

## Cycles in the Collatz-problem

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(minor edits 2011-10-14      First version: 10'2004)

*Abstract: The possibility of occurrence of cycles in the Collatz-problem is discussed. Here I use my approach to the collatz-problem with the means of an exponential diophantine expression. Although I don't arrive at a proof or disproof of cycles I find some strong arguments on a path of rational approximation, which also shows a connection to an unsolved aspect in the Waring-problem. My discussion is based on the consideration of a compressed version of the Collatz-transformation, which reduces to the consideration of odd numbers only.*

*The approach here can easily be extended to connected cycles, analogously to that of m-cycles in [Steiner] and [de Weger], though I didn't append that formulas yet. This will be continued in the next days/weeks.*

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## 1. Notation/Basic Definitions

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### 1.1. The forward-transformation $T()$

In the following I rewrite the Collatz-statement in a compressed form. Instead of

$$1.1.1. \quad \text{Collatz}(a) \rightarrow b \quad := \quad b = \begin{cases} 3a+1 & \text{if } a \text{ is odd} \\ \frac{a}{2} & \text{if } a \text{ is even} \end{cases}$$

I write in all the following:

$$1.1.2. \quad \textbf{forward-transformation:}$$

$$T(a; A) \rightarrow b \quad := \quad b = \frac{3a+1}{2^A} \quad a, b > 0, a \text{ and } b \text{ are odd, } A > 0 \text{ is integer}$$

such, that  $A$  is the highest exponent of 2, keeping the condition  $a$  and  $b$  are odd integers.

The use of the parameter  $A$  may seem to be of no use, since it is completely determined by the value of  $a$ . But this notation allows then to discuss a structure of an iterated transformation using  $a$  as an unknown variable by means of its exponents only.

$$1.1.3. \quad \text{In any equation for members of an iterated transformation only positive odd integers are assumed.}$$

An **iterated forward transformation** is then written as:

$$1.1.4. \quad T(a; A, B) := T(T(a; A); B) = \frac{\frac{3a+1}{2^A} * 3 + 1}{2^B} \quad \text{of any finite number of parameters } A, B, \dots$$

### 1.2. The backward-transformation $C()$

The use of exponential parameters allows then to discuss the reverse operation with the same notational scheme:

$$1.2.1. \quad C(b; A) \rightarrow a \quad := \quad a = \frac{b * 2^A - 1}{3} \quad a, b > 0, a \text{ and } b \text{ are odd}$$

and

$$1.2.2. \quad \textbf{backward-transformation:}$$

$$C(b; B, A) \rightarrow a \quad := \quad a = \frac{\frac{b * 2^A - 1}{3} * 2^B - 1}{3} \quad a, b > 0, a \text{ and } b \text{ are odd}$$

where now the exponential parameters  $A$  and  $B$  are free parameters even if  $b$  is given (though with some modular restrictions).

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### 1.3. Restrictions on the parameters of $T(a;A,B,\dots,H)$

The exponents  $A,B,\dots$  are restricted by the Collatz-definition and its domain to

$$1.3.1. \quad 1 \leq A, B, C, \dots$$

(but note, that this restriction can easily be omitted, when generalizing the problem into different characteristics like allowing negative numbers or different iteration formulae)

and  $T()$  has then the basic characteristic:

1.3.2. for an exponent  $A=1$  the  $T()$ -transformation is ascendent

1.3.3. for all exponents  $A>1$  the  $T()$  - transformation is descendent, except if  $a=1$  and  $A=2$ , where it loops<sup>1</sup>.

### 1.4. Reformulation of the Collatz-conjecture in terms of $T()$

With this notation the Collatz-conjecture for  $T()$  is

1.4.1. For all positive odd integers  $a$  there exists a finite set of exponents  $A,B,C,\dots,Z$  such that  $T(a; A, B, C, \dots, Z) = 1$

and conversely for the backward  $C()$ -transformation :

1.4.2. all  $a$  of the domain can be computed by the inverse transformation  $C()$  with a finite set of exponents  $A,B,\dots,Z$  starting from  $a=1$   
 $C(1; A, B, C, \dots, Z) = a$

meaning: each odd integer is constructable by an appropriate set of exponents starting from  $1$  using the iterated  $C()$ -transformation.

<sup>1</sup> Note, that extending the domain to negative integers, we also have  $a=-1$  and  $A=1$  as another loop and few other known loops using negative  $a$ .

### 1.5. A "canonical" form

An iterated transformation

$$b=T(a;A,B,C,D...H)$$

with  $N$  terms and the sum

$$S=A+B+C+...+H$$

can explicitly be written as:

$$1.5.1. \quad T(a; A, B, C, D, \dots, H) = \frac{3^N}{2^S} a + \frac{3^{N-1} + 3^{N-2}2^A + 3^{N-3}2^{A+B} + \dots + 3^1 2^{A+B+\dots+F} + 2^{A+B+\dots+G}}{2^S}$$

or

$$1.5.2. \quad T(a; A_1, A_2, A_3, A_4, \dots, A_N) = \frac{3^N}{2^S} a + \frac{\sum_{k=1}^N 3^{N-k} * 2^{\sum_{j=0}^{k-1} A_j}}{2^S} \quad \text{setting } A_0 = 0$$

We could call this a "canonical" form, where the most right term is independent of the variable  $a$  is then

1.5.3. **canonical form** of  $T()$

$$T(a; A, B, C, D, \dots, H) = \frac{3^N}{2^S} a + T(0; A, B, C, D, \dots, H)$$

and of the reverse transformation

1.5.4. **canonical form** of  $C()$

$$C(a; H, \dots, D, C, B, A) = \frac{2^S}{3^N} a + C(0; H, \dots, D, C, B, A)$$

and in (1.8) we see, that given a fixed set of exponents, infinitely many  $a$ 's can be transformed by this characteristic transformation  $T(a; \dots)$  as far as they have the same residue ( $\text{mod } 2^S$ ).

## 2. Applications

### 2.1. some simple observations looking at exponents with indeterminate first element a

#### \* A first curious observation

Eq. (1.8) says, that given a certain transformation, say:

$$2.1.1. \quad b=T(a;A,B,C,D) \quad \text{where } N=4, S=A+B+C+D$$

we can find a minimal solution  $(a_1, b_1)$  in terms of a modular class of  $2^S$ :

$$2.1.2. \quad b_1 = \frac{3^N}{2^S} a_1 + T(0; A, B, C, D)$$

where the canonical transformation  $T(0;A,B,C,D)$  is independent of  $a$  and defines a unique residue class modulo  $2^S$ .

The next possible solution  $a_2, b_2$  in the domain is then the same residue-class:

$$2.1.3. \quad \begin{aligned} b_1 &= \frac{3^N}{2^S} a_1 + T(0; A, B, C, D) \\ b_2 &= \frac{3^N}{2^S} a_2 + T(0; A, B, C, D) \quad \text{where } a_2 = a_1 + 2 * 2^S \end{aligned}$$

and then:

$2.1.4. \quad \begin{aligned} b_2 &= b_1 + 2 * 3^N & a_2 &= a_1 + 2 * 2^S \\ b_k &= b_1 + 2 * (k-1) * 3^N & a_k &= a_1 + 2 * (k-1) * 2^S \end{aligned}$
---

*Example<sup>1</sup>:*

For instance, for the transformation  $b=T(a;1,2,3,4)$  we find after a first solution  $(a_1, b_1)$  infinitely many variants  $(a_k, b_k)$  as

$$2.1.5. \quad \begin{aligned} b = T(a;1,2,3,4) & \Rightarrow (a_0 \rightarrow b_0) = (11 & \rightarrow 1) \\ & (a_1 \rightarrow b_1) = (2059 & \rightarrow 163) \\ & \dots \\ & (a_k \rightarrow b_k) = (11 + 2^{10} * 2k & \rightarrow 1 + 3^4 * 2k) \\ & k = 0..inf \end{aligned}$$

The term  $k$  is needed with cofactor 2, using  $2*k$ , since the result  $b_k$  must be odd to fall into the domain of  $T()$ .

```

1 \\ An example of Pari/GP - code
2 \\ =====
3 \\ finds (rational) T0 from canonical a1 = T(0, [A,B,C,...])
4 T0 (m=[1])= local(lae,a,v); \
5     lae = matsize(m)[2]; \
6     a = 0 ; for(k=1,lae, a =(3*a+1)/2^m[k]);
7 return (a);
8
9 \\ finds values a->b for b=T(a;[A,B,C,...])
10 \\ if k=1, return this values, if k>1 return k'th next value
11 TFind_ab(v=[1],k=1)=local(t0,d,e,lae,w,a,b); \
12     t0 = T0(v); d=denominator(t0); e=numerator(t0); lae=length(v); \
13     w = - e / 3^lae % d; \
14     a = w; if( T(a,v) % 2 == 0 , a += d); \
15     b = (( a + 2*(k-1)* d ) * 3^lae + e)/d; \
16 return ([a,b])

```

**\* A second curious observation**

With that tool we can construct transformations of arbitrary length, where all intermediate members of the transform are  $a_1, a_2, a_3, \dots, a_n > a_0$ , (called "glides" for instance in *Lagarias*): just select appropriate exponents.

A sequence of exponents  $e_k$  containing only 1 and 2, which follows the rule, that in the product

$$\frac{3}{2} * \frac{3}{4} * \frac{3}{2} * \frac{3}{4} * \frac{3}{2} * \frac{3}{4} * \frac{3}{4} * \frac{3}{2} * \frac{3}{4} * \frac{3}{2} * \frac{3}{4} * \frac{3}{4} * \frac{3}{2} * \frac{3}{4} * \frac{3}{4} * \dots = \prod_{k=1}^N \frac{3}{2^{e_k}} = p_N$$

all partial products  $p_k > 1$  can serve as glide-generator.

For each denominator 2 we choose an exponent 1 and for each denominator 4 an exponent 2:

$$a_n = T(a_0, 1, 2, 1, 2, 1, 2, 2, 1, 2, 1, 2, 2, 1, 2, 1, 2, 1, 2, 2, \dots) \Rightarrow a_k > a_0$$

for all  $k <= n$

which reflects also the convergents of the continued fraction of  $\log(3)/\log(2)^1$ .

Thus any length of a glide can be constructed simply by setting exponents, and the smallest pair  $(a, b)$  satisfying such a glide of a specific length can then be determined by solving the modulus-conditions. Note however, that allowing any exponent this does not say, whether there are glides of the same length with smaller pair of  $(a, b)$ , so this method alone does not construct glide-records. I did not investigate this yet.

**\* A third curious observation**

Another curious observation is the following:

Using a segmentation of the set of positive integers into classes of the following form shows, that for each initial value of  $a$  a specific first exponent in the transformation is required and the result belongs to one of two residues classes (mod 6), though further analyses did not provide useful results. Note that the constant terms in *structure* reflect the two sequences

$$S3: (3, 13, 53, \dots, (10 \cdot 4^k - 1)/3) \quad \text{and}$$

$$S1: (1, 5, 21, \dots, (4^k - 1)/3, \dots)$$

whose first  $T()$ -transforms are all 5 resp. all 1:

classnr	structure of a	exponent A	result	
0	$2i$	–	–	not allowed
1	$4i + 3$	1	$6i + 5$	ascending
2	$8i + 1$	2	$6i + 1$	descending
3	$16i + 13$	3	$6i + 5$	descending
4	$32i + 5$	4	$6i + 1$	descending
5	$64i + 53$	5	$6i + 5$	descending
6	$128i + 21$	6	$6i + 1$	descending
...	...	...	...	...

<sup>1</sup> The relation becomes obvious, if we recursively denote a partial sequence 1,2 as  $a_0$ , 1,2,2 as  $b_0$ , then the occurring two types of sequences of  $a_0, b_0$  as  $a_1, b_1$  and so on. The lengths of  $a_0, a_1, a_2, \dots$  reflect then the coefficients of the continued fraction of  $\log(3)/\log(2)$ .

That the given segmentation into classes covers the whole set of positive integers can be shown by induction.

A table-view

classes 0..inf covering all positive numbers          split (mod 2) into:	class 0 ▼ covering 0 (mod 2)	note: class 0 need not be considered in the compressed transformation						
	classes 1..inf covering 1 (mod 2)          split (mod 4) into:	classes 2..inf ▼ covering 1 (mod 4)          split (mod 8) into:   ▼ (in all split- groups T() is descending)	class 2 ▼♂ covering 1 (mod 8)	note: class 2 contains the trivial loop, T(1;2,2...) is neither ▼ nor ▲				
				classes 3..inf covering 5 (mod 8)          split (mod 16) into:	classes 4..inf covering 5 (mod 16)          split (mod 32) into:	class 4 ▼♂ 5 (mod 32)	cl 5..inf 21 (mod 32)	cl 6 ▼♂
						class 3 ▼♀ 13 (mod 16)		...
				class 1 ▲♀ covering 3 (mod 4)	note: class 1 is the sole ascending transformation T()			

▼: T() is descending      ♂: T()≡1 (mod 6)  
 ▲: T() is ascending      ♀: T()≡5 (mod 6)

The benefit of this table occurs, if we consider a certain number  $a$ , which may be described by a certain class:

Say,  $a = 32i + 5$ , then  $b = 6i + 1$ , which is smaller than  $a$  because of the common parameter  $i$ , and on the other hand must have the structure of one of the classes. If it is, for instance, of class 1, so  $b = 4j + 3$ , then  $j > i$ , and  $c$  will be  $c = 6j + 5$  which is obviously  $c > b$ . If then  $c$  is of the class  $128k + 21$  then  $d$  is  $d = 6k + 1$  and obviously smaller than  $c$ .

I evaluated such modular transformation-tables for more than one step, so involving (mod 18) instead of (mod 6) (considering 2-step-transformations) and (mod 54) (considering 3-step-transformations), but with not much new insight: for all possible combinations of transformations one seems to find possible candidates, and the Fermat-method of *infinite descent*, constructing a contradiction seems to not work on any level of complexity.

### 3. The general loop

#### 3.1. Overview

For a general transformation, where "general" means: no special restrictions for the exponents, we may write:

$$3.1.1. \quad b = T(a; A, B, C, D, \dots, H) \quad \text{with } N \text{ exponents and } S = \text{sum of exponents}$$

For a general loop  $b$  must equal  $a$ , this means

$$3.1.2. \quad a = T(a; A, B, C, D, \dots, H)$$

Using (1.8) this is

$$3.1.3. \quad a = \frac{3^N}{2^S} a + \frac{T(0; A, B, C, D, \dots, H)}{2^S}$$

$$a = \frac{T(0; A, B, C, D, \dots, H)}{2^S - 3^N} = \frac{3^{N-1} + 3^{N-2}2^A + 3^{N-3}2^{A+B} + \dots + 3^1 2^{A+B+\dots+F} + 2^{A+B+\dots+G}}{2^S - 3^N}$$

From this for a given set of exponents we'll find exactly one solution, which may or may not be in the allowed domain of positive odd integers ( $a$  might be negative and/or – in most cases – rational).

Note, that  $A=B=C=\dots=2$  gives  $a=1$  and we have the "trivial" loop:

$$1 = T(1; 2, 2, 2, \dots, 2) \quad \text{for arbitrary many exponents}$$

and allowing negative integers for  $a$  we have besides two known others:

$$-1 = T(-1; 1, 1, 1, \dots, 1) \quad \text{for arbitrary many exponents}$$

For all greater  $A$  with  $N$  occurrences ( $N$  steps) we get rational solutions due to the formula

$$a = T(a; A, A, A, \dots, A) = a \frac{3^N}{2^{AN}} + \frac{3^N - 2^{AN}}{2^{AN} - 3^N}$$

$$a \left( \frac{2^{AN} - 3^N}{2^{AN}} \right) = \frac{3^N - 2^{AN}}{(3 - 2^A) 2^{AN}}$$

Finally

$$a = \frac{1}{2^A - 3} \leq 1$$

Modular arguments against a general loop based on this expression were not found yet, and variants of the collatz-problem, say the  $5x+1$  variant, having the same structure in their canonical expression, actually do have loops in the allowed domain.

### 3.2. Application

But the above formula can be investigated in terms of *approximation*, and some results, which exclude several small lengths of general loops are achievable already with this tool.

Let's write all intermediate steps of a hypothetical loop  $a = T(a;A,B,C,D)$  as

$$3.2.1. \quad b=T(a;A), c=T(b;B), d=T(c;C), a = T(d;D)$$

Recall their meaning as one-step-transformations:

$$3.2.2. \quad b = \frac{3a+1}{2^A} \quad c = \frac{3b+1}{2^B} \quad d = \frac{3c+1}{2^C} \quad a = \frac{3d+1}{2^D}$$

Multiply to build the trivial product:

$$3.2.3. \quad bcda = \frac{3a+1}{2^A} * \frac{3b+1}{2^B} * \frac{3c+1}{2^C} * \frac{3d+1}{2^D}$$

Rearrange the lhs and the denominators

$$3.2.4. \quad 2^{A+B+C+D} = \frac{3a+1}{a} * \frac{3b+1}{b} * \frac{3c+1}{c} * \frac{3d+1}{d}$$

write  $S = A+B+C+D$  and cancel:

$$3.2.5. \quad 2^S = \left(3 + \frac{1}{a}\right) * \left(3 + \frac{1}{b}\right) * \left(3 + \frac{1}{c}\right) * \left(3 + \frac{1}{d}\right)$$

is required to allow a loop. This formula can easily be extended to any length  $N$ .

### 3.3. Example: disproof of 2-step general-loop

*Theorem:*

$$3.3.1. \quad \text{A 2-step-loop cannot exist.}$$

Assume the contrary. Then we have

$$3.3.2. \quad 2^S = \left(3 + \frac{1}{a}\right) * \left(3 + \frac{1}{b}\right)$$

But the range of results of the rhs are

$$3.3.3. \quad 3^2 = 9 = \left(3 + \frac{1}{\inf}\right) * \left(3 + \frac{1}{\inf}\right) < \left(3 + \frac{1}{a}\right) * \left(3 + \frac{1}{b}\right) < \left(3 + \frac{1}{1}\right) * \left(3 + \frac{1}{1}\right) = 4^2$$

So, for the smallest  $a=1$  we had the rhs=16, and for increasing a the rhs converges to 9, so the range for the rhs is  $9 < rhs < 16$ .

The only perfect power of  $2 \cdot 2^S > 9$  is  $16=2^4$ , so for any  $a>1$  the rhs is nearer to  $3^2$  than the lhs (and no other solution is possible, so  $a=1, A=2$ , and the loop is already  $1=T(1;2)$ ).

Let's call

$$3.3.4. \quad PC_2(3^N) := 2^M$$

*the smallest M'th perfect power of 2 greater than  $3^N$*

for the following.

The focus is here the goodness of approximation of rhs to  $3^N$ , which is empirically much better for several  $N$  than that of the lhs (the  $PC_2()$ -expression), especially with increasing  $a$ .

So the characteristics of the lhs and rhs may contradict and thus may make the inequality (even more the final attempted equality) impossible - at least in certain ranges.

### 3.4. Example: disproof of 3-step-general loop

Example with  $N=3$ . The following equation must be satisfied, if a 3-step-loop exists:

$$3.4.1. \quad 2^S = \left(3 + \frac{1}{a}\right) * \left(3 + \frac{1}{b}\right) * \left(3 + \frac{1}{c}\right)$$

Now for the rhs we have the bounds

$$3.4.2. \quad \left(3 + \frac{1}{\inf}\right) * \left(3 + \frac{1}{\inf}\right) * \left(3 + \frac{1}{\inf}\right) = 27..64 = \left(3 + \frac{1}{1}\right) * \left(3 + \frac{1}{1}\right) * \left(3 + \frac{1}{1}\right)$$

or, rewritten, we have some range for the equation 3.4.1:

$$\begin{aligned} (3 + 0)(3 + 0)(3 + 0) &= 27 \text{ //lower bound} \\ \dots \\ 2^S &= (3+1/a)(3+1/b)(3+1/c) = 32 \\ \dots \\ (3 + 1)(3 + 1)(3 + 1) &= 64 \text{ // upper bound, trivial loop} \end{aligned}$$

and we had two possible solutions,  $3^3 < 2^5 = 2^5$  and  $2^5 = 64 = 2^6$ . where the latter would again define the trivial loop, which we won't discuss here.

Here we see, that for  $N=3$  the powerceil-function  $PC_2()$  on the lhs has a good approximation to  $3^N$ , so good, that we may find a solution for  $a, b, c$  on the rhs, which are possibly in a reasonable range.

So we search for a solution of the only admissible equation

$$3.4.3. \quad \left(3 + \frac{1}{a}\right) * \left(3 + \frac{1}{b}\right) * \left(3 + \frac{1}{c}\right) = 32$$

For a first estimate assume first  $a=b=c$  then we had

$$\begin{aligned} (3+1/b)^3 &= 32 = 2^5 \\ (3+1/b) &= 2^{5/3} \\ 1/b &= 2^{5/3} - 3 \\ 3.4.4. \quad b &= 1/(2^{5/3} - 3) \sim 5.72 \end{aligned}$$

and  $a$  should be smaller and  $c$  should be greater than  $b$ .

Although for higher  $N$  the following consideration is not of much use, it shows another general reasoning about bounds, so I introduce it here:

Since we want a loop of different elements, all elements must be odd and cannot be divisible by 3, we could choose the smallest possible numbers to insert on the rhs, recalling that this makes the **worst** approximation to  $3^N$  so that the  $PC_2()$ -expression could be a **better** approximation, which is required from the above formula for the existence of a loop. The smallest possible set of  $(a, b, c)$  would be

$$(a, b, c) = (5, 7, 11)$$

and we insert that in the previous formula:

$$3.4.5. \quad \left(3 + \frac{1}{5}\right) * \left(3 + \frac{1}{7}\right) * \left(3 + \frac{1}{11}\right) \sim 31.086$$

which is already a better approximation to  $3^N=27$  than the lhs with the  $PC_2(3^N)$ -term which were  $PC_2(3^3)=32$ .

So for even the smallest possible selection for  $a,b,c$  the inequality

$$[PC_2(3^3) \leq] \quad 2^S = \left(3 + \frac{1}{a}\right) * \left(3 + \frac{1}{b}\right) * \left(3 + \frac{1}{c}\right)$$

cannot be satisfied since actually

$$3.4.6. \quad [PC_2(3^3) \leq] \quad 2^S > \left(3 + \frac{1}{a}\right) * \left(3 + \frac{1}{b}\right) * \left(3 + \frac{1}{c}\right) \text{ for all } (a \geq 5, b \geq 7, c \geq 11)$$

This way of arguing is the core of the following:

the bad approximation of  $2^S$  (or more precisely  $PC_2(3^N)$ ) to  $3^N$  compared to the product in the rhs, especially, if the values of the elements of the projected loop are known to be high for other reasons (for instance  $a,b,c > 2^{58}$  because of computational checks) prevents the existence of many projected loops.

From here we can derive some general conditions, which the exponents and/or the members of a loop must allow. First observe

- 1) the product of the parentheses at rhs  $P(N)$  will not exceed  $4^N$ ,  
it equals  $4^N$  if all members  $a,b,c,\dots=1$
- 2) the product  $P(N)$  will be greater than  $3^N$ .

then the following conditions must be satisfied by selection of exponents or members:

- 3) the product  $P(N)$  must be a perfect power of 2  $\geq PC_2(3^N)$
- 4) the sum  $S$  of exponents cannot be arbitrarily constructed  
but is bounded by  $\log(3)/\log(2) < S/N < 2$

to allow a loop in the collatz-problem.

Since  $P(N)$  must be a perfect power of 2 and must be greater than  $3^N$ , its minimal value must be equal or greater than  $PC_2(3^N)$  ( $= \text{powerceil}2(3^N)$ ), which means the next perfect power  $2^M$  greater than  $3^N$

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### 3.5. Generalization, the critical inequality for general loops

We can restate the formula using this bounds, where the obvious generalization from the 3-step-example to the  $N$ -step is made:

*Critical inequality for the general loop (complete form):*

$$3.5.1. \quad 2^N < PC_2(3^N) \leq 2^S = \prod_{k=1}^N \left( 3 + \frac{1}{a_k} \right) < 4^N$$

where the "critical condition" in the view of approximation is the less or equal -relation with the product only:

*Critical inequality for the general loop (short form):*

$$3.5.2. \quad PC_2(3^N) \leq 2^S = \prod_{k=1}^N \left( 3 + \frac{1}{a_k} \right)$$

which -if it cannot be satisfied using a certain length  $N$ - excludes the possibility of a loop of this length.

See for instance the excerpt of a table for  $N=1$  to 200 which is fully documented in Appendix 1. For ease of documentation the formula is normed by the  $3^N$ -term so we have

$$3.5.3. \quad \frac{PC_2(3^N)}{3^N} \leq \frac{2^S}{3^N} = \prod_{k=1}^N \left( 1 + \frac{1}{3a_k} \right)$$

and restated

$$3.5.4. \quad \text{ratio} = \frac{PC_2(3^N)}{3^N \prod_{k=1}^N \left( 1 + \frac{1}{3a_k} \right)} \leq 1 \quad \text{to make a loop possible from approximation arguments}$$

$$\text{ratio} = \frac{2^h * PC_2(3^N)}{3^N \prod_{k=1}^N \left( 1 + \frac{1}{3a_k} \right)} = 1 \quad \text{to actually establish a loop with some integer } h \geq 0$$

$a_1$  is taken as smallest element  $a_1=5$

all following elements  $a_k$  are taken as the next possible odd integer,  $6i+1$  or  $6i-1$

The following table lists results for the tests for general loops for lengths up to  $N=200$ .

$N :=$  looplevelth

$ug := PC_2(3^N)/3^N$

$prod := (3+1/a_1)(3+1/a_2).../3^N$

$ratio := ug/prod$  must be  $\leq 1$  to make this loop possible by satisfying the critical equation

n	$PC_2(3^N)/3^N$	prod	ratio	ug < prod: loop can exist
1	1.333333	1.066667	1.250000	-false-
2	1.777778	1.117460	1.590909	-false-
3	1.185185	1.151323	1.029412	-false-
4	1.580247	1.180844	1.338235	-false-

5	1.053498	1.203998	0.875000	
6	1.404664	1.225120	1.146552	-false-
7	1.872885	1.242876	1.506897	-false-
8	1.248590	1.259447	0.991379	
9	1.664787	1.273924	1.306818	-false-
(...)				
195	1.908332	1.772149	1.076846	-false-
196	1.272221	1.773152	0.717492	
197	1.696295	1.774149	0.956118	
198	1.130864	1.775143	0.637055	
199	1.507818	1.776130	0.848934	
200	1.005212	1.777116	0.565642	

The twofold value of this formula is,

*for small N: empirically the product is far nearer to the N'th power of 3 than the appropriate powerceil2-function and this condition for a general loop can easily be verified for a sample of small lengths N.*

*for high values of a: the approximation of the product  $P(N)$  to  $3^N$  is extremely good, far too good to be worse than the  $PC_2()$ -approximation for small N. Knowing from empirical research that no  $a < 2^{58}$  is actually a member of a loop, we can estimate, up to which N no general loop can exist.*

### 3.6. Lower bounds for N, given a minimal member a of a loop.

Assume the (unrealistic) assumption for the members of the hypothesized loop, that they all are in the range  $a, a+2, a+k*2, \dots, a+2*n+d$  where also the numbers  $a+k*2 \equiv 0 \pmod{3}$  are excluded and  $d$  reflects the overhead given by this additional restriction, then we have according to

$$3.6.1. \quad PC_2(3^N) \leq 2^S = \prod_{k=0}^{n-1} \left( 3 + \frac{1}{a_k} \right)$$

and to have a rough generous limit let's set all  $a_k = a = 2^{58}$ , (where our additional assumption gives then bounds for an unrealistically short loop), then we have (or: is required)

$$PC_2(3^N) \leq \left( 3 + \frac{1}{2^{58}} \right)^N$$

$$\frac{PC_2(3^N)}{3^N} \leq \left( 1 + \frac{1}{3 * 2^{58}} \right)^N$$

$$\lg_3 \left( \frac{PC_2(3^N)}{3^N} \right) \leq N \lg_3 \left( 1 + \frac{1}{3 * 2^{58}} \right)$$

$$\lg_3 \left( \frac{PC_2(3^N)}{3^N} \right) \leq \frac{N}{\ln(3)} \left( \frac{1}{3 * 2^{58}} - \frac{1}{2 * 9 * 2^{116}} + \frac{1}{27 * 2^{174}} - \dots \right)$$

$\ln(3)$  is about 1.098 and approximating it with 1 increases the bound at the rhs, thus makes it again easier for a loop to exist. So we have to satisfy, with a certain N at least the bound:

$$\lg_3 \left( \frac{PC_2(3^N)}{3^N} \right) \leq N \left( \frac{1}{3 * 2^{58}} - \frac{1}{2 * 9 * 2^{116}} + \frac{1}{81 * 2^{174}} - \dots \right)$$

From empirical evaluation of the continued fraction of  $\log(3)/\log(2)$  we find that even the best approximations in the lhs of the above equation actually are poor. Empirically I got (best approximation depending on  $N$ ), using the convergents of the continued fractions for  $N$  up to  $\sim 10^{19}$ , best approximations of about  $1/N$  to  $1$  by  $PC_2(3^N)/3^N$  :

24	1.00000000000000010635580339	0.9999999999999999477187338
	$2^{*9115015689657667}$	$2^{*9881527843552324}$
	-----	-----
	$3^{*5750934602875680}$	$3^{*6234549927241963}$
25	1.000000000000000179327108	0.9999999999999999656514447
	$2^{*206745572560704147}$	$2^{*216627100404256471}$
	-----	-----
	$3^{*130441933147714940}$	$3^{*136676483074956903}$
26	1.00000000000000015168663	0.9999999999999999835841555
	$2^{*630118245525664765}$	$2^{*423372672964960618}$
	-----	-----
	$3^{*397560349370386783}$	$3^{*267118416222671843}$
27	1.0000000000000002696849	0.999999999999999987528186
	$2^{*7354673373747273033}$	$2^{*6724555128221608268}$
	-----	-----
	$3^{*4640282259296926456}$	$3^{*4242721909926539673}$
28	1.0000000000000001012431	0.999999999999999998315582
	$2^{*43497921996957973433}$	$2^{*36143248623210700400}$
	-----	-----
	$3^{*27444133206411171953}$	$3^{*22803850947114245497}$
29	1.000000000000000340444	0.99999999999999999328013
	$2^{*123139092617126647266}$	$2^{*79641170620168673833}$
	-----	-----
	$3^{*77692117359936589403}$	$3^{*50247984153525417450}$

A guess, which is a lower bound for the remaining fraction as function of  $N$ , based on plots of this approximations is the following:

$$3.6.2. \quad 1 + \frac{1}{9N \lg_3(N)} < \frac{PC_2(3^N)}{3^N}$$

taking logarithms this is

$$3.6.3. \quad \log(3) \left( \frac{1}{9N \lg_3(N)} - \frac{1}{2(9N \lg_3(N))^2} + \frac{1}{3(9N \lg_3(N))^3} \dots \right) < \lg_3 \left( \frac{PC_2(3^N)}{3^N} \right)$$

so the lhs in the above equation must be smaller than the rhs in the equation before and we get for  $N$  a lower bound:

$$3.6.4. \quad \frac{1}{9N \lg_3(N)} < \lg_3 \frac{PC_2(3^N)}{3^N} \leq N \left( \frac{1}{3 * 2^{58}} - \frac{1}{2 * 9 * 2^{116}} + \frac{1}{27 * 2^{174}} - \dots \right)$$

Taking the lhs and rhs only we get a rough estimate for  $N$ :

$$3.6.5. \quad \frac{1}{9N^2 \lg_3(N)} \leq \frac{1}{3 * 2^{58}} - eps$$

Numerical computation says then that by

$$58 \ln(2) - \ln(3) + \ln(\ln(3)) \leq 2 \ln(N) + \ln(\ln(N))$$

$$N \geq 76260075$$

the length of a cycle must be at least  $N=76\ 260\ 075$ , or  $N > 1.14 * 2^{26}$  or  $N > 1.77 * 3^{16}$  or  $N > 7.63 * 10^7$  to allow a general loop. The minimal length of a general loop given at eq 2.26 in [lagarias] is  $N > 275000 = 2.75 * 10^5$  based on the assumption of  $a \geq 2^{40}$ ; my formula gives  $N > 182317$  for this.

### 3.7. A loose end here

For the derivation of this formula I assumed the unrealistic structure of the loop, that all members  $a_k$  equal  $a_0$  ; actually they must all be greater and also at least increasing by 2 or 4, depending on the forbidden numbers divisible by 3. So for an assumed loop of length  $N=10^8$  the last member  $a_{n-1}$  is in fact at least about  $a_n \sim a_0+3*10^8$  and the rhs of the critical equation decreases again by something according to the new estimated formula

$$3.7.1. \quad PC_2(3^N) \leq 2^S = \prod_{k=0}^{n-1} \left( 3 + \frac{1}{a_0 + 3k} \right) = 3^N \prod_{k=0}^{n-1} \left( 1 + \frac{1}{3a_0 + 9k} \right)$$

and taking logarithms this is

$$3.7.2. \quad \frac{1}{9N \lg_3(N)} < \lg_3 \left( \frac{PC_2(3^N)}{3^N} \right) \leq \sum_{k=0}^{N-1} \lg_3 \left( 1 + \frac{1}{3a_0 + 9k} \right)$$

$$\frac{1}{9N \lg_3(N)} < \sum_{k=0}^{N-1} \lg_3 \left( 1 + \frac{1}{3a_0 + 9k} \right)$$

Generally the given fomula has its value in allowing the following formulation:

*without loss of generality a can be assumed to be the smallest element of the loop.*

*If the critical condition cannot be satisfied for a certain a, then no higher a can satisfy the critical condition.*

This theorem allows to exclude all search for greater  $a$  once for a certain  $a$  this hypothetic loop was disproven - so we don't need to look at  $a+2$ , if the condition is already not satisfied for  $a$ .

## 4. The primitive loop (1-cycle)

### 4.1. Definitions

With the considerations of the previous chapter it was not possible to finally to exclude the possibility of a general loop, but at least one finds an estimate for a lower bound for the length, depending on the value of its smallest number  $a_0$ .

To simplify things one could next look at loops of special forms; the most primitive the one, which has only ascending steps and then one single descending step. One may call such a loop a "*1-peak-loop*". Such a loop has the form:

$$4.1.1. \quad a = T(a; 1, 1, 1, \dots, 1, A) \text{ with length } N, N-1 \text{ ones and } S=N-1+A$$

Before studying the 1-peak-loop let's introduce some more convenient notations.

Write a "*1-peak-transformation*", not necessarily forming a loop:

$$4.1.2. \quad PT(a, N : A) := b = T(a; 1, 1, 1, \dots, 1, A) \\ \text{with length } N, \text{ the first } N-1 \text{ exponents being } 1 \text{ and } S=N-1+A$$

Write concatenations of several such "*1-peak-transformations*" as "*m-peak-transformation*":

$$4.1.3. \quad b = PT(a; N_1 : A_1, N_2 : A_2, \dots, N_m : A_m, ) \\ \text{with overall length } N=N_1+N_2+\dots+N_m, S=(N-m)+A_1+A_2+\dots+A_m$$

and a "*m-peak-loop*" then equalling  $b=a$ :

$$4.1.4. \quad a = PT(a; N_1 : A_1, N_2 : A_2, \dots, N_m : A_m)$$

It is obvious, that any general loop can be understood as a "*m-peak-loop*" where possibly some of the partial "*1-peak-transformations*" are allowed to be degenerate, meaning they have the length 1 and only one exponent  $A > 1$ .

This type of loop was also studied by several researchers; using the notation "*1-cycle*" and "*m-cycle*" and indeed for this type of loops definitive results could be proven:

- \* *There is no 1-cycle of any length except the trivial one (Ray Steiner, 1978)*
- \* *There is no 2-cycle of any length (John Simons, 1996)*
- \* *There are no 3..68-cycles (Benne de Weger/John Simons, 2002)*
- \* *several  $m > 68$  m-cycles are also excluded (by similar approximation arguments to mine) (deWeger/Simons, 2002)*

We shall see, that the "critical equation", as stated in the chapter about the general loop provides sharp bounds, which cannot be satisfied by the quality of approximation of  $PC_2(3^N)/3^N$ . (and an interesting relation to a still open detail in the problem of sums of like powers by E. Waring<sup>1</sup> occurs).

The canonical form of a 1-peak-loop is:

$$4.1.5. \quad a = T(a; 1, 1, 1, 1, \dots, 1, A) = PT(a; N : A) \\ = \frac{3^N}{2^S} a + \frac{3^{N-1} + 3^{N-2} 2^1 + \dots + 3^1 2^{N-2} + 2^{N-1}}{2^S} \\ = \frac{3^N}{2^S} a + \frac{3^N - 2^N}{2^S}$$

<sup>1</sup> (see mathworld, mentioned in the entry powerfraction)

This transformation can be separated into two steps: an only-ascending step, involving only the  $l$ -exponents, and the final step involving the  $A$ -exponent.

Rewritten using  $L=N-l$  this is:

$$4.1.6. \quad \begin{aligned} b = T(a; 1, 1, 1, 1, \dots, 1) &= \frac{3^L}{2^L} a + \frac{3^{L-1} + 3^{L-2} 2^1 + \dots + 3^1 2^{L-2} + 2^{L-1}}{2^L} \\ a = T(b; A) &= \frac{3a+1}{2^A} \end{aligned}$$

The structure of  $b$  and  $a$  from the first of these equations can be written in terms of a common free parameter  $k$ . First rearrange:

$$4.1.7. \quad \begin{aligned} b &= \frac{3^L}{2^L} a + \frac{3^L - 2^L}{2^L} = \frac{3^L(a+1)}{2^L} - 1 \\ b+1 &= \frac{3^L(a+1)}{2^L} \\ \frac{b+1}{3^L} &= \frac{a+1}{2^L} \end{aligned}$$

and from this follows, that the numerators must be equal multiples of their denominators:

$$4.1.8. \quad \begin{aligned} b &= k * 3^L - 1 & a &= k * 2^L - 1 \\ \frac{(k * 3^L - 1) + 1}{3^L} &= \frac{(k * 2^L - 1) + 1}{2^L} \\ k &= k \end{aligned}$$

Since  $b$  must be odd,  $k$  must be even here and the first theorem for  $l$ -peak-transformations is

$$4.1.9. \quad \begin{aligned} &\text{Given a } l\text{-peak-transformation } b=PT(a; L; 1) \text{ then it follows for } a \text{ and } b, \text{ that} \\ &a = k * 2 * 2^L - 1 \quad \text{and} \\ &b = k * 2 * 3^L - 1 \end{aligned}$$

This also means that the intermediate members of the purely ascending part of a  $l$ -peak-transformation are

$$4.1.10. \quad (a_0, a_1, a_2, \dots, a_L) = 2 * k * (2^L, 2^{L-1} * 3^1, 2^{L-2} * 3^2, \dots, 3^L) - 1$$

So for a three-step-transformation  $b=T(a; 1, 1, 1)$  we have  $L=3$  and the first three solutions

$$\begin{aligned} (k=0; (a_0, a_1, a_2, a_3) &= 0 (8, 12, 18, 27.) - 1 = (-1, -1, -1, -1) \text{ --- not in the domain)} \\ k=1; (a_0, a_1, a_2, a_3) &= 2 (8, 12, 18, 27.) - 1 = (15, 23, 35, 53) \\ k=2; (a_0, a_1, a_2, a_3) &= 4 (8, 12, 18, 27.) - 1 = (31, 47, 71, 107) \end{aligned}$$

To form a  $l$ -peak-loop, a single descending transformation with exponent  $A>l$  must be appended.

$$a = T(b; A)$$

since  $b=2k*3^L - 1$  the structure of  $a$  must also be:

$$4.1.11. \quad a = \frac{3(2k * 3^L - 1) + 1}{2^A} = \frac{2k * 3^N - 2}{2^A} = \frac{k * 3^N - 1}{2^{A-1}}$$

Here occurs, that  $k$  must be odd such that the numerator is divisible by the denominator and we can complete the description of the structure of  $a$ :

$$4.1.12. \quad \begin{aligned} a &= k * 2^N - 1 \\ a &= \frac{k * 3^N - 1}{2^{A-1}} \quad \text{simultaneously, } k \text{ odd, } >0 \end{aligned}$$

From here the critical equation for the  $l$ -peak-loop can be derived:

by equalling both structure-descriptions of  $a$ :

$$4.1.13. \quad k * 2^N - 1 = \frac{k * 3^N - 1}{2^{A-1}}$$

and finally to allow a  $l$ -peak-loop, we need a length  $N$  of the loop, such with a free odd positive parameter  $k$  and  $A > l$  the following equality holds:

$$4.1.14. \quad 2^{A-1} = \frac{k * 3^N - 1}{k * 2^N - 1} \quad A > l, \quad k \text{ odd } > 0$$

It is interesting, that  $k$  cannot even be  $l$  in this formula.

Proof: The term  $3^N - 1$  contains powers of  $2$  in a systematic form; if  $N$  is odd, then  $A=2$ ; but  $A$  must also be related to  $N$  by  $A \sim N * (\log(3)/\log(2) - 1)$ , so for odd  $N$  there is no further solution except one ( $A=2, N=1, a=1$ ).

So  $N$  must be even. But then in the denominator the form  $2^N - 1$  contains the primefactor  $3$ , but not the numerator.

So we get a noninteger result for all  $N \neq 1$

To relate this result to the critical inequality for general loops note, that for a  $l$ -peak-loop the transformation is  $a = PT(a; N: A)$ , thus  $S = N - l + A = N * (A - l)$  and multiplying with  $2^N$  gives:

$$4.1.15. \quad 2^S = 2^N \frac{k * 3^N - 1}{k * 2^N - 1}$$

and finally it must be solvable for  $N, S$  and a free parameter  $k > 0$ , odd,

$$4.1.16. \quad \frac{PC_2(3^N)}{3^N} \leq \frac{2^S}{3^N} = \frac{2^N}{3^N} \frac{k * 3^N - 1}{k * 2^N - 1}$$

### 4.2. A short analysis of the approximation using a table of irregular fractions

Let's discuss the equation 4.1.14

4.2.1. 
$$2^{A-1} = \frac{k * 3^N - 1}{k * 2^N - 1} \quad A > 1, \quad k \text{ odd} > 0$$

where heuristics show, that the lhs are always greater than the rhs, and we may formulate as a proposal, which denies the possibility of a primitive loop:

conjecture:

4.2.2. 
$$2^{A-1} > \frac{k * 3^N - 1}{k * 2^N - 1} \quad A > 1, \quad k \text{ odd} > 0$$

First observe the bounds for the rhs in terms of  $k$ :

4.2.3. 
$$\frac{3^N}{2^N} > \frac{k * 3^N - 1}{k * 2^N - 1} > \frac{-1}{-1}$$

for  $k \rightarrow \text{inf}$ ,  $k$ ,  $k=0$  respectively. Since it is required that  $k > 0$ , the smallest rhs in 4.2.2

We may build a table for the empirical values of the middle term. Let's denote

$$d_k = \left\lfloor \frac{k * 3^N - 1}{k * 2^N - 1} \right\rfloor \quad \frac{p_k}{q_k} = \left\{ \frac{k * 3^N - 1}{k * 2^N - 1} \right\} \text{ the integer and fractional part}$$

$$d_{oo} = \left\lfloor \frac{3^N}{2^N} \right\rfloor \quad \frac{p}{q} = \frac{p_{oo}}{q_{oo}} = \left\{ \frac{3^N}{2^N} \right\} \text{ the integer and fractional part, } k \rightarrow \infty$$

$d = \min(d_k)$ , and the fractional part possibly nonregular if there exists one pair  $d_k \ll d_j$

The table below focuses the question, whether the (irregular) fractional part can become integer (or zero), given integer  $d$ . Here increasing  $N$  define the rows and increasing  $k$  define the columns. The digit  $d$  is taken out of each entry, because empirically occurs, that it doesn't change when  $k$  varies from 1 to infinity. Also the entry for  $k=0$  was inserted, however as irregular fraction with negative fractional part to have the same  $d$ :

$N$	$d$	$k=0$	1	2	3	4	5	6	...	$\infty$
1	1+	$\frac{0}{-1}$	$\frac{1}{1}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{4}{7}$	$\frac{5}{9}$	$\frac{6}{11}$	...	$\frac{1}{2}$
2	2+	$\frac{1}{-1}$	$\frac{2}{3}$	$\frac{3}{7}$	$\frac{4}{11}$	$\frac{5}{15}$	$\frac{6}{19}$	$\frac{7}{23}$	...	$\frac{1}{4}$
3	3+	$\frac{2}{-1}$	$\frac{5}{7}$	$\frac{8}{15}$	$\frac{11}{23}$	$\frac{14}{31}$	$\frac{17}{39}$	$\frac{20}{37}$	...	$\frac{3}{8}$
4	5+	$\frac{4}{-1}$	$\frac{5}{15}$	$\frac{6}{31}$	$\frac{7}{47}$	$\frac{8}{63}$	$\frac{9}{79}$	$\frac{10}{95}$	...	$\frac{1}{16}$
5	7+	$\frac{6}{-1}$	$\frac{25}{31}$	$\frac{44}{63}$	$\frac{63}{95}$	$\frac{82}{127}$	$\frac{101}{159}$	$\frac{120}{191}$	...	$\frac{19}{32}$
...	...	...	...	...	...	...	...	...	...	...

The proof of R.Steiner<sup>1</sup> that there is no 1-peak-loop, was successful by proving that the approximation of the rhs in 4.2.1) has a certain bad degree and thus an integer-solution is not possible - which means translated to the following table, that  $d$  plus the fractional part  $p/q$  is never a power of 2 (which could also only happen if the fractional part  $p/q$  degenerates to become integer)

<sup>1</sup> personal communication, see Sec. 5.3

Already in this small snippet one can nicely see, that

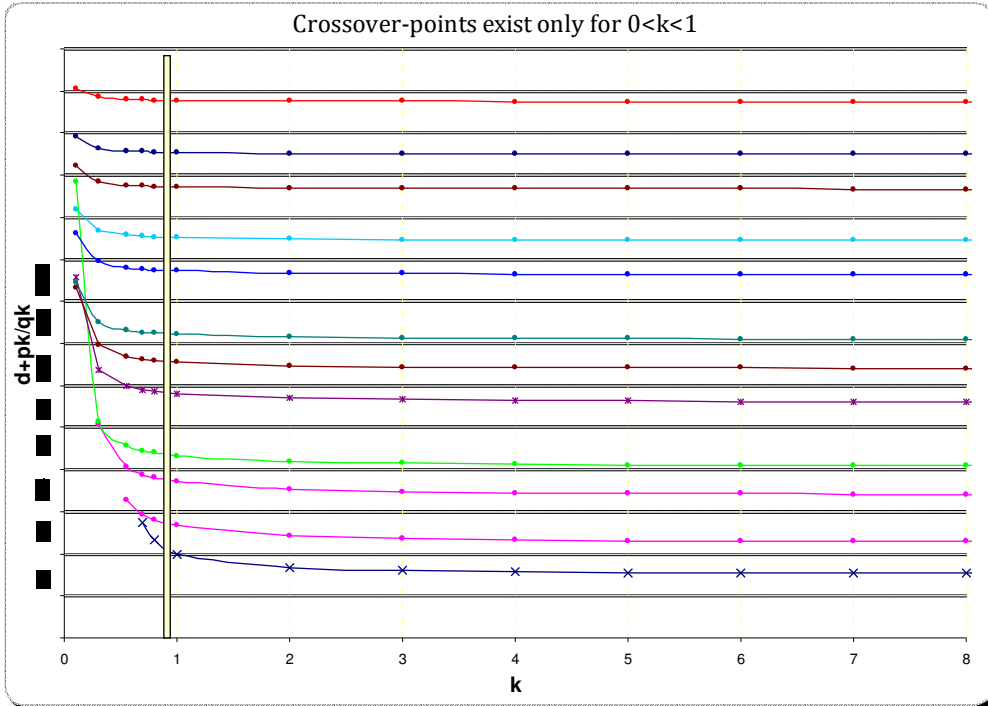
a) the progression of numerators and denominators are indicated by the inf-term

b)  $d=d_k$  for  $k>0$  and  $N>1$ , so we don't have **an irregular fraction crossing an integer** for  $k>0$

If b) can be shown to be valid for all  $N$ , then the *1-peak*-loop is also disproven.

It might be of interest to plot a graph with interpolated  $k$ . Such graph exhibits empirically that the crossing of an integer occurs only between  $0<k<1$ .<sup>1</sup>

When plotting this interpolated table for the  $3x+1$ -problem, moreover allowing also negative  $k$ , then we find integer crossing in the negative domain, and also the *1-peak*-loop residing there.



The reason, that there are no integer-crossings for  $k>=1$  depends on the numerator-value  $p=p_{oo}$  of the fraction for  $k->oo$ . Since it is by construction  $p_0 = d - 1$ , for to have no integer crossings we need, that  $p_0+k*p_{oo}<-1+k*q_{oo}$  or  $p_0+1 < k * (q_{oo}-p_{oo})$ . or  $d < k * (q_{oo}-p_{oo})$ , which - if it is already true for  $k=1$  - is obviously true for all  $k>1$ . For  $k=1$  the last expression is

$$d < 1 (q_{oo}-p_{oo})$$

$$d+p_{oo} < q_{oo}$$

or, using the expressions for  $3x+1$ , this means for any row  $N$

$$\left\lfloor \frac{3^N - 1}{2^N - 1} \right\rfloor + \left\{ \frac{3^N}{2^N} \right\} * 2^N < 2^N$$

$$\left[ 3^N \left( \frac{1}{2^N - 1} \right) - \frac{1}{2^N - 1} \right] < 2^N \left( 1 - \left\{ \frac{3^N}{2^N} \right\} \right)$$

$$\left[ 3^N \left( \frac{1}{2^N} + \frac{1}{2^{2N}} + \frac{1}{2^{3N}} + \dots \right) - \left( \frac{1}{2^N} + \frac{1}{2^{2N}} + \frac{1}{2^{3N}} + \dots \right) \right] < 2^N \left( 1 - \left\{ \frac{3^N}{2^N} \right\} \right)$$

<sup>1</sup> A property which is special to the  $3x+1$ -question. Other parameters,  $ax+b$  may have such crossings also in the table, and also having appropriate numbers in  $d$ , so that a *1-peak*-loop is possible with such parameters (see some examples in the appendix at 5.2)

$$\left\lfloor \frac{3^N - 1}{2^N} + \frac{3^N - 1}{4^N} + eps \right\rfloor < 2^N \left( 1 - \left\{ \frac{3^N}{2^N} \right\} \right)$$

and for large  $N$ : we can formulate:

$$a: \quad \text{if } \left\lfloor \frac{3^N}{2^N} \right\rfloor < 2^N \left( 1 - \left\{ \frac{3^N}{2^N} \right\} \right)$$

then there is no *1-peak*-loop for large  $N$ .

This can be rewritten in two ways:

$$b: \quad \text{if } \left\lfloor \frac{3^N}{2^N} \right\rfloor (2^N - 1) > 3^N - 2^N$$

$$c: \quad \text{or if } \left\{ \frac{3^N}{2^N} \right\} - \frac{1}{2^N} \left\{ \frac{3^N}{2^N} \right\} < 1 - \frac{3^N}{4^N}$$

then there is no *1-peak*-loop for large  $N$ .

Formulation c) occurs in a sharper conjecture

$$c.1: \quad \text{For } N > 2 \text{ we have } \left\{ \frac{3^N}{2^N} \right\} < 1 - \frac{3^N}{4^N}$$

in a detail of the Waring-problem

(see mathworld/powerfrac <http://mathworld.wolfram.com/PowerFractionalParts.html> )

where it is mentioned, that this detail is not yet solved, nevertheless assumed to be true.

But *if* it is true, then also c) is true and the *1-peak*-loop is not possible due to missing crossing-points in the above table.

Kurt Mahler approached the same problem in terms of his *z-numbers*, and was able to prove, that at most finitely many *z*-numbers, and thus solutions for *c.1*, can exist.

The above table seems to be a useful type of display. In the appendix I have documented similar tables for Collatz-like problem using different parameters, so  $(5x+1)/2$  ,  $(5x+1)/3$  ,  $(11x+1)/2$  ,  $(11x+1)/3$ . There are crossing points for some of these versions and thus loops in that problem-configurations cannot be excluded by the investigated properties here.

#### 4.3. Concatenated *m-peak*-loops -> general loop

*(much more material needs to be inserted; so far only a sketch:)*

The discussion is a simple generalization of the previous. Assume two "primitive transformations" (each single one not being a cycle):

$$b = PT(a; N_1; A) \quad a = PT(b; N_2; B)$$

then

$$a = k2^{N_1} - 1 \quad b = \frac{k3^{N_1} - 1}{2^{A-1}}$$

$$b = j2^{N_2} - 1 \quad a = \frac{j3^{N_2} - 1}{2^{B-1}}$$

$$ab = (k2^{N_1} - 1)(j2^{N_2} - 1) = \left(\frac{k3^{N_1} - 1}{2^{A-1}}\right)\left(\frac{j3^{N_2} - 1}{2^{B-1}}\right)$$

Then

$$1. \quad 2^{A-1+B-1} = \left(\frac{k3^{N_1} - 1}{j2^{N_2} - 1}\right)\left(\frac{j3^{N_2} - 1}{k2^{N_1} - 1}\right)$$

and also, by rotating denominators

$$2. \quad 2^{A-1+B-1} = \left(\frac{k3^{N_1} - 1}{k2^{N_1} - 1}\right)\left(\frac{j3^{N_2} - 1}{j2^{N_2} - 1}\right)$$

Here in *1.* each parenthese must be a perfect power of *2*, which imposes restrictions on *j* and *k* for modular reasons (for instance the two most simple ones: both must be odd, if they are simultaneously *1* they form the trivial cycle)

In *2.* we recognize, that increasing *k* (resp *j*) decreases the parentheses down to a perfect power of *3/2* which is always noninteger. The clue here were that we want to be able to show, that they also do not cross an integer bound while increasing *k* from 1 to infinity. But this is open to be proved...

It is obviously generalizable to any number of partial primitive transformations ("m-cycles"). Note, that based on the form *2.* only, Simons/deWeger could show, that up to *72* concatenations there is no such *m*-cycle, and that by increasing the number *m* of partial cycles the limit condition becomes too weak for the disprove, and integer-crossings cannot be excluded. Here a reintroduction of the modular arguments in the form of *1.* might be helpful (each term in *1.* must be a perfect power of *2*)

#### 4.4. Some loose ends

(not yet inserted)

## 5. Appendix

### 5.1. Approximation table for the $(3x+1)/2$ -problem

The following table lists results for the tests for general loops for lengths up to  $N=200$ .

$N :=$  looplevelth

$ug :=$  powerceil $2(3)/3^N$

$prod := (3+1/a)(3+1/b).../3^N$

$ratio := ug/prod$  must be  $\leq 1$  to make this loop possible by satisfying  
the critical equation

n	ug	prod	ug/prod	ug <= prod: loop can exist
1	1.333333	1.066667	1.250000	-false-
2	1.777778	1.117460	1.590909	-false-
3	1.185185	1.151323	1.029412	-false-
4	1.580247	1.180844	1.338235	-false-
5	1.053498	1.203998	0.875000	-false-
6	1.404664	1.225120	1.146552	-false-
7	1.872885	1.242876	1.506897	-false-
8	1.248590	1.259447	0.991379	-false-
9	1.664787	1.273924	1.306818	-false-
10	1.109858	1.287622	0.861944	-false-
11	1.479811	1.299885	1.138416	-false-
12	1.973081	1.311596	1.504336	-false-
13	1.315387	1.322259	0.994803	-false-
14	1.753850	1.332509	1.316201	-false-
15	1.169233	1.341960	0.871288	-false-
16	1.558977	1.351089	1.153868	-false-
17	1.039318	1.359586	0.764437	-false-
18	1.385758	1.367826	1.013110	-false-
19	1.847677	1.375554	1.343224	-false-
20	1.231785	1.383070	0.890616	-false-
21	1.642379	1.390163	1.181429	-false-
22	1.094920	1.397079	0.783720	-false-
23	1.459893	1.403638	1.040078	-false-
24	1.946524	1.410048	1.380467	-false-
25	1.297683	1.416152	0.916344	-false-
26	1.730243	1.422127	1.216659	-false-
27	1.153496	1.427838	0.807861	-false-
28	1.537994	1.433438	1.072941	-false-
29	1.025329	1.438807	0.712625	-false-
30	1.367106	1.444077	0.946699	-false-
31	1.822808	1.449144	1.257852	-false-
32	1.215205	1.454124	0.835696	-false-
33	1.620274	1.458923	1.110596	-false-
34	1.080182	1.463644	0.738009	-false-
35	1.440243	1.468204	0.980956	-false-
36	1.920324	1.472694	1.303954	-false-
37	1.280216	1.477038	0.866746	-false-
38	1.706955	1.481319	1.152321	-false-
39	1.137970	1.485469	0.766068	-false-
40	1.517293	1.489561	1.018618	-false-
41	1.011529	1.493533	0.677273	-false-
42	1.348705	1.497453	0.900666	-false-
43	1.798274	1.501263	1.197840	-false-
44	1.198849	1.505026	0.796564	-false-
45	1.598465	1.508688	1.059507	-false-
46	1.065644	1.512306	0.704648	-false-
47	1.420858	1.515831	0.937346	-false-
48	1.894477	1.519316	1.246928	-false-
49	1.262985	1.522714	0.829430	-false-
50	1.683980	1.526076	1.103471	-false-
51	1.122653	1.529358	0.734068	-false-
52	1.496871	1.532605	0.976684	-false-
53	1.995828	1.535778	1.299555	-false-
54	1.330552	1.538919	0.864602	-false-
55	1.774069	1.541990	1.150506	-false-
56	1.182713	1.545032	0.765494	-false-
57	1.576951	1.548009	1.018696	-false-
58	1.051300	1.550957	0.677840	-false-
59	1.401734	1.553845	0.902106	-false-
60	1.868978	1.556707	1.200598	-false-
61	1.245986	1.559512	0.798959	-false-
62	1.661314	1.562292	1.063383	-false-

63	1.107543	1.565018	0.707687	
64	1.476724	1.567721	0.941956	
65	1.968965	1.570374	1.253819	-false-
66	1.312643	1.573004	0.834482	
67	1.750191	1.575587	1.110818	-false-
68	1.166794	1.578149	0.739343	
69	1.555725	1.580666	0.984221	
70	1.037150	1.583163	0.655113	
71	1.382867	1.585618	0.872131	
72	1.843823	1.588053	1.161058	-false-
73	1.229215	1.590449	0.772873	
74	1.638954	1.592826	1.028960	-false-
75	1.092636	1.595165	0.684967	
76	1.456848	1.597487	0.911962	
77	1.942463	1.599772	1.214212	-false-
78	1.294976	1.602041	0.808328	
79	1.726634	1.604276	1.076270	-false-
80	1.151089	1.606495	0.716522	
81	1.534786	1.608680	0.954065	
82	1.023191	1.610851	0.635186	
83	1.364254	1.612991	0.845792	
84	1.819006	1.615116	1.126238	-false-
85	1.212670	1.617211	0.749853	
86	1.616894	1.619292	0.998519	
87	1.077929	1.621344	0.664837	
88	1.437239	1.623384	0.885335	
89	1.916319	1.625395	1.178986	-false-
90	1.277546	1.627395	0.785025	
91	1.703394	1.629367	1.045433	-false-
92	1.135596	1.631328	0.696118	
93	1.514128	1.633263	0.927057	
94	1.009419	1.635187	0.617311	
95	1.345892	1.637086	0.822126	
96	1.794522	1.638974	1.094906	-false-
97	1.196348	1.640839	0.729108	
98	1.595131	1.642693	0.971046	
99	1.063421	1.644524	0.646643	
100	1.417894	1.646345	0.861237	
101	1.890526	1.648145	1.147063	-false-
102	1.260350	1.649934	0.763879	
103	1.680467	1.651703	1.017415	-false-
104	1.120311	1.653462	0.677555	
105	1.493749	1.655200	0.902458	
106	1.991665	1.656930	1.202021	-false-
107	1.327777	1.658640	0.800521	
108	1.770369	1.660341	1.066268	-false-
109	1.180246	1.662023	0.710126	
110	1.573661	1.663697	0.945882	
111	1.049107	1.665352	0.629961	
112	1.398810	1.667000	0.839118	
113	1.865080	1.668629	1.117732	-false-
114	1.243387	1.670251	0.744431	
115	1.657849	1.671855	0.991622	
116	1.105233	1.673452	0.660451	
117	1.473643	1.675032	0.879770	
118	1.964858	1.676605	1.171927	-false-
119	1.309905	1.678162	0.780560	
120	1.746540	1.679711	1.039786	-false-
121	1.164360	1.681245	0.692558	
122	1.552480	1.682772	0.922573	
123	1.034987	1.684284	0.614497	
124	1.379982	1.685789	0.818597	
125	1.839977	1.687280	1.090499	-false-
126	1.226651	1.688764	0.726360	
127	1.635535	1.690234	0.967638	
128	1.090356	1.691697	0.644534	
129	1.453809	1.693147	0.858643	
130	1.938412	1.694590	1.143882	-false-
131	1.292274	1.696020	0.761945	
132	1.723032	1.697444	1.015075	-false-
133	1.148688	1.698855	0.676154	
134	1.531584	1.700260	0.900794	
135	1.021056	1.701653	0.600038	
136	1.361408	1.703040	0.799399	
137	1.815211	1.704414	1.065006	-false-
138	1.210141	1.705783	0.709434	
139	1.613521	1.707140	0.945160	
140	1.075681	1.708492	0.629608	
141	1.434241	1.709832	0.838820	
142	1.912321	1.711167	1.117554	-false-
143	1.274881	1.712490	0.744460	
144	1.699841	1.713808	0.991850	
145	1.133227	1.715116	0.660729	

146	1.510970	1.716418	0.880304	
147	1.007313	1.717709	0.586428	
148	1.343084	1.718996	0.781319	
149	1.790779	1.720272	1.040986	-false-
150	1.193853	1.721544	0.693478	
151	1.591804	1.722805	0.923960	
152	1.061202	1.724062	0.615525	
153	1.414937	1.725308	0.820107	
154	1.886582	1.726550	1.092689	-false-
155	1.257721	1.727783	0.727940	
156	1.676962	1.729011	0.969897	
157	1.117975	1.730229	0.646143	
158	1.490633	1.731443	0.860919	
159	1.987510	1.732648	1.147094	-false-
160	1.325007	1.733849	0.764200	
161	1.766676	1.735041	1.018233	-false-
162	1.177784	1.736228	0.678358	
163	1.570379	1.737407	0.903863	
164	1.046919	1.738582	0.602168	
165	1.395892	1.739748	0.802353	
166	1.861189	1.740910	1.069090	-false-
167	1.240793	1.742063	0.712255	
168	1.654391	1.743213	0.949047	
169	1.102927	1.744355	0.632284	
170	1.470569	1.745493	0.842495	
171	1.960759	1.746623	1.122600	-false-
172	1.307173	1.747749	0.747918	
173	1.742897	1.748867	0.996586	
174	1.161931	1.749982	0.663968	
175	1.549242	1.751088	0.884731	
176	1.032828	1.752192	0.589449	
177	1.377104	1.753288	0.785441	
178	1.836138	1.754380	1.046602	-false-
179	1.224092	1.755465	0.697304	
180	1.632123	1.756547	0.929166	
181	1.088082	1.757621	0.619065	
182	1.450776	1.758692	0.824918	
183	1.934368	1.759756	1.099225	-false-
184	1.289579	1.760817	0.732375	
185	1.719438	1.761870	0.975916	
186	1.146292	1.762921	0.650223	
187	1.528390	1.763965	0.866451	
188	1.018926	1.765005	0.577294	
189	1.358569	1.766039	0.769274	
190	1.811425	1.767070	1.025100	-false-
191	1.207616	1.768095	0.683004	
192	1.610155	1.769116	0.910147	
193	1.073437	1.770131	0.606417	
194	1.431249	1.771143	0.808093	
195	1.908332	1.772149	1.076846	-false-
196	1.272221	1.773152	0.717492	
197	1.696295	1.774149	0.956118	
198	1.130864	1.775143	0.637055	
199	1.507818	1.776130	0.848934	
200	1.005212	1.777116	0.565642	

### 5.2. Residual tables for other problem-parameters

The red lines mark the cross-over-points, which -if are integer- allow an *l*-cycle for that parameters (if some other conditions are also met, but this is not the focus of these tables). The leading integer of a row is the integral part of  $(3^n * i - 1) / (2^n * i - 1)$  and the following fractions the respective fractional parts for  $i >= 0$ , where *n* refers to the *n*'th row.

(irregular) Fraction or digits of $(3^n * i - 1) / (2^n * i - 1)$ vertical N=1..maxn, horizontal i=0.5, oo						
0	1	2	3	4	5	1
1	-1	3	5	7	9	2
2	1	2	3	4	5	6
2	-1	3	7	11	15	19
3	2	5	8	11	14	17
3	-1	7	15	23	31	39
4	5	6	7	8	9	1
5	-1	15	31	47	63	79
6	25	44	63	82	101	19
7	-1	31	63	95	127	159
10	35	60	85	110	135	25
11	-1	63	127	191	255	319
16	27	38	49	60	71	11
17	-1	127	255	383	511	639
24	185	346	507	668	829	161
25	-1	255	511	767	1023	1279
37	264	491	718	945	1172	227
38	-1	511	1023	1535	2047	2559
56	737	1418	2099	2780	3461	681
57	-1	1023	2047	3071	4095	5119

(irregular) Fraction or digits of $(5^n * i - 1) / (2^n * i - 1)$ vertical N=1..maxn, horizontal i=0.5, oo						
1	2	3	4	5	6	1
2	-1	1	3	5	7	9
5	6	7	8	9	10	1
6	-1	3	7	11	15	19
14	19	24	29	34	39	5
15	-1	7	15	23	31	39
38	39	40	41	42	43	1
39	-1	15	31	47	63	79
96	117	138	159	180	201	21
97	-1	31	63	95	127	159
243	252	261	270	279	288	9
244	-1	63	127	191	255	319
609	654	699	744	789	834	45
610	-1	127	255	383	511	639
1524	1749	1974	2199	2424	2649	225
1525	-1	255	511	767	1023	1279
3813	4170	4527	4884	5241	5598	357
3814	-1	511	1023	1535	2047	2559
9535	10296	11057	11818	12579	13340	761
9536	-1	1023	2047	3071	4095	5119

(irregular) Fraction or digits of  $(5^n \cdot i - 1) / (3^n \cdot i - 1)$   
 vertical N=1..maxn, horizontal i=0..5, oo

	0	2	4	6	8	10	2
1	---	---	---	---	---	---	---
	-1	2	5	8	11	14	3
2	---	---	---	---	---	---	---
	1	8	15	22	29	36	7
2	---	---	---	---	---	---	---
	-1	8	17	26	35	44	9
4	---	---	---	---	---	---	---
	3	20	37	54	71	88	17
4	---	---	---	---	---	---	---
	-1	26	53	80	107	134	27
7	---	---	---	---	---	---	---
	6	64	122	180	238	296	58
7	---	---	---	---	---	---	---
	-1	80	161	242	323	404	81
12	---	---	---	---	---	---	---
	11	220	429	638	847	1056	209
12	---	---	---	---	---	---	---
	-1	242	485	728	971	1214	243
21	---	---	---	---	---	---	---
	20	336	652	968	1284	1600	316
21	---	---	---	---	---	---	---
	-1	728	1457	2186	2915	3644	729
35	---	---	---	---	---	---	---
	34	1614	3194	4774	6354	7934	1580
35	---	---	---	---	---	---	---
	-1	2186	4373	6560	8747	10934	2187
59	---	---	---	---	---	---	---
	58	3584	7110	10636	14162	17688	3526
59	---	---	---	---	---	---	---
	-1	6560	13121	19682	26243	32804	6561
99	---	---	---	---	---	---	---
	98	4606	9114	13622	18130	22638	4508
99	---	---	---	---	---	---	---
	-1	19682	39365	59048	78731	98414	19683
165	---	---	---	---	---	---	---
	164	22704	45244	67784	90324	112864	22540
165	---	---	---	---	---	---	---
	-1	59048	118097	177146	236195	295244	59049

(irregular) Fraction or digits of  $(11^n \cdot i - 1) / (3^n \cdot i - 1)$   
 vertical N=1..maxn, horizontal i=0..5, oo

	2	4	6	8	10	12	2
3	---	---	---	---	---	---	---
	-1	2	5	8	11	14	3
13	---	---	---	---	---	---	---
	12	16	20	24	28	32	4
13	---	---	---	---	---	---	---
	-1	8	17	26	35	44	9
49	---	---	---	---	---	---	---
	48	56	64	72	80	88	8
49	---	---	---	---	---	---	---
	-1	26	53	80	107	134	27
180	---	---	---	---	---	---	---
	179	240	301	362	423	484	61
180	---	---	---	---	---	---	---
	-1	80	161	242	323	404	81
662	---	---	---	---	---	---	---
	661	846	1031	1216	1401	1586	185
662	---	---	---	---	---	---	---
	-1	242	485	728	971	1214	243
2430	---	---	---	---	---	---	---
	2429	2520	2611	2702	2793	2884	91
2430	---	---	---	---	---	---	---
	-1	728	1457	2186	2915	3644	729
8910	---	---	---	---	---	---	---
	8909	9910	10911	11912	12913	13914	1001
8910	---	---	---	---	---	---	---
	-1	2186	4373	6560	8747	10934	2187
32671	---	---	---	---	---	---	---
	32670	37120	41570	46020	50470	54920	4450
32671	---	---	---	---	---	---	---
	-1	6560	13121	19682	26243	32804	6561
119796	---	---	---	---	---	---	---
	119795	122818	125841	128864	131887	134910	3023
119796	---	---	---	---	---	---	---
	-1	19682	39365	59048	78731	98414	19683
439252	---	---	---	---	---	---	---
	439251	472504	505757	539010	572263	605516	33253
439252	---	---	---	---	---	---	---
	-1	59048	118097	177146	236195	295244	59049

### 5.3. The Steiner-proof of 1977

[Steiner]:

Briefly, my 1977 proof runs as follows. I will just give the steps, not the details here.

1). Any circuit for the  $3x+1$  problem corresponds to an integer solution  $k, l, h$ , of the Diophantine equation

$$(2^{k+l} - 3^k) h = 2^l - 1 \quad (*)$$

2) To show that the only integer solution of (\*) is  $1,1,1$ .

First, reduce this to a problem in linear forms in logarithms:

$$0 < |l/k - \log_2 3/2| < 1/(k \cdot \ln 2 \cdot (2^k - 1))$$

3). This shows that if  $k > 4$  then  $l/k$  must be a convergent in the continued fraction expansion of  $\log_2(3/2)$ .

4). By using a lemma of LEGENDRE, one can prove that a partial quotient of this CF must exceed  $10^{4690}$ .

5). Using BAKER'S, or RHIN'S theorem one finds a reasonable upper bound for  $k$  and the denominators of all convergents in this range are all smaller than 2500.

The Steiner-formula is identical to my critical condition for  $1$ -peak-loops, by few rearrangements.

First, to relate variables of his formula (\*) and of mine (4.1.14), I translate:

$$(h, l, k) \quad \rightarrow (k, A-1, N)$$

$$2^{A-1} k 2^N - k 3^N = 2^{A-1} - 1$$

$$\text{Then } 2^{A-1} (k 2^N - 1) = k 3^N - 1$$

$$2^{A-1} = \frac{k 3^N - 1}{k 2^N - 1}$$