}\frac{k*3^N - 1}{k*2^N - 1}$$

4.2. A short analysis of the approximation using a table of irregular fractions

Let's discuss the equation *4.1.14*

4.2.1.	$2^{A-1} = \dfrac{k*3^N - 1}{k*2^N - 1}$	*A>1, k odd >0*

where heuristics show, that the lhs are always greater than the lhs, and we may formulate as a proposal, which denies the possibility of a primitive cycle:

conjecture:

4.2.2.	$2^{A-1} > \dfrac{k*3^N - 1}{k*2^N - 1}$	*A>1, k odd >0*

First observe the bounds for the rhs in terms of *k*:

4.2.3.	$\dfrac{3^N}{2^N} > \dfrac{k*3^N - 1}{k*2^N - 1} > \dfrac{-1}{-1}$

for *k->inf , k , k=0* respectively. Since it is required that k>0, the smallest rhs in 4.2.2

We may build a table for the empirical values of the middle term. Let's denote

$$d_k = \left\lfloor \frac{k*3^N - 1}{k*2^N - 1} \right\rfloor \qquad \frac{p_k}{q_k} = \left\{ \frac{k*3^N - 1}{k*2^N - 1} \right\}$$ *the integer and fractional part*

$$d_{oo} = \left\lfloor \frac{3^N}{2^N} \right\rfloor \qquad \frac{p}{q} = \frac{p_{oo}}{q_{oo}} = \left\{ \frac{3^N}{2^N} \right\}$$ *the integer and fractional part,k ->oo*

d = min(d_k), and the fractional part possibly nonregular if there exists one pair d_k<>d_j

The table below focuses the question, whether the (irregular) fractional part can become integer (or zero), given integer *d*. Here increasing *N* define the rows and increasing *k* define the columns. The digit *d* is taken out of each entry, because empirically occurs, that it doesn't change when *k* varies from *1* to *infinity*. Also the entry for *k=0* was inserted, however as irregular fraction with negative fractional part to have the same *d*:

N	d	$k=0$	1	2	3	4	5	6	...	∞
1	$1+$	$\frac{0}{-1}$	$\frac{1}{1}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{4}{7}$	$\frac{5}{9}$	$\frac{6}{11}$		$\frac{1}{2}$
2	$2+$	$\frac{1}{-1}$	$\frac{2}{3}$	$\frac{3}{7}$	$\frac{4}{11}$	$\frac{5}{15}$	$\frac{6}{19}$	$\frac{7}{23}$		$\frac{1}{4}$
3	$3+$	$\frac{2}{-1}$	$\frac{5}{7}$	$\frac{8}{15}$	$\frac{11}{23}$	$\frac{14}{31}$	$\frac{17}{39}$	$\frac{20}{37}$		$\frac{3}{8}$
4	$5+$	$\frac{4}{-1}$	$\frac{5}{15}$	$\frac{6}{31}$	$\frac{7}{47}$	$\frac{8}{63}$	$\frac{9}{79}$	$\frac{10}{95}$		$\frac{1}{16}$
5	$7+$	$\frac{6}{-1}$	$\frac{25}{31}$	$\frac{44}{63}$	$\frac{63}{95}$	$\frac{82}{127}$	$\frac{101}{159}$	$\frac{120}{191}$		$\frac{19}{32}$
...	...	...								

The proof of R.Steiner[1] that there is no *1-peak*-cycle, was successful by proving that the approximation of the rhs in 4.2.1) has a certain bad degree and thus an integer-solution is not possible - which means translated to the following table, that *d* plus the fractional part *p/q* is never a power of *2* (which could also only happen if the fractional part *p/q* degenerates to become integer)

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[1] personal communication, see Sec. 5.3

Already in this small snippet one can nicely see, that

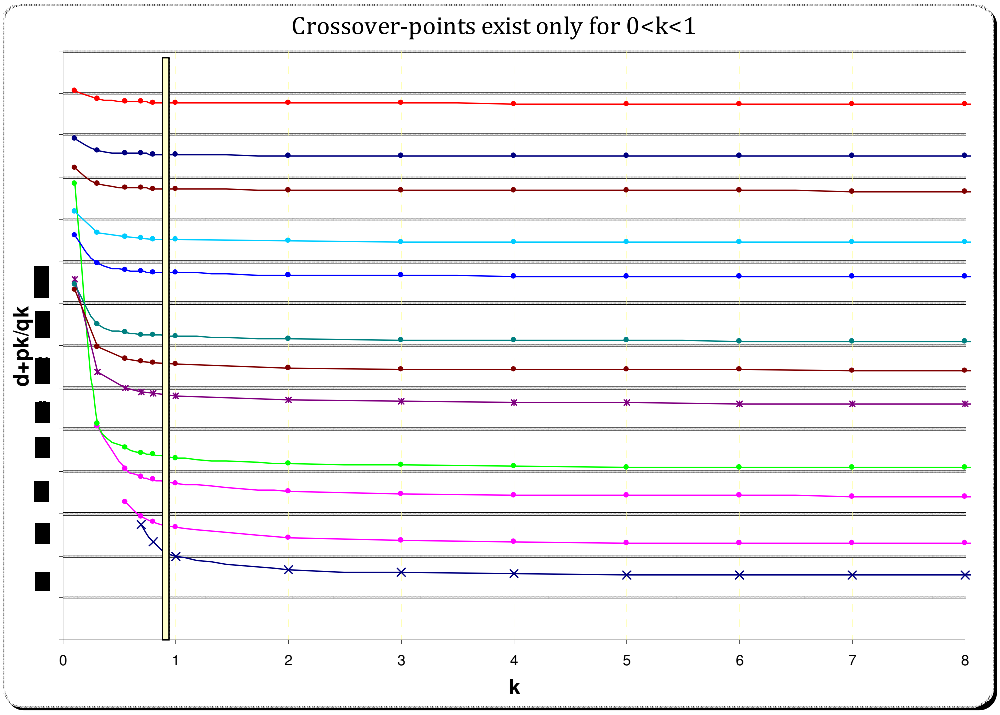
a) the progression of numerators and denominators are indicated by the inf-term

*b) $d=d_k$ for $k>0$ and $N>1$, so we don't have **an irregular fraction crossing an integer** for $k>0$*

If *b)* can be shown to be valid for all *N*, then the *1-peak*-cycle is also disproven.

It might be of interest to plot a graph with interpolated *k*. Such graph exhibits empirically that the crossing of an integer occurs only between $0<k<1$.[1]

When plotting this interpolated table for the *3x+1*-problem, moreover allowing also negative *k*, then we find integer crossing in the negative domain, and also the *1-peak*-cycle residing there.



The reason, that there are no integer-crossings for $k>=1$ depends on the numerator-value $p=p_{oo}$ of the fraction for $k\text{->}oo$. Since it is by construction $p_0 = d - 1$, for to have no integer crossings we need, that $p_0+k*p_{oo}<-1+k*q_{oo}$ or $p_0+1 < k * (q_{oo}-p_{oo})$. or $d < k * (q_{oo}-p_{oo})$, which - if it is already true for $k=1$ - is obviously true for all $k>1$. For $k=1$ the last expression is

$d < 1\ (q_{oo}-p_{oo})$
$d+p_{oo} < q_{oo}$

or, using the expressions for *3x+1*, this means for any row *N*

$$\left\lfloor \frac{3^N-1}{2^N-1} \right\rfloor + \left\{ \frac{3^N}{2^N} \right\} * 2^N < 2^N$$

$$\left\lfloor 3^N\left(\frac{1}{2^N-1}\right) - \frac{1}{2^N-1} \right\rfloor < 2^N\left(1-\left\{\frac{3^N}{2^N}\right\}\right)$$

o

[1] A property which is special to the *3x+1*-question. Other parameters, *ax+b* may have such crossings also in the table, and also having appropriate numbers in *d*, so that a *1-peak*-cycle is possible with such parameters (see some examples in the appendix at 5.2)

$$\left\lfloor 3^N\left(\frac{1}{2^N}+\frac{1}{2^{2N}}+\frac{1}{2^{3N}}+\dots\right)-\left(\frac{1}{2^N}+\frac{1}{2^{2N}}+\frac{1}{2^{3N}}+\dots\right)\right\rfloor < 2^N\left(1-\left\{\frac{3^N}{2^N}\right\}\right)$$

$$\left\lfloor \frac{3^N-1}{2^N}+\frac{3^N-1}{4^N}+eps\right\rfloor < 2^N\left(1-\left\{\frac{3^N}{2^N}\right\}\right)$$

and for large N: we can formulate:

a: *if* $\left\lfloor \frac{3^N}{2^N}\right\rfloor < 2^N\left(1-\left\{\frac{3^N}{2^N}\right\}\right)$

then there is no *1-peak*-cycle for large N.

This can be rewritten in two ways:

b: *if* $\left\lfloor \frac{3^N}{2^N}\right\rfloor (2^N-1) > 3^N-2^N$

c: *or if* $\left\{\frac{3^N}{2^N}\right\}-\frac{1}{2^N}\left\{\frac{3^N}{2^N}\right\} < 1-\frac{3^N}{4^N}$

then there is no *1-peak*-cycle for large N.

Formulation c) occurs in a sharper conjecture

c.1: *For N>2 we have* $\left\{\frac{3^N}{2^N}\right\} < 1-\frac{3^N}{4^N}$

in a detail of the Waring-problem
(see mathworld/powerfrac http://mathworld.wolfram.com/PowerFractionalParts.html)
where it is mentioned, that this detail is not yet solved, nevertheless assumed to be true.

But ***if*** it is true, then also *c)* is true and the *1-peak*-cycle is not possible due to missing crossing-points in the above table.

Kurt Mahler approached the same problem in terms of his *z-numbers*, and was able to prove, that at most finitely many *z*-numbers, and thus solutions for *c.1,* can exist.

The above table seems to be a useful type of display. In the appendix I have documented similar tables for Collatz-like problem using different parameters, so *(5x+1)/2 , (5x+1)/3, (11x+1)/2, (11x+1)/3*. There are crossing points for some of these versions and thus cycles in that problem-configurations cannot be excluded by the investigated properties here.

4.3. Concatenated m-peak-cycles -> general cycle

(much more material needs to be inserted; so far only a sketch:)

The discussion is a simple generalization of the previous. Assume two "primitive transformations" (each single one not being a cycle):

$b=PT(a;N_1:A)$ $a = PT(b;N_2:B)$

then we know a structural description for a and b from the earlier discussion of $b=PT(a;N_1:A)$

$$a = k2^{N_1} - 1 \qquad b = \frac{k3^{N_1} - 1}{2^{A-1}}$$

and as well for a and b by the second transformation $a=PT(b;N_2:B)$

$$b = j2^{N_2} - 1 \qquad a = \frac{j3^{N_2} - 1}{2^{B-1}}$$

which must also agree (be simultaneously true). Then we can again build the trivial equation of the products

$$ab = \underbrace{(k2^{N_1} - 1)}_{a}\underbrace{(j2^{N_2} - 1)}_{b} = \underbrace{(\frac{k3^{N_1} - 1}{2^{A-1}})}_{b}\underbrace{(\frac{j3^{N_2} - 1}{2^{B-1}})}_{a}$$

Then

$$1. \qquad 2^{A-1+B-1} = (\frac{k3^{N_1} - 1}{j2^{N_2} - 1})(\frac{j3^{N_2} - 1}{k2^{N_1} - 1})$$

and also, by rotating denominators

$$2. \qquad 2^{A-1+B-1} = (\frac{k3^{N_1} - 1}{k2^{N_1} - 1})(\frac{j3^{N_2} - 1}{j2^{N_2} - 1})$$

Here in *1.* each parenthese must be a perfect power of *2* , which imposes restrictions on j and k for modular reasons (for instance the two most simple ones: both must be odd, if they are simultaneously *1* they form the trivial cycle)

In *2.* we recognize, that increasing k (resp j) decreases the parentheses down to a perfect power of *3/2* which is always noninteger. The clue here were that we want to be able to show, that they also do not cross an integer bound while increasing k from *1* to infinity. But this is open to be proved...

It is obviously generalizable to any number of partial primitive transformations ("m-cycles"). Note, that based on the form *2.* only, Simons/deWeger could show, that up to *72* concatenations there is no such m-cycle. He also showed that -by increasing the number m of partial cycles- the limit condition becomes too weak for the disprove, and integer-crossings cannot be excluded. Here a reintroduction of the modular arguments in the form of *1.* might be helpful (each term in *1.* must be a perfect power of *2*)

4.4. Some loose ends

(not yet inserted)

5. Appendix

5.1. Approximation table for the (3x+1)/2 -problem

The following table lists results for the tests for general cycles for lengthes up to N=200 .

N:= cyclelength
ug:= powerceil2(3)/3^N
prod:= (3+1/a)(3+1/b).../3^N
ratio:= ug/prod must be <=1 to make this cycle possible by satisfying the critical equation

n	ug	prod	ug/prod	ug <= prod: cycle can exist
1	1.333333	1.066667	1.250000	-false-
2	1.777778	1.117460	1.590909	-false-
3	1.185185	1.151323	1.029412	-false-
4	1.580247	1.180844	1.338235	-false-
5	1.053498	1.203998	0.875000	
6	1.404664	1.225120	1.146552	-false-
7	1.872885	1.242876	1.506897	-false-
8	1.248590	1.259447	0.991379	
9	1.664787	1.273924	1.306818	-false-
10	1.109858	1.287622	0.861944	
11	1.479811	1.299885	1.138416	-false-
12	1.973081	1.311596	1.504336	-false-
13	1.315387	1.322259	0.994803	
14	1.753850	1.332509	1.316201	-false-
15	1.169233	1.341960	0.871288	
16	1.558977	1.351089	1.153868	-false-
17	1.039318	1.359586	0.764437	
18	1.385758	1.367826	1.013110	-false-
19	1.847677	1.375554	1.343224	-false-
20	1.231785	1.383070	0.890616	
21	1.642379	1.390163	1.181429	-false-
22	1.094920	1.397079	0.783720	
23	1.459893	1.403638	1.040078	-false-
24	1.946524	1.410048	1.380467	-false-
25	1.297683	1.416152	0.916344	
26	1.730243	1.422127	1.216659	-false-
27	1.153496	1.427838	0.807861	
28	1.537994	1.433438	1.072941	-false-
29	1.025329	1.438807	0.712625	
30	1.367106	1.444077	0.946699	
31	1.822808	1.449144	1.257852	-false-
32	1.215205	1.454124	0.835696	
33	1.620274	1.458923	1.110596	-false-
34	1.080182	1.463644	0.738009	
35	1.440243	1.468204	0.980956	
36	1.920324	1.472694	1.303954	-false-
37	1.280216	1.477038	0.866746	
38	1.706955	1.481319	1.152321	-false-
39	1.137970	1.485469	0.766068	
40	1.517293	1.489561	1.018618	-false-
41	1.011529	1.493533	0.677273	
42	1.348705	1.497453	0.900666	
43	1.798274	1.501263	1.197840	-false-
44	1.198849	1.505026	0.796564	
45	1.598465	1.508688	1.059507	-false-
46	1.065644	1.512306	0.704648	
47	1.420858	1.515831	0.937346	
48	1.894477	1.519316	1.246928	-false-
49	1.262985	1.522714	0.829430	
50	1.683980	1.526076	1.103471	-false-
51	1.122653	1.529358	0.734068	
52	1.496871	1.532605	0.976684	
53	1.995828	1.535778	1.299555	-false-
54	1.330552	1.538919	0.864602	
55	1.774069	1.541990	1.150506	-false-
56	1.182713	1.545032	0.765494	
57	1.576951	1.548009	1.018696	-false-
58	1.051300	1.550957	0.677840	
59	1.401734	1.553845	0.902106	
60	1.868978	1.556707	1.200598	-false-
61	1.245986	1.559512	0.798959	
62	1.661314	1.562292	1.063383	-false-

63	1.107543	1.565018	0.707687	
64	1.476724	1.567721	0.941956	
65	1.968965	1.570374	1.253819	-false-
66	1.312643	1.573004	0.834482	
67	1.750191	1.575587	1.110818	-false-
68	1.166794	1.578149	0.739343	
69	1.555725	1.580666	0.984221	
70	1.037150	1.583163	0.655113	
71	1.382867	1.585618	0.872131	
72	1.843823	1.588053	1.161058	-false-
73	1.229215	1.590449	0.772873	
74	1.638954	1.592826	1.028960	-false-
75	1.092636	1.595165	0.684967	
76	1.456848	1.597487	0.911962	
77	1.942463	1.599772	1.214212	-false-
78	1.294976	1.602041	0.808328	
79	1.726634	1.604276	1.076270	-false-
80	1.151089	1.606495	0.716522	
81	1.534786	1.608680	0.954065	
82	1.023191	1.610851	0.635186	
83	1.364254	1.612991	0.845792	
84	1.819006	1.615116	1.126238	-false-
85	1.212670	1.617211	0.749853	
86	1.616894	1.619292	0.998519	
87	1.077929	1.621344	0.664837	
88	1.437239	1.623384	0.885335	
89	1.916319	1.625395	1.178986	-false-
90	1.277546	1.627395	0.785025	
91	1.703394	1.629367	1.045433	-false-
92	1.135596	1.631328	0.696118	
93	1.514128	1.633263	0.927057	
94	1.009419	1.635187	0.617311	
95	1.345892	1.637086	0.822126	
96	1.794522	1.638974	1.094906	-false-
97	1.196348	1.640839	0.729108	
98	1.595131	1.642693	0.971046	
99	1.063421	1.644524	0.646643	
100	1.417894	1.646345	0.861237	
101	1.890526	1.648145	1.147063	-false-
102	1.260350	1.649934	0.763879	
103	1.680467	1.651703	1.017415	-false-
104	1.120311	1.653462	0.677555	
105	1.493749	1.655200	0.902458	
106	1.991665	1.656930	1.202021	-false-
107	1.327777	1.658640	0.800521	
108	1.770369	1.660341	1.066268	-false-
109	1.180246	1.662023	0.710126	
110	1.573661	1.663697	0.945882	
111	1.049107	1.665352	0.629961	
112	1.398810	1.667000	0.839118	
113	1.865080	1.668629	1.117732	-false-
114	1.243387	1.670251	0.744431	
115	1.657849	1.671855	0.991622	
116	1.105233	1.673452	0.660451	
117	1.473643	1.675032	0.879770	
118	1.964858	1.676605	1.171927	-false-
119	1.309905	1.678162	0.780560	
120	1.746540	1.679711	1.039786	-false-
121	1.164360	1.681245	0.692558	
122	1.552480	1.682772	0.922573	
123	1.034987	1.684284	0.614497	
124	1.379982	1.685789	0.818597	
125	1.839977	1.687280	1.090499	-false-
126	1.226651	1.688764	0.726360	
127	1.635535	1.690234	0.967638	
128	1.090356	1.691697	0.644534	
129	1.453809	1.693147	0.858643	
130	1.938412	1.694590	1.143882	-false-
131	1.292274	1.696020	0.761945	
132	1.723032	1.697444	1.015075	-false-
133	1.148688	1.698855	0.676154	
134	1.531584	1.700260	0.900794	
135	1.021056	1.701653	0.600038	
136	1.361408	1.703040	0.799399	
137	1.815211	1.704414	1.065006	-false-
138	1.210141	1.705783	0.709434	
139	1.613521	1.707140	0.945160	
140	1.075681	1.708492	0.629608	
141	1.434241	1.709832	0.838820	
142	1.912321	1.711167	1.117554	-false-
143	1.274881	1.712490	0.744460	
144	1.699841	1.713808	0.991850	
145	1.133227	1.715116	0.660729	

146	1.510970	1.716418	0.880304	
147	1.007313	1.717709	0.586428	
148	1.343084	1.718996	0.781319	
149	1.790779	1.720272	1.040986	-false-
150	1.193853	1.721544	0.693478	
151	1.591804	1.722805	0.923960	
152	1.061202	1.724062	0.615525	
153	1.414937	1.725308	0.820107	
154	1.886582	1.726550	1.092689	-false-
155	1.257721	1.727783	0.727940	
156	1.676962	1.729011	0.969897	
157	1.117975	1.730229	0.646143	
158	1.490633	1.731443	0.860919	
159	1.987510	1.732648	1.147094	-false-
160	1.325007	1.733849	0.764200	
161	1.766676	1.735041	1.018233	-false-
162	1.177784	1.736228	0.678358	
163	1.570379	1.737407	0.903863	
164	1.046919	1.738582	0.602168	
165	1.395892	1.739748	0.802353	
166	1.861189	1.740910	1.069090	-false-
167	1.240793	1.742063	0.712255	
168	1.654391	1.743213	0.949047	
169	1.102927	1.744355	0.632284	
170	1.470569	1.745493	0.842495	
171	1.960759	1.746623	1.122600	-false-
172	1.307173	1.747749	0.747918	
173	1.742897	1.748867	0.996586	
174	1.161931	1.749982	0.663968	
175	1.549242	1.751088	0.884731	
176	1.032828	1.752192	0.589449	
177	1.377104	1.753288	0.785441	
178	1.836138	1.754380	1.046602	-false-
179	1.224092	1.755465	0.697304	
180	1.632123	1.756547	0.929166	
181	1.088082	1.757621	0.619065	
182	1.450776	1.758692	0.824918	
183	1.934368	1.759756	1.099225	-false-
184	1.289579	1.760817	0.732375	
185	1.719438	1.761870	0.975916	
186	1.146292	1.762921	0.650223	
187	1.528390	1.763965	0.866451	
188	1.018926	1.765005	0.577294	
189	1.358569	1.766039	0.769274	
190	1.811425	1.767070	1.025100	-false-
191	1.207616	1.768095	0.683004	
192	1.610155	1.769116	0.910147	
193	1.073437	1.770131	0.606417	
194	1.431249	1.771143	0.808093	
195	1.908332	1.772149	1.076846	-false-
196	1.272221	1.773152	0.717492	
197	1.696295	1.774149	0.956118	
198	1.130864	1.775143	0.637055	
199	1.507818	1.776130	0.848934	
200	1.005212	1.777116	0.565642	

5.2. Residual tables for other problem-parameters

The red lines mark the cross-over-points, which -if are integer- allow an *1*-cycle for that parameters (if some other conditions are also met, but this is not the focus of these tables). The leading integer of a row is the integral part of $(3^n*i - 1)/(2^n*i-1)$ and the following fractions the respective fractional parts for $i>=0$, where n refers to the n'th row.

(irregular) Fraction or digits of (3^n*i-1) / (2^n*i-1)
vertical N=1..maxn, horizontal i:=0..5 , oo

1	0/-1	1/1	2/3	3/5	4/7	5/9	...	1/2
2	1/-1	2/3	3/7	4/11	5/15	6/19	...	1/4
3	2/-1	5/7	8/15	11/23	14/31	17/39	...	3/8
5	4/-1	5/15	6/31	7/47	8/63	9/79	...	1/16
7	6/-1	25/31	44/63	63/95	82/127	101/159	...	19/32
11	10/-1	35/63	60/127	85/191	110/255	135/319	...	25/64
17	16/-1	27/127	38/255	49/383	60/511	71/639	...	11/128
25	24/-1	185/255	346/511	507/767	668/1023	829/1279	...	161/256
38	37/-1	264/511	491/1023	718/1535	945/2047	1172/2559	...	227/512
57	56/-1	737/1023	1418/2047	2099/3071	2780/4095	3461/5119	...	681/1024

(irregular) Fraction or digits of (5^n*i-1) / (2^n*i-1)
vertical N=1..maxn, horizontal i:=0..5 , oo

2	1/-1	2/1	3/3	4/5	5/7	6/9	...	1/2
6	5/-1	6/3	7/7	8/11	9/15	10/19	...	1/4
15	14/-1	19/7	24/15	29/23	34/31	39/39	...	5/8
39	38/-1	39/15	40/31	41/47	42/63	43/79	...	1/16
97	96/-1	117/31	138/63	159/95	180/127	201/159	...	21/32
244	243/-1	252/63	261/127	270/191	279/255	288/319	...	9/64
610	609/-1	654/127	699/255	744/383	789/511	834/639	...	45/128
1525	1524/-1	1749/255	1974/511	2199/767	2424/1023	2649/1279	...	225/256
3814	3813/-1	4170/511	4527/1023	4884/1535	5241/2047	5598/2559	...	357/512
9536	9535/-1	10296/1023	11057/2047	11818/3071	12579/4095	13340/5119	...	761/1024

(irregular) Fraction or digits of (5^n*i-1) / (3^n*i-1)
vertical N=1..maxn, horizontal i:=0..5 , oo

1	0/-1	2/2	4/5	6/8	8/11	10/14	...	2/3
2	1/-1	8/8	15/17	22/26	29/35	36/44	...	7/9
4	3/-1	20/26	37/53	54/80	71/107	88/134	...	17/27
7	6/-1	64/80	122/161	180/242	238/323	296/404	...	58/81
12	11/-1	220/242	429/485	638/728	847/971	1056/1214	...	209/243
21	20/-1	336/728	652/1457	968/2186	1284/2915	1600/3644	...	316/729
35	34/-1	1614/2186	3194/4373	4774/6560	6354/8747	7934/10934	...	1580/2187
59	58/-1	3584/6560	7110/13121	10636/19682	14162/26243	17688/32804	...	3526/6561
99	98/-1	4606/19682	9114/39365	13622/59048	18130/78731	22638/98414	...	4508/19683
165	164/-1	22704/59048	45244/118097	67784/177146	90324/236195	112864/295244	...	22540/59049

(irregular) Fraction or digits of (11^n*i-1) / (3^n*i-1)
vertical N=1..maxn, horizontal i:=0..5 , oo

3	2/-1	4/2	6/5	8/8	10/11	12/14	...	2/3
13	12/-1	16/8	20/17	24/26	28/35	32/44	...	4/9
49	48/-1	56/26	64/53	72/80	80/107	88/134	...	8/27
180	179/-1	240/80	301/161	362/242	423/323	484/404	...	61/81
662	661/-1	846/242	1031/485	1216/728	1401/971	1586/1214	...	185/243
2430	2429/-1	2520/728	2611/1457	2702/2186	2793/2915	2884/3644	...	91/729
8910	8909/-1	9910/2186	10911/4373	11912/6560	12913/8747	13914/10934	...	1001/2187
32671	32670/-1	37120/6560	41570/13121	46020/19682	50470/26243	54920/32804	...	4450/6561
119796	119795/-1	122818/19682	125841/39365	128864/59048	131887/78731	134910/98414	...	3023/19683
439252	439251/-1	472504/59048	505757/118097	539010/177146	572263/236195	605516/295244	...	33253/59049

5.3. Paraphrase of the Steiner-proof of 1977

[Steiner]:

Briefly, my 1977 proof runs as follows. I will just give the steps, not the details here.

1). Any circuit for the *3x+1* problem corresponds to an integer solution *k, l, h,* of the Diophantine equation

$$(2^{k+l} - 3^k)\, h = 2^l - 1 \qquad (*)$$

2) To show that the only integer solution of (*) is *1,1,1*.

First, reduce this to a problem in linear forms in logarithms:

$$0 < | l/k - \log_2 3/2 | < 1/(k*\ln 2 * (2^k - 1))$$

3). This shows that if $k > 4$ then l/k must be a convergent in the continued fraction expansion of $\log_2 (3/2)$.

4). By using a lemma of LEGENDRE, one can prove that a partial quotient of this CF must exceed 10^{4690}.

5). Using BAKER'S, or RHIN'S theorem one finds a reasonable upper bound for *k* and the denominators of all convergents in this range are all smaller than *2500*.

The Steiner-formula is identical to my critical condition for *1-peak*-cycles, by few rearrangements.

First, to relate variables of his formula (*) and of mine *(4.1.14)*, I translate:

(h,l,k) *->(k, A-1, N)*

$$2^{A-1} k 2^N - k 3^N = 2^{A-1} - 1$$

Then

$$2^{A-1}(k 2^N - 1) = k 3^N - 1$$

$$2^{A-1} = \frac{k 3^N - 1}{k 2^N - 1}$$
