Exponential diophantine problems

Collatz-conjecture

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Exercises in "proving" - Nonexistence of *1-cycle* in the Collatz-problem

Intro:

It has already been proved that there exist no nontrivial 1-cycle; the first proof has been found by R. Steiner [St,77], on which J. Simons and B. de Weger based in 2003 their own proof for nonxistence of the 2-cycle, and after that by some new techniques the proof that moreover nontrivial 3- to 79-cycles cannot exist. [Si,07]

For me the final step in that proofs has always been a riddle, although I've arrived at the required formulae a couple of years ago myself naively (not knowing the Steiner and Simons/de Weger results) and later casually attempted to understand the mentioned proofs. The Steiner proof uses results from transcendental-number theory which lower-bound rational approximations for linear forms of logarithms and suffice to prove the nonexistence of the 1-cycle. A question in the discussion forum math.stackexchange.com[1] asked for an explanation of and a reference to such a proof and this motivated me to try this myself again with over the years improved experience.

I assume I've understood now the technique for the last step, and it seems as if my description is even a bit simpler to read - I don't need to refer to the theory of continued fractions.

Criticism and error-reporting is much appreciated.

Gottfried Helms, Kassel, 12'2017 (minor textual edits over 10'2017)

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[1] https://math.stackexchange.com/q/2485128 22.10.2017

1. The transformation, notated in the Syracuse-style

We use the following general style[2] for the notation of one step of the Collatz-iteration for a given odd number *a>0* :

(1.1) $$b=\frac{3a+1}{2^A}$$ *(a and b odd>0, A chosen to fit this requirement)*

$b=\frac{3a+1}{2^A}$

For iterated transformations[3] we can write:

(1.2) $$a_2=\frac{3a_1+1}{2^{A_1}} \qquad a_3=\frac{3a_2+1}{2^{A_1}} \qquad \ldots \qquad a_{N+1}=\frac{3a_N+1}{2^{A_N}}$$

We denote the ***[N]umber of iterations*** *N* and the ***[S]um of exponents*** A_k as *S*.

N, S, A_k, B

Note: for memorizability I denote values in the exponents with capital letters and members of a transformation and general indeterminates with small letters.

Note: the notations in Simons[Si,07] is "*K*" for *N* here, "*K+L*" for *S* here and "*L*" for *S-N=B* here; for the use of letter *B* see subsection "*1-cycle*" below ($B=A_N-1$).

Note: in Eric Rosendaal's webpage[Ro,09] the letter "*N*" denotes the first element of a trajectory, which is here a_1 . The number of all elementary transformations (on odd and on even numbers) until the final value $a_{N+1}=1$ is reached (often named as *"total stopping time"*), is called function "Delay" $D(a_1)$ ("$D(N)$" in the original notation) which, if $a_{N+1}=1$ is assumed, were $D(a_1) = S + N$ here.

There is a useful product notation involving all elements of an iterated transformation in expanded and in unexpanded form[4]:

(1.3) $$a_2a_3\ldots a_Na_{N+1}=\left(\frac{3a_1+1}{2^{A_1}}\right)\left(\frac{3a_2+1}{2^{A_1}}\right)\ldots\left(\frac{3a_N+1}{2^{A_N}}\right)$$

which, when a bit harmonized and then rearranged,

$$a_1a_2a_3\ldots a_N=\frac{a_1}{a_{N+1}}\left(\frac{3a_1+1}{2^{A_1}}\right)\left(\frac{3a_2+1}{2^{A_1}}\right)\ldots\left(\frac{3a_N+1}{2^{A_N}}\right)$$

(1.3 a) $$2^S=\frac{a_1}{a_{N+1}}\left(3+\frac{1}{a_1}\right)\left(3+\frac{1}{a_2}\right)\ldots\left(3+\frac{1}{a_N}\right)$$

leads to a somehow characteristic equation:

(1.4) $$\frac{2^S}{3^N}=\frac{a_1}{a_{N+1}}\left(1+\frac{1}{3a_1}\right)\left(1+\frac{1}{3a_2}\right)\ldots\left(1+\frac{1}{3a_N}\right)$$

$\frac{2^S}{3^N}=\frac{a_1}{a_{N+1}}\prod_{k=1}^{N}\left(1+\frac{1}{3a_k}\right)$

Note: in [Ro,09] there is a function *$RES(a_1)$* introduced ("*RES(N)*" in the original text) which is the product of the parentheses on the rhs: *$RES(a_1)=(1+1/(3a_1))(1+1/(3a_2))\ldots(1+1/(3a_N))$*.

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[2] This notation can be found for instance in Wikipedia as "Syracuse"-notation [WP], a version of the Collatz-iteration as used when the problem was circulating in the Syracuse university (New York, USA) in the 1950s. It was employed early in the study of the Collatz-problem, for instance by R. Crandall [Cr,78], who was one of the first pioneers in the study of the cycle problem in the Collatz-problem.

[3] Also called "*trajectory*" or "*orbit*" depending on some authors preferences.

[4] Seen already in Crandall [Cr,78] but widely exploited in [Si,04] and [Si,07]

2. On the (general) cycle-problem

To have a cycle it is required that $a_{N+1}=a_1$ and for some chosen N (and the subsequent S) it must exist a solution in positive odd integers a_k, not divisible by 3 (so all $a_k \equiv \pm 1 \pmod 6$) for the equation (1.3 a) with $a_{N+1} = a_1$:

(2.1) $$2^S = \left(3+\frac{1}{a_1}\right)\left(3+\frac{1}{a_2}\right)...\left(3+\frac{1}{a_N}\right)$$

In a more "canonical" form:

(2.2) $$\frac{2^S}{3^N} = \left(1+\frac{1}{3a_1}\right)\left(1+\frac{1}{3a_2}\right)...\left(1+\frac{1}{3a_N}\right)$$

$$\frac{2^S}{3^N} = \prod_{k=1}^{N}\left(1+\frac{1}{3a_k}\right)$$

Here obviously S must be such, that $2^S > 3^N$, and we can find the smallest S by

(2.3) $$S = \lceil N \log_2(3) \rceil$$

$$S = \lceil N \log_2(3) \rceil$$

Note: By this equation, given N, the odd integer elements a_k , and prominently a_1 , become the unknowns now for which we need to solve equation *(2.1)* to find a cycle.

Note: If we chose $2^S<3^N$ we need a_k from the negative integers, and there exist cycles.

With a little heuristic we can observe two contradicting properties: on one hand 2^S and 3^N are usually fairly distant, so their ratio is not very near to 1, but on the other hand for large a_k the rhs is very near to 1. To look deeper at this we logarithmize:

$$\log\left(\frac{2^S}{3^N}\right) = \log\left(1+\frac{1}{3a_1}\right)+\log\left(1+\frac{1}{3a_2}\right)+...+\log\left(1+\frac{1}{3a_N}\right)$$

$$S\log 2 - N\log 3 = \frac{1}{3}\left(\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_N}\right) - \frac{1}{2}\frac{1}{3^2}\left(\frac{1}{a_1^2}+\frac{1}{a_2^2}+...+\frac{1}{a_N^2}\right)+\frac{1}{3}\frac{1}{3^3}(...)-...$$

and (neglecting small trailing terms) we find an inequality, which shall become productive below:

(2.4) $$S\log 2 - N\log 3 < \frac{1}{3}\left(\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_N}\right)$$

The reason, why this becomes productive for the disproof of possibility of certain cycles is, that there is a formula which ***lower-bounds*** the absolute difference on the lhs ***functionally dependend* on** N. Simons[Si,07] refers[5] to a result of G. Rhin [Rh,86] such that

$$|S\log 2 - N\log 3| > \frac{1}{457N^{13.3}}$$

(2.4 a) $$|S\log 2 - N\log 3| > e^{-13.3(0.46057+\log N)} \sim \frac{1}{457N^{13.3}}$$

Putting this to the *lhs* of the above inequality we find that

$$\frac{1}{457N^{13.3}} < S\log 2 - N\log 3 < \frac{1}{3}\left(\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_N}\right)$$

which - in a more compact form - sets an ***upper bound*** for the a_k to allow existence of a cycle:

(2.4 b) $$\frac{1}{457N^{13.3}} < \frac{1}{3}\left(\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_N}\right)$$

[5] in [Si,07], pg. 5: "*The best result today is proved by Rhin[18] i.e.* $\Lambda > e^{-13.3(0.46057+\log(K)}$ (9) "

Note: It is worth to note that in formula (2.1) $2^S=\left(3+\frac{1}{a_1}\right)\left(3+\frac{1}{a_2}\right)\ldots\left(3+\frac{1}{a_N}\right)$ we can guess a rough mean value a_m for the a_k. When we assume all $a_k = a_m$ then we can determine an *upper bound* for the leading element a_1 (which shall be assumed to be the minimal one and thus must actually be *smaller* than a_m) for the assumed cycle:

(2.5) $$2^S=\left(3+\frac{1}{a_m}\right)^N \Rightarrow a_m=\frac{1}{2^{S/N}-3} \qquad \Rightarrow \qquad a_1<\frac{1}{2^{S/N}-3}$$

$a_1<\frac{1}{2^{S/N}-3}$

So if we assume there is a cycle with *100* steps, then $N=100$ and $S=159$. From this we get then $a_1<95.3$ - and if we know (by searches in small numbers a_1) that no number $1<a_1<100$ is a member of a cycle, then we have already proven that there is no cycle possible with $N=100$, $S=159$ (or "K"$=100$,"$K+L$"$=159$ in Simons' notation).[6]

3. On the *1-cycle* problem: no *1-cycle* possible (except the trivial one)

The *1-cycle* is a construct meant to be better analyzable than the cycle in general. Here we look at hypothesized[7] cycles whose N members form a ***strictly increasing*** sequence (all the $A_k=1$ for $k=1$ *to* $N-1$) and are followed by a single decreasing step $a_{N+1}=(3a_N+1)/2^{A_N}$ where $A_N>1$.

up up ... up down

We have then not only $S=\text{ceil}(N\log_2(3))\sim1.58\cdot N$ but also $A_N=S-(N-1)\sim0.58\cdot N$.

Note: in [Si,07] this value $S-N=A_N-1$ is denoted as "L"

By that *1-cycle*-definition we have an exploitable regular pattern in the relation between consecutive a_k in the (increasing) sequence: by the observations that we can rewrite those transformations, each with $A_k=1$,

$$a_2=\frac{3a_1+1}{2}=\frac{3a_1+3-2}{2}=\frac{3}{2}(a_1+1)-1$$
$$a_3=\frac{3}{2}(a_2+1)-1=\frac{3}{2}(\frac{3}{2}(a_1+1)-1+1)-1=\left(\frac{3}{2}\right)^2(a_1+1)-1$$
$$\vdots$$

we can (after induction) extract the general relation

(3.1 a) $$a_k=\left(\frac{3}{2}\right)^{k-1}(a_1+1)-1$$

and can then state for the final element a_N:

(3.1 b) $$a_N=\left(\frac{3}{2}\right)^{N-1}(a_1+1)-1$$

$a_N=\left(\frac{3}{2}\right)^{N-1}(a_1+1)-1$

By the next iteration this must fall back to a_1 to form a cycle:

$$a_1=\frac{3\left(\left(\frac{3}{2}\right)^{N-1}(a_1+1)-1\right)+1}{2^{A_N}}=\frac{3\left(\frac{3}{2}\right)^{N-1}(a_1+1)-2}{2^{A_N}}$$
$$=\frac{\left(\frac{3}{2}\right)^{N}(a_1+1)-1}{2^{A_N-1}}=\frac{\frac{3^N(a_1+1)-2^N}{2^N}}{2^{A_N-1}}$$

and we get:

(3.2) $$a_1=\frac{3^Na_1}{2^S}+\frac{3^N-2^N}{2^S}$$

[6] More examples and heuristics for a larger overview and general trend using convergents of the continued fraction of $\log_2(3)$ for values up to $N=10^{1\,000\,000}$ can be found in [He,14]

[7] We shall arrive at the conclusion, that such cycles are not possible in the Collatz-iteration. However, allowing negative numbers a_k, such an *1-cycle* is actually possible, (as well as in modifications like the $3x+d$ or the $5x+1$-problem), and can be found by solving (of course by applying the other restrictions by the inequalities), so it might be justified to speak of "hypothesized *1-cycles*"

From this a unique determination for a_1 can be derived:

$$a_1(2^S - 3^N) = 3^N - 2^N$$

which gives then

(3.3 a) $$a_1 = \frac{3^N - 2^N}{2^S - 3^N}$$ *or*

$a_1 = \frac{3^N - 2^N}{2^S - 3^N}$

(3.3 b) $$\frac{a_1 + 1}{2^N} = \frac{2^B - 1}{2^S - 3^N}$$ *(introducing $B = S - N = A_N - 1$)*

$B = S - N$
$= A_N - 1$

From *(3.3 b)* we can also conclude some more knowledge about the required structure of the first element a_1, which is independent from the values S or B (which are somehow jittering with varying N). Because the denominator in the *lhs* is coprime to that on the *rhs*, a_1+1 must be a -say- k'th multiple of that perfect power 2^N, so we know

(3.4) $$a_1 = 2^N k - 1$$

$a_1 = 2^N k - 1$

with some positive (odd)[8] k where $2<k<2^B$ and thus $a_1<2^S$.

After we have this description of the element a_1 in terms of N we proceed now to simplify the parenthese in *(2.4 b)*: $\frac{1}{457N^{13.3}} < \frac{1}{3}\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_N}\right)$. By *(3.1 a)* we know, that $a_k = (3/2)^{k-1}(a_1+1)-1$ We rewrite the parenthese first as an exact expression:

(3.5 a) $$\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_N}\right) = \left(\frac{1}{a_1}+\frac{1}{1.5(a_1+1)-1}+\frac{1}{1.5^2(a_1+1)-1}+\ldots+\frac{1}{1.5^{N-1}(a_1+1)-1}\right)$$

For large a_1 this approximates a partial geometric series with quotient $q = 1/(1.5) = 2/3$. We collect the difference to such an expression in a small epsilon-error term:

(3.5 b) $$\begin{aligned}\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_N}\right) &= \left(\frac{1}{a_1}+q\frac{1}{a_1}+q^2\frac{1}{a_1}+\ldots+q^{N-1}\frac{1}{a_1}\right)-3\varepsilon_{a_1,N}\\ &= \frac{1}{a_1}\cdot\frac{1-q^N}{1-q}-3\varepsilon_{a_1,N}\\ &= \frac{3}{a_1}\left(1-\left(\frac{2}{3}\right)^N\right)-3\varepsilon_{a_1,N}\end{aligned}$$

Inserting this into *(2.4 b)* gives

$$\frac{1}{457N^{13.3}} < \frac{1}{3}\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_N}\right) = \frac{1}{3}\cdot\left(\frac{3}{a_1}\left(1-\left(\frac{2}{3}\right)^N\right)-3\varepsilon_{a_1,N}\right)$$

and by this

(3.5 c) $$\frac{1}{457N^{13.3}} < \frac{1}{a_1}\left(1-\left(\frac{2}{3}\right)^N\right)-\varepsilon_{a_1,N}$$

For an initial rough estimate we ignore the epsilon and focus on that element which shall give a contradiction:

(3.5.d) $$\frac{1}{457N^{13.3}} < \frac{1}{a_1}\frac{3^N-2^N}{3^N}$$

$$a_1 < 457N^{13.3}\frac{3^N-2^N}{3^N}$$

[8] This and some more properties of k are derived in more detail in [He,06]. See a short overview *Note 2* in the "**Notes**"-appendix.

On the other hand, we know by *(3.4)* that a_1 must have the form $a_1 = 2^N k - 1$. Inserting this we get

$$(3.6) \qquad 2^N k - 1 < 457 N^{13.3} \left(1 - \left(\frac{2}{3} \right)^N \right)$$

We compare now the logarithm of both sides, changing $2^N k-1$ to $2^N k(1 - 1/2^N/k)$ and using $\log(457) \sim 6.13$:

$$(3.6\ a) \qquad N \log 2 + \log k - (\tfrac{1}{2^N k}) - \tfrac{1}{2}(\tfrac{1}{2^N k})^2 - \tfrac{1}{3}(\tfrac{1}{2^N k})^3 - \ldots < 6.13 + 13.3 \log N - (\tfrac{2}{3})^N - \tfrac{1}{2}(\tfrac{2}{3})^{2N} - \ldots$$

Here we get -on both sides- for the logarithms of the parentheses their Mercator-series with very small terms which already for small numbers N become relatively neglibile. Ignoring them for the moment this gives:

$$(3.7) \qquad N \log 2 - 13.3 \log N \quad < \quad 6.13 - \log k$$

which must be solved for some integer $N>1$ and $k>1$ to make a nontrivial *1-cycle* possible.

Because $N \log 2$ grows much faster than $13.3 \log N$ we'll have values larger than *6.31* already with fairly small N. We assume also the smallest possible value for k, namely $k=3$. Then the first contradiction occurs for $N = 95$:

$$(3.8) \qquad 95 \log 2 - 13.3 \log 95 \quad \not< \quad 6.13 - \log 3 \qquad \textit{for } N=95$$

The consequence is:

(3.9 a) ***Lemma 1:*** A *1-cyle* with a length of $N>=95$ cannot exist.

Next, for all $N<95$ the existence of cycles can simply be tested by the formula inserting N and its related S into the formula *(3.3. a)*:

$$a_1 = \frac{3^N - 2^N}{2^S - 3^N} \qquad \text{==>} \quad \textit{for all } N = 2 \textit{ to } 95 \qquad a_1 \textit{ is non-integer}$$

and we find, that none of that values for N gives a positive odd integer value in a_1 except for $N=1$, $S = A_1 = 2$, $a_1 = 1$ which of course we knew already. The consequence is:

(3.9 b) ***Lemma 2:*** A *1-cyle* with a length of $1<N<95$ does not exist.

So combining ***Lemma 1*** and ***Lemma 2*** we have proved:

(3.10) ***Theorem:* There is no nontrivial *1-cycle* in the Collatz *3x+1* - problem**

No 1-cycle!

Notes:

***Note 1*:** (1-cycles) the equation *(3.9 b)* gives for $N=1,S=1$ the value $a_1=-1$ which means $-1 = (3 \cdot -1 + 1)/2^1$ forming a *1-cycle* - which may, however, be seen as degenerated. A non-degenerated *1-cycle* is with $N=2, S=3$ giving $-5 = (3^2 - 2^2)/(2^3 - 3^2) = 5/-1$ indicating the cyclic transformation $-5 \to -7 \to -5$. Such *1-cycles* are existent also in the generalized $3x+d$ -problem for some parameters d and also in some $mx + 1$ - problems. One can thus conclude that the result for the $3x+1$-problem of Collatz cannot be proven by algebraical / structural analysis alone but that any such attempted proof needs explicite reference to the specific problem-parameters 3 and 1.

***Note 2*:** (1-cycles) Properties of the indeterminate k in $a_1=2^N k -1$ are derived in [He,06]. In short we have the following. If we iterate with all $A_k=1$ for $k=1..N-1$ then $a_N = (3^{N-1}k - 1)/2$. To have the numerator divisible by 2 the indeterminate k must be odd. By a short analysis, if we write $3^N = n\, 2^B + r$ where $r<2^B$ we find that as well it must be that $k< 2^B$ and moreover $k=1/r \pmod{2^B}$.
That $k =/= 1$ follows from $2^{S-N}=(3^N k-1)/(2^N k-1)=(3^N-1)/(2^N-1)$ which has no solution for $N>1$. This was already known to the medieval mathematician N. Oresme (see [WP]-entry).

***Note 3*:** (1-cycles) A tremendously simple proof for the nonexistence of the *1-cycle* were possible if we could show that the structure of 3^N cannot be $(2^N-1)2^B + r$ (*where* $B=S-N$ *and* r *is the residue* $3^N \pmod{2^B}$) ! A bit more visually that means, that 3^N displayed as bitstring cannot have N leading "*1*" and only B (jittering) digits as residue.

It is easy to derive the necessity of that structure for the solution for an *1-cycle* starting at *(3.9 b)*:

$$
\begin{array}{lll}
(a_1+1)(2^S - 3^N) & = (2^B - 1)\, 2^N & \text{//this is rerrangement of (3.9 b)} \\
k \cdot 2^N(2^S - 3^N) & =(2^B - 1)\, 2^N & \text{// this uses } a_1 = 2^N k - 1 \\
k \cdot (2^S - 3^N) & =(2^B - 1) & \\
k \cdot (2^{N+B} - (n2^B+r)) & =(2^B - 1) & \\
k \cdot 2^B(2^N - n) - 2^B & = kr - 1 & \\
k \cdot (2^N - n) & = (kr - 1)/2^B + 1 <= k &
\end{array}
$$

Since the *rhs* is at most equal to k the parenthese in the *lhs* can at most equal 1. The empirical observation, that in $3^N= n\, 2^B+r$ there is ***never*** $n=(2^N-1)$ is perhaps even more suggestive than that of the empirical observations along the line of the above eq *(3.9 b)*, but I've never seen an attempt to prove that that empirical observation is true for all N and 3^N.

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