Special Functionss,
Exercises and heuristics

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Sums of like powers of logarithms

Abstract: This is an exercise in application of the concept of Carleman-matrices for the question of getting a finite expression for the problem of summing a function at consecutive arguments $s(x,n) = f(x)+f(x+1)+f(x+2)+ \dots + f(x+n-1)$. I've experimented earlier with the problem of sums-of-like-powers in the Faulhaber and Bernoulli-sense. That problem led to the solution by finite polynomials, now known as Bernoulli-polynomials.

Here I try, whether we can import the rationale, which I'd employed in that question, for the current problem where $f(x)$ is the log-function, and more generally, for $s_p(x,n) = f_p(x) + f_p(x+1) + \dots + f_p(x+n-1)$ where $f_p(x)$ is the p'th power of $\log(x)$. We do not arrive at polynomials for that problem, but must work with infinite series; however we get some much interesting reflection of the zeta-function and its derivatives.

Temporary remark: there is one (minor) unsolved problem with it, where we want to have the constant for a power series being zeta(0)' but get it by the difference of 1; I'm still working on this. It does not affect the solution for the sums of consecutive logarithms with finitely many terms; only for the infinite series this difference occurs. Also there is no problem at all if we work the same scheme out for the alternating sums.

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1. Sums of like powers of logarithms

1.1. Intro

Is there a polynomial or series-expression for the sum of consecutive logarithms

$$log(1) + log(2) + log(3) + ... + log(n) \quad ?$$

Or even, with some integer exponent p

$$log(1)^p + log(2)^p + log(3)^p + ... + log(n)^p \quad ?$$

A bit background: In my treatize "Sums of like powers" [Helms2009] I have successfully explored the old and venerable problem of the possibility to find a polynomial expression for the sum of like powers

$$s_p(n) = 1^p + 2^p + 3^p + ... + n^p$$

when I was working with the *Carleman-matrix ansatz* and I've rediscovered this way the Bernoulli-polynomials but - more precisely - their integrals. There the coefficients of the polynomials appear organized in a matrix, which I called ***ZETA***-matrix and which contains the original Faulhaber-solutions for that summing problem (plus a small, but significant, extension).

This ansatz/method does then not only allow to generalize the above sum-of-like-powers to any contiguous segment

$$s_p(a,n) = a^p + (a+1)^p + (a+2)^p + ... + (a+n-1)^p$$

but moreover, by evaluation of the Bernoulli-/Zeta-polynomials, allows meaningful generalizations to

- non-integer first value a and even to
- non-integer differences of the first and the last element of the sum.

  (The latter thus introducing somehow "fractional bounds" for the summation[1]).
- and even to non-natural exponents p *(but then gives power series instead of polynomials).*

The path to the solution via the Bernoulli-polynomials resp. the ***ZETA***-matrix has some basic and much interesting aspects for the student of number theory - so I asked myself, whether we can apply the same ideas as employed in that treatize to the question here of sums of logarithms (and hopefully of powers of logarithms), and whether we shall arrive at polynomials or at least at functionally similar power series for such generalized sums.

[1] It might be of interest that there exists a serious example of the use of such "fractional index": it occurs already in a treatize of L. Euler [Euler1813], when he considered some generalization which was then equivalent to denote summation with a fractional bound: $\sum_{k=1}^{-\frac{1}{2}} \frac{1}{k} = -2\ln(2)$ (This is due to a remark in [Muller2007])

1.2. The problem for which we search for the formula

So we want to find a general solution for the proposed function:

(1) $S_p(a,b) = \log(a+1)^p + \log(a+2)^p + \log(a+3)^p + \log(a+4)^p + \dots + \log(b)^p$

$$= \sum_{k=a}^{b-1} \log(1+k)^p$$

First we should check whether we can meaningfully introduce a helper function in the sense of an analogon to the Hurwitz-Zeta-modification:

(2) $H_p(a) = \log(a+1)^p + \log(a+2)^p + \log(a+3)^p + \log(a+4)^p + \dots$

$$= \sum_{k=a}^{\infty} \log(1+k)^p$$

from which we expect then the general solution for our problem:

$$S_p(a,b) = H_p(a) - H_p(b+1) \qquad = S_p(a, oo) - S_p(b+1, oo)$$

$$= \sum_{k=a}^{\infty} \log(1+k)^p - \sum_{k=b+1}^{\infty} \log(1+k)^p = \sum_{k=a}^{b} \log(1+k)^p$$

Clearly we should begin with the convergent cases where $p < -1$ but we shall see, that the generalization to general integer exponents p (with perhaps exceptions) gives (heuristically) meaningful results.

1.3. The method of "indefinite summation"

We try this by the method of *indefinite summation*[2].

For this we need a function which performs the increment for the argument

t: $\log(x) \to \log(x+1)$

Much obviously this can be solved by the function

$t(x) = \log(1+\exp(x))$

A power series for this can be obtained simply using Pari/GP to get the following leading coefficients:

(2) $t(x) = \log(2) + 1/2\, x + 1/8\, x^2 - 1/192\, x^4 + 1/2880\, x^6 - \dots$

Examination of the pattern in that coefficients give the most plausible explanation in terms of the Dirichlet-η "eta"-function (which is also called as "alternating zeta"[see mathworld]) at its integer arguments from 1 down to $-infinity$ which are

$\eta(1)=$	log(2) ~ 0.69314
$\eta(0)=$	½
$\eta(-1)=$	¼
$\eta(-2)=$	0
$\eta(-3)=$	-1/8
$\eta(-4)=$	0
$\eta(-5)=$	¼
...	...

Thus if that initial heuristic holds, then the given coefficients for $t(x)$ have the form

[2] See for instance "indefinite summation" at wikipedia [wp:indefinite]

~ 0.69314 = log(2)	$=\eta(1)/0!$
1/2	$=\eta(0)/1!$
1/8	$=\eta(-1)/2!$
0	$=\eta(-2)/3!$
-1/192	$=\eta(-3)/4!$
0	$=\eta(-4)/5!$
1/12880	$=\eta(-5)/6!$
...	...

and thus (at the time of the current edition this was likely been proved, see footnote[3]) we can write the power series for the function $t(x)$ as

$$(3) \qquad t(x)=\sum_{k=0}^{\infty}\frac{\eta(1-k)}{k!}x^k$$

1.4. Asymptotic solution by the Neumann-series of a Carleman-matrix

We use this function to define the "*matrixoperator*" (or: *Carlemanmatrix*[4]) ***T***. With this we have

$$V(\log(x)) \cdot T = V(\log(1+x))$$

Note, that the size of the vectors ***V*** and the matrix ***T*** is assumed as infinite here and also, that ***T*** is ***not*** triangular[5]; its top-left aspect is

1	0.693147	0.480453	0.333025	0.230835	0.160003
0	1/2	0.693147	0.720680	0.666049	0.577088
0	1/8	0.423287	0.700030	0.887192	0.976834
0	0	1/8	0.384930	0.706913	1.01685
0	-1/192	0.00840472	0.118734	0.360535	0.713057
0	0	-1/192	0.0126071	0.112468	0.343120
0	1/2880	-0.000820731	-0.00416026	0.0147313	0.106971
0	0	1/2880	-0.00123110	-0.00311219	0.0158164

where we see in the second column the coefficients of the power series for $t(x)$ and in the following columns that of $t(x)^2$, $t(x)^3$ and so on which is due to its property being a Carlemanmatrix.

1.4.1. The Neumann-series of *T*

The matrix-based method of the idea of indefinite summation requires now that for the sum with infinitely many terms

$$V(\log(x)) \cdot (I + T + T^2 + T^3 + \dots) = V(\log(x)) + V(\log(1+x)) + V(\log(2+x)) + V(\log(3+x)) + \dots$$

the involved matrix-series $(I + T + \dots)$ is a meaningful expression and that the analogon to the geometric series in the cases of scalar arguments holds equivalently[6]

$$I + T + T^2 + T^3 + \dots = (I - T)^{-1}$$

where ***I*** is the identity-matrix

[3] There was a seemingly correct proof for this in an answer to my question in the discussionboard math.stackexchange , see http://math.stackexchange.com/questions/307274/how-can-i-prove-my-conjecture-for-the-coefficients-in-tx-log1-expx

[4] See for instance "Carleman matrix" in wikipedia [wp:carleman]; also known in variants as Bell-, Jabotinsky- or Sonnenscheinmatrix. In my own earlier articles I did not yet know that names, I've simply called them "matrixoperators" and didn't yet adapt that all denotations to the canonical one, which likely shall become "Carlemanmatrix"

[5] This is different from the example with the ***ZETA***-matrix/Bernoulli-polynomials in the [2009]-treatize

[6] This is known as *Neumann-series*; an example, and criteria for convergence/applicability, are for instance in Wikipedia

or can be made to hold by some regularization. Unfortunately the parenthese is not immediately invertible so we need some workaround.

1.4.2. A pseudo-inversion by omitting the empty first column

A method, which I've discovered and successfully applied in my 2009-treatize on the Pascalmatrix ***P***, and which has also been earlier invented by some authors[7] is the following, which I call here "pseudo-inversion". I'll apply and describe it, in a short overview, ignoring at the moment that there might be some sophisticated bias-effect in the empirical asymptotics[8].

First, by the construction of a matrix ***C*** as Carlemanmatrix we'll have systematically that in ***(I - C)*** the first column is zero, so for that matrix the inversion is not possible.

To get some -hopefully meaningful- pseudoinverse we introduce the upper square submatrix of ***(I - C)*** from which the first column is removed, with some size $n \times n$, call that submatrix Q_n and invert it to get Q_n^{-1} . We repeat this process with increasing n and thus increasing matrixsize, checking whether the entries in the resulting matrix Q_n^{-1} converge to some fixed values. If we arrive at such a matrix-convergent, we prepend an empty rowvector at the top and remove the bottom row - just to get a squarematrix again and assume this matrix as a reasonable approximation to the true pseudoinverse of ***(I - C)*** also for the case of infinite size. The empty top row shall get some meaningful values which I'll explain later.

In our case with ***T*** in the position of the matrix ***C*** in the above, we indeed seem to get such a matrix convergent valid in the first *8* to *10* digits already if the dimension is only $n=32$ or $n=64$; let's call it ***H*** because it it the associated matrix to the function $H(x)$ with which we want to work here, and denote it as H^* because the top row has not yet been populated with meaningful values.

Having $n=32$ we get H_{32}^* with its top-left aspect as

H^* =

0	0	0	0	0	0
1.00000	-0.577216	0.145632	0.0290711	-0.00821534	-0.0116269
0.500000	0.533859	-0.634570	0.185811	0.0605025	-0.00355547
0.166667	0.325579	0.386859	-0.671304	0.210493	0.0920360
0.0416667	0.125274	0.240711	0.312757	-0.695810	0.227607
0.00833333	0.0337257	0.0991620	0.191897	0.267035	-0.713651
0.00138889	0.00685935	0.0284730	0.0814665	0.160312	0.235667
0.000198413	0.00117261	0.00592379	0.0245251	0.0689118	0.138200

With this we get empirically

$V(\log(2))-V(\log(1)) \cdot H^*$ =	1.0000000	0	0	0	...
$V(\log(3))-V(\log(1)) \cdot H^*$=	2.0000000	0.69314718	0.48045301	0.33302465	...
$V(\log(4))-V(\log(1)) \cdot H^*$=	3.0000000	1.7917595	1.6874020	1.6589936	...
$V(\log(4))-V(\log(3)) \cdot H^*$=	1.0000000	1.0986123	1.2069490	1.3259690	...

The approximations to the expected values are already very well (and many more examples come out successful with good approximations to the expected values.

Thus we might -with good evidence- assume that the desired general expression holds by convergence of the power series in some range of its arguments a,b :

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[7] I found it some years ago at [Robbins2005], who had pointed also to an article of Peter Walker[Walker1991]

[8] I've analyzed the occuring matrices by means of the ***LDU***-decomposition into triangular, even in the case of infinite size invertible, matrices in the context of tetration and the "superlog", finding one possibly nontrivial systematic aberration by the truncation which might not be overcome by increasing the matrixsize. Also P. Walker mentioned the possibility of some aberration in his use of that concept but without giving it a deeper consideration. But here I've found also an analytic description which allows to compute the matrix ***H*** without going via this "Pseudo-inversion".

$V(\log(b+1))-V(\log(a)) \cdot \mathbf{H}^*_{,1} \;=\; \sum_{k=a}^{b} \log(k) = S_1(a,b)$

where c indicates the column and begins with index 0

1.4.3. The function $H^*_c(x)$ and their completions $H_c(x)$

A closer look at the (primarily interesting) 2^{nd} column $\mathbf{H}^*_{,1}$ suggests, that it just contains the coefficients for the composite function $\log(\Gamma(\exp(x)))$; Pari/GP gives us the Taylor series for this as

$$\begin{aligned}\log(\Gamma(\exp(x))) = \; & - 0.57721566\, x \\ & + 0.53385920\, x^2 \\ & + 0.32557879\, x^3 \\ & + 0.12527414\, x^4 \\ & + 0.033725651\, x^5 \\ & + 0.0068593536\, x^6 \\ & + O(x^7)\end{aligned}$$

which is close enough to make us confident that we are in principle on the right track here. *(For an estimate of the range of convergence see appendix 2.3).*

Let us, before generalization to all columns of $\mathbf{H}^*$, first look at the function $H_1^*(x)$ which is defined by the power series with coefficients from column $c=1$, so we have

$$H_1^*(x) = \sum_{k=1}^{oo} h^*_{k,1} \cdot x^k \qquad = \qquad \log(\Gamma(\exp(x)))$$

and for a logarithm of a natural number as argument we have for instance

$$\begin{aligned}&H_1^*(\log(4)) = \log(\Gamma(4)) = \log(1\cdot 2\cdot 3) \quad = \log(1)+\log(2)+\log(3)\\ &H_1^*(\log(6)) = \log(\Gamma(6)) = \log(1\cdot 2\cdot 3\cdot 4\cdot 5) \quad = \log(1)+\log(2)+\log(3)+\log(4)+\log(5)\\ &H_1^*(\log(6)) - H_1^*(\log(4)) = \log(4\cdot 5) \quad = \log(4)+\log(5)\\ &\qquad = S_1(4,5)\end{aligned}$$

Our goal with that function $H_1(x)$ was however, to get some analogy to the Hurwitz-zeta function, which provides an infinite series of function-values of consecutive arguments ***beginning*** at its own argument, so we want actually

$$H_1(\log(4)) = \log(4)+\log(5)+\log(6)+ \ldots$$

and this means, that we should have the function at $x = \log(1) = 0$ as

$$H_1(\log(1)) = H_1(0) = \log(1) + \log(2) + \log(3) + \ldots$$

from which we would then substract

$$\begin{aligned}H_1(\log(a)) &= H_1(0) - H_1^*(\log(a))\\ &= \log(1) + \log(2) + \log(3) + \ldots\\ &\quad - (\log(1) + \log(2) + \ldots + \log(a-1))\\ &= \log(a) + \log(a+1) + \log(a+2) + \ldots\end{aligned}$$

so $H_1(0)$ should give us the value for the series of consecutive logarithms of ***all*** consecutive natural numbers. In this case the argument x for the power series equals zero and the result must thus been contained in the first coefficient $h^*_{0,1}$ only, which is that for x^0.

Now there is a possible closed-form expression for this infinite series: this is by zeta-regularization and namely by the first derivative of the zeta at zero:

$\zeta(0)' \;=_R\; \log(1/1) + \log(1/2) + \log(1/3) + \ldots$ // the symbol $=_R$ means here: by regularization

$-\zeta(0)' =_R \log(1) + \log(2) + \log(3) + ...$
$\sim 0.918939...$

Our function $H_1(x)$ should thus contain that value as constant coefficient, and the completed power series for $H_1(x)$ should look finally like

$$H_1(x) = (-\zeta(0)') - \log(\Gamma(\exp(x)))$$
$$= 0.918939 + 0.57721566\, x - 0.53385920\, x^2 - 0.32557879\, x^3 - 0.12527414\, x^4 \;...$$

Similar considerations are to be done for the functions $H^*_c(x)$ and $H_c(x)$ for the columns of $\boldsymbol{H}^*$ with higher index c, except the only significant variations that

- we need the c'th derivatives of the $\zeta(0)$, and
- analogies like the $\log(\Gamma(\exp(x)))$ for higher c; unfortunately such functions are not in the set of commonly used and named functions[9].

For the completion of the matrix $\boldsymbol{H}^*$ to arrive at the final matrix $\boldsymbol{H}$ this means, that we fill into the entries of the first row the (appropriately signed) c'th derivatives of the zeta at zero and have thus

$h_{r,c} = (-1)^c \cdot \zeta(0)^{(c)}$ for $r=0$
$h_{r,c} = - h^*_{r,c}$ for $r>0$

That first few signed derivatives (beginning with order zero) are approximately[10]

$[-1/2, 0.91893853, -2.0063565, 6.0047112, -23.997103, 120.00023]$

With this we have the definitions for all completed functions $H_c(x)$ at hand:

$$H_c(x) = (-1)^c \cdot \zeta(0)^{(c)} - H_c^*(x) = \sum_{r=0}^{oo} h_{r,c} x^r$$

and the sought functions $S_p(a,b)$ are then

$$S_p(a,b) = \log(a)^p + \log(a+1)^p + ... + \log(b)^p$$

$$=_R H_p(\log(a)) - H_p(\log(b+1))$$

$$=_R \sum_{k=1}^{oo} \left(\log(a)^k - \log(b+1)^k\right) \cdot h_{k,p}$$

For a visual impression: the top-left aspect of completed matrix $\boldsymbol{H}$ (computed with size $32x32$) looks like

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[9] I've asked in the MSE-community without getting an answer besides some interesting comments. See http://math.stackexchange.com/questions/207455

[10] There is some ***optimized numerical computation*** for the derivatives of the zeta at 0. That derivatives are expressible as infinite power series; a well converging one is

$$\zeta^{(p)}(s)_{|s=0} = (-1)^p \left(\sum_{k=0}^{\inf} \frac{\gamma_{p+k}}{k!} \right) - p!$$

where the coefficients γ_k are the Stieltjes-constants beginning with $\gamma_0=0.5772156...$ (which is also known as "Euler-Mascheroni-constant"). Only the first two of that numbers have "simple" representations; we have for instance

$-0.91893853... = -\frac{1}{2} \log(2\pi)$

$$H_{32} = \begin{bmatrix} -0.500000 & 0.918939 & -2.00636 & 6.00471 & -23.9971 & 120.000 \\ -1.00000 & 0.577216 & -0.145632 & -0.0290711 & 0.00821534 & 0.0116269 \\ -0.500000 & -0.533859 & 0.634570 & -0.185811 & -0.0605025 & 0.00355547 \\ -0.166667 & -0.325579 & -0.386859 & 0.671304 & -0.210493 & -0.0920360 \\ -0.0416667 & -0.125274 & -0.240711 & -0.312757 & 0.695810 & -0.227607 \\ -0.00833333 & -0.0337257 & -0.0991620 & -0.191897 & -0.267035 & 0.713651 \\ -0.00138889 & -0.00685935 & -0.0284730 & -0.0814665 & -0.160312 & -0.235667 \\ -0.000198413 & -0.00117261 & -0.00592379 & -0.0245251 & -0.0689118 & -0.138200 \end{bmatrix}$$

1.5. An analytical description of the entries of the matrix *H*

Reconsidering the Carleman-representation of the problem of the expansion to sums of logarithms from single Carleman-operator arguments gave the key for the ***analytic*** description of the entries in ***H***.

The Carleman-expression for the transfer *log(x)->log(1+x)* can also be written as[11]

$$V(\log(x)) \cdot \mathbf{fS2F} \cdot \mathbf{P} \cdot \mathbf{fS1F} = V(\log(x+1))$$

because

$$\begin{aligned} V(\log(x)) \cdot \mathbf{fS2F} &= V(\exp(\log(x))-1) = V(x-1) \\ V(\ x-1\ ) \cdot \mathbf{P} &= V(x-1+1) = V(x) \\ V(\ x\ ) \cdot \mathbf{fS1F} &= V(\log(x+1)) \end{aligned}$$

where the Carleman-matrices ***fS2f***, ***P***, ***fS1F*** perform the functional composition of the transferfunction t_1 in that three steps.

To arrive at a result of a sum of two vectors we can simply introduce another matrix ***P***

$$V(\log(x)) \cdot \mathbf{fS2F} \cdot (\mathbf{P} + \mathbf{P}^2) \cdot \mathbf{fS1F} = V(\log(x+1)) + V(\log(x+2))$$

and to keep also the identity to let the sum begin at the argument *log(x)* itself we introduce the identity matrix (which is also $\mathbf{P}^0$):

$$V(\log(x)) \cdot \mathbf{fS2F} \cdot (\mathbf{I} + \mathbf{P} + \mathbf{P}^2) \cdot \mathbf{fS1F} = V(\log(x)) + V(\log(x+1)) + V(\log(x+2))$$

Now, to make this an infinite series in the sense of the Hurwitz-analogon we need the Neumann-series for the argument ***P***, informally we want some inverse

$$\mathbf{Q} = (\mathbf{I} - \mathbf{P})^{-1}$$

but which again has no immediate solution. However, in my 2009-treatize[12] I've already developed a meaningful solution for this problem, namely the ***ZETA***-matrix such that we can probably insert:

$$V(\log(x)) \cdot \mathbf{fS2F} \cdot \mathbf{ZETA} \cdot \mathbf{fS1F} = V(\log(x)) + V(\log(x+1)) + V(\log(x+2)) + \ldots$$

Here, the ***ZETA***-matrix contains zeta-values and binomial-coefficients and more precisely the coefficients of the integrals of the Bernoulli-polynomials[13] such that for instance we have the linear expansion of two vectorial arguments into a finite sum of vectors:

$$(V(a) - V(b+1)) \cdot \mathbf{ZETA} = V(a) + V(a+1) + V(a+2) + \ldots + V(b)$$

With this the last part of the product in

$\mathbf{A} = \mathbf{ZETA} \cdot_R \mathbf{fS1F}$ *where the symbol "$\cdot_R$" means Ramanujan-summation in the dot-products*

[11] see the display and the description of all used matrices and of their application in formulae in chapter 3

[12] see [Helms2009]: "Sums of like powers - a matrix approach..."

[13] in terms of the Faulhaber-polynomials; we'll use it here in the upper (near) triangular form

can actually be done analytically using regularization; we have to assign values to strongly divergent series like

$$\sum_{k=1}^{oo} (-1)^k \frac{\zeta(1-k)}{k}$$

but which can actually be done/be defined by methods of regularization. This way each entry in ***A*** results from a formula which can/must analytically be solved in the above way.

That sums can/must be done by Ramanujan-summation; to have a visual impression of the result, I've computed here the top-left segment of the expected ***A*** (denoted as ***A**** here) by the Carleman-formulae:

H = fS2F · (ZETA ·$_R$ fS1F) *where the symbol "·$_R$" means Ramanujan-summation in the dot-products*

A = fS2F^{-1} · H = ZETA ·$_R$ fS1F*

$$A^* = \begin{bmatrix} -0.500000 & 0.918939 & -2.00636 & 6.00471 & -23.9971 & 120.000 \\ -1.00000 & 0.577216 & -0.145632 & -0.0290711 & 0.00821534 & 0.0116269 \\ 0 & -0.822467 & 0.707386 & -0.171276 & -0.0646102 & -0.00225796 \\ 0 & 0.400686 & -1.06997 & 0.847425 & -0.147252 & -0.0917158 \\ 0 & -0.270581 & 0.957674 & -1.48277 & 0.954035 & -0.0892009 \\ 0 & 0.207386 & -0.852677 & 1.75743 & -1.97496 & 1.00717 \\ 0 & -0.169557 & 0.770027 & -1.88314 & 2.82599 & -2.49135 \\ 0 & 0.144050 & -0.704121 & 1.93872 & -3.49938 & 4.16021 \end{bmatrix}$$

	ζ(0)	-ζ(0)'	ζ(0)"	-ζ(0)$^{(3)}$	ζ(0)$^{(4)}$	-ζ(0)$^{(5)}$
		γ/1				
		-ζ(2)/2				
		ζ(3)/3				
		-ζ(4)/4				

As an overview, the composition of the coefficients in matrix ***A**** is by

- Stirling-numbers first kind,
- factorials,
- Stieltjes constants, where the γ_0 is also known as Euler-/Mascheroni constant
- derivatives of the *ζ()* -function of orders from zero up to the column-index.

After that we need the composition with Stirlingnumbers of the second kind due to the other part of the above product

H = fS2F · A

but which requires then only finite compositions of the entries along the columns in ***A*** because ***fS2F*** is lower triangular. In ***H*** the above mentioned coefficients are combined only with further coefficients from the

- Stirlingnumbers second kind.

For instance let's look at the function $H_1(x)$. First, the coefficients in the second column $A_{,1}$ are defined by the regularization of the (divergent) dotproducts

$A_1 =_R$ ***ZETA · fS1F***$_{,1}$

and we have for the vector of entries:

$$a_0 = \sum_{k=1}^{oo} \frac{(-1)^k \zeta(-k)}{k} \underset{R}{=} -\zeta'(0) - 1 \approx -0.0810614662531$$

$$a_1 = \sum_{k=1}^{oo} \binom{k}{1} \frac{(-1)^{2k-1} \zeta(-(k-1))}{k} \underset{R}{=} \gamma_0 \approx 0.577$$

where γ_0 is the first Stieltjes constant (or "Euler-Mascheroni-constant")

and the following entries

$$a_c = \frac{1}{c-1} + (-1)^c \sum_{k=c}^{oo} \binom{k}{c} \frac{\zeta(-(k-c))}{k} \underset{R}{=} \frac{\zeta(c)}{c}$$

so the vector looks like

A_1 = [-(ζ(0)'-1) , γ_0 , - ζ(2)/2, ζ(3)/3, -ζ(4)/4, ...]

After that, taking the premultiplication by ***fS2F*** we get ***H_1***

$H_{,1}$ = fS2F · $A_{,1}$

and

$$H_1(x) = \sum_{r=0}^{oo} h_{r,1} x^r$$

Here we have then

$$\begin{aligned}
h_{0,1} &= \frac{1}{0!}(\quad \zeta^{(1)}(0) \quad) \\
h_{1,1} &= \frac{1}{1!}(\quad 1\gamma_0 \quad) \\
h_{2,1} &= \frac{1}{2!}(\quad 1\gamma_0 \quad -1\zeta(2) \quad) \\
h_{3,1} &= \frac{1}{3!}(\quad 1\gamma_0 \quad -3\zeta(2) \quad +2\zeta(3) \quad) \\
h_{4,1} &= \frac{1}{4!}(\quad 1\gamma_0 \quad -7\zeta(2) \quad +12\zeta(3) \quad -6\zeta(4) \quad) \\
&\dots \\
h_m &= \frac{1}{m!}\left(-s_{2:m,1}\gamma_0 + \sum_{k=2}^{m} \left(\tfrac{k!}{(-1)^k k} s_{2:m,k} \zeta(k) \right) \right)
\end{aligned}$$

where s_2 are the Stirling numbers 2nd kind and γ the Euler-Mascheroni-constant
also used in Ramanujan's replacement for the summation of the ζ(1)
(see also the appendix 2.2)

This decomposition (except the leading one for $h_{0,1}$) can be crosschecked using the software Mathematica at wolframalpha.com when the decomposition of the terms in in the function *-lngamma(exp(x))* is queried; however, due to the ambivalence of expressing the zetas at even arguments by even powers of the number *π* and thus higher zetas as powers of the lower ones one must do manual work to get this memorizable pattern.

For the documentation of the full matrix ***H*** see appendix 3.2.

An interesting additional type of expression can be derived if we evaluate the full expression with different association. Having ***A = ZETA · fS1F*** we can then proceed:

*V(x) · **fS2F** · **$A_{,1}$** = V(x) · (**fS2F** · [ζ(0)' , -γ, ζ(2)/2, -ζ(3)/3, ζ(4)/4, ...]~)*

and may change order of summation:

= (V(x) · fS2F) · [ζ(0)' , -γ, ζ(2)/2, -ζ(3)/3, ζ(4)/4, ...]~
= [1, e^x-1, $(e^x-1)^2$, $(e^x-1)^3$, ...] · [ζ(0)' , -γ, ζ(2)/2, -ζ(3)/3, ζ(4)/4, ...]~

If we assume *ln(1+x)* instead of *x* (which is, what we initially want in this article) we can then write

$$(V(\ln(1+x)) \cdot fS2F) \cdot A_{,1} = [1, x, x^2, x^3, \dots] \cdot [\zeta(0)', -\gamma_0, \zeta(2)/2, -\zeta(3)/3, \zeta(4)/4, \dots]\sim$$

$$= (-x)\cdot\gamma_0 + \sum_{k=2}^{oo} (-x)^k \frac{\zeta(k)}{k}$$

which is then another expression for the function $H_1(\log(1+x))$

1.6. Generalizations/Conclusion

Just like with the bernoulli/zeta-polynomials we have the option to generalize this to fractional values of a and b, and also to non-integer/non-natural differences between a and b.

However, different from the Bernoulli-polynomials for the sums of like powers, in the sums-of-like-powers-of-logarithms we do not get polynomials (which have finite number of coefficients) but infinite series, which prevents simple "exact" expressions or even rational numbers or expressions in simple terms of known constants so far.

Gottfried Helms 6.9.2016 (minor textual edits)

8.11.2014 (7.11.2010)

2. Appendices

2.1. Pari/Gp Code

1) Computation of the Carlemanmatrix for *t(x)* of size *nxn*

```
  \\ transferfunktion
t(x)= log(1+exp(x))   \\ t:  log(x) -> log(1+x)

n = 32
T = carleman(vector(n,r,aeta(2-r)/(r-1)!))  \\"aeta" und "carleman" are user-
                                            \\ defined functions

\\ 2) computation of the pseudoinverse S of T
   tmp1 = matid(n+1)-carleman(vector(n+1,r,aeta(2-r)/(r-1)!)
   tmp2 = matrix(n,n,r,c,tmp1[r,c+1])
   tmp3 = tmp2^-1

S    = matrix(n,n,r,c,if(r>1,tmp3[r-1,c]))
                  \\ here in S the first row is zero and must be filled later
```

2) Computation of the Stieltjes numbers for the derivatives of the *zeta(0)*

```
\\ fill the first row of S with derivatives of zeta(0)
   \\ a) compute Stieltjes-numbers to high precision
       default(seriesprecision,n)
       pse = sumalt(k=0,(-1)^k*1/(1+k)^x)  \\ powerseries for aeta(x)
       psz = 1/(1-2^(1-x))*pse             \\ powerseries for zeta(x)
       pcz = polcoeffs(psz,n)              \\ coeffs for zeta(x)
             \\ the Taylor/Laurent-series-coefficients of a power series
             \\ are its derivatives at x=0, scaled by factorials, so we can
             \\ simply use them for the derivatives of zeta(0):

       S_0 = vector(n,c,(-1)^(c-1)*pcz[c]*(c-1)!)
                                           \\ equals derivatives of zeta at 0

    S[1,] = S_0            \\ insert derivtives to allow correct results for
%box >tst VE(S,n,8)     \\ display the top-left-segment
```

3) Check something more with the derivatives of the zeta at zero

```
\\ derivatives of the zeta based on the decomposition
\\ zeta(s) = zetaInc(s) - 1/(1-s)   where ZetaInc is an "incomplete Zeta"
\\ check: (compare)  abs(S_0[1+k] == abs(zeta_d(0,k))

        pszInc = psz + 1/(1-x)              \\ powerseries for zetaInc(x)
        pczInc = polcoeffs(pszInc,n)        \\ coeffs for zetaInc(x)

zetaInc(x) = sum(k=0,#pczInc-1,x^k*pczInc[1+k])
{ zetaInc_d(x=2,d=0) = local(a);  \\ d'th deriv. of zeta at small x
      a = sum(k=0,n-1-d,x^k*pczInc[1+k+d]*binomial(d+k,d));
    return(a*d!) ; }

{ zeta_d(x=2,d=0) = local(a,b,c);  \\ d'th deriv. of zeta at small x
      a = zetaInc_d(x,d);
      b = d!/(1-x)^(d+1) ;    \\ d'th derivative of 1/(1-x)
      c = a - b;
    return(c) ; }

\\ alternate form:
 zeta_d_at0(d) = (-1)^d*sum(k=0,n-1, Stieltjes[1+d+k]/k!) - d!

 %box >tst ESum(0.0)*(dV(log(1))-dV(log(4)))*VE(S,n,8)
 vector(6,r, log(1)^(r-1)+log(2)^(r-1)+log(3)^(r-1))  \\ compare
```

2.2. Range of convergence for *S*-function

The radius of convergence of $s_1(x)$ seems to be π; this can be seen when we multiply each k'th coefficient of its powers series with π^k by $^dV(\pi) \cdot S[,2]$ and check, whether the resulting numbers decrease, increase or stay constant. We find, that they roughly stay constant. To see this even better, we separate the π^k-scaled coefficients into 4 groups and notice, that the sequences along each column remain in some order of magnitude:

coefficient at term k multiplied by π^k

4k	4k+0	4k+1	4k+2	4k+3
0	0	1.81338	-5.26898	-10.0950
4	-12.2028	-10.3207	-6.59451	-3.54162
8	-1.73814	-0.672985	-0.142892	-0.0810845
16	-0.107840	0.0164950	0.0837811	-0.0317088
20	-0.0855998	0.0366779	0.0876456	-0.0384842
24	-0.0909447	0.0369494	0.0946102	-0.0322223
28	-0.0976286	0.0246398	0.0989107	-0.0148964
32	-0.0974852	0.00409236	0.0927276	0.00634826
36	-0.0845708	-0.0148914	0.0736231	0.0202013
40	-0.0611351	-0.0214660	0.0487870	0.0186391
44	-0.0383312	-0.0125084	0.0311753	0.00454470
48	-0.0280286	0.00344328	0.0287250	-0.00975679
56	-0.0322845	0.0132222	0.0371995	-0.0134470
60	-0.0418530	0.0108430	0.0449354	-0.00642419
64	-0.0457314	0.00146129	0.0442041	0.00288276
68	-0.0408765	-0.00582265	0.0365824	0.00706039
72	-0.0321879	-0.00673591	0.0283755	0.00527336
76	-0.0255399	-0.00319998	0.0237931	0.00100185
80	-0.0230389	0.000953170	0.0230655	-0.00243063
84	-0.0236186	0.00330984	0.0244421	-0.00355830
88	-0.0252950	0.00321634	0.0259607	-0.00239162
96	-0.0262628	0.00125295	0.0260883	-0.0000124348
100	-0.0254093	-0.00110746	0.0242921	0.00191561
104	-0.0228853	-0.00229061	0.0213873	0.00220356
108	-0.0200028	-0.00171849	0.0188986	0.000971317
112	-0.0181725	-0.000134452	0.0178402	-0.000623243
116	-0.0178422	0.00116959	0.0180654	-0.00142987
120	-0.0183716	0.00139611	0.0186314	-0.00111384
124	-0.0187445	0.000664794	0.0186458	-0.000163678
128	-0.0183400	-0.000304128	0.0178539	0.000673493
136	-0.0172096	-0.000823348	0.0165975	0.000894812
140	-0.0159131	-0.000723632	0.0153776	-0.00740972

Even more, if we multiply also the current index into the number $({}^dZ(-1) \cdot {}^dV(\pi) \cdot S[,2])$ we get the constance of the magnitude in the columns even more visible:

coefficient at term k multiplied by $(k+1)\cdot\pi^k$

4k	4k+0	4k+1	4k+2	4k+3
0	0	3.62675	-15.8069	-40.3799
4	-61.0142	-61.9243	-46.1616	-28.3330
8	-15.6433	-6.72985	-1.57182	-0.973014
16	-1.40192	0.230929	1.25672	-0.507340
20	-1.45520	0.660203	1.66527	-0.769684
24	-1.90984	0.812887	2.17603	-0.773335
28	-2.44072	0.640634	2.67059	-0.417099
32	-2.82707	0.122771	2.87456	0.203144
36	-2.79084	-0.506307	2.57681	0.727248
40	-2.26200	-0.815709	1.90269	0.745566
44	-1.57158	-0.525352	1.34054	0.199967
48	-1.26129	0.158391	1.35008	-0.468326
56	-1.58194	0.661109	1.89718	-0.699243
60	-2.21821	0.585523	2.47144	-0.359755
64	-2.60669	0.0847547	2.60804	0.172965
68	-2.49346	-0.361004	2.30469	0.451865
72	-2.09221	-0.444570	1.90116	0.358588
76	-1.76225	-0.223999	1.68931	0.0721335
80	-1.68184	0.0705346	1.72991	-0.184728
84	-1.81863	0.258167	1.93093	-0.284664
88	-2.04890	0.263740	2.15474	-0.200896
96	-2.23234	0.107754	2.26968	-0.00109426
100	-2.26142	-0.0996711	2.21058	0.176236
104	-2.12833	-0.215317	2.03180	0.211542
108	-1.94027	-0.168412	1.87096	0.0971317
112	-1.83543	-0.0137141	1.83754	-0.0648172
116	-1.87343	0.123977	1.93300	-0.154426
120	-2.00250	0.153572	2.06809	-0.124750
124	-2.11813	0.0757865	2.14427	-0.0189867
128	-2.14578	-0.0358871	2.12462	0.0808191
136	-2.08236	-0.100448	2.04149	0.110957
140	-1.98914	-0.0911777	1.95296	-0.948444

2.3. List of used matrices

All vectors and matrices are assumed of infinite size because they contain the coefficients of the involved power series. For the empirical computations using the software Pari/GP they are truncated to some size *n* resp *nxn* and the empirical results are approximative and also often improved by Euler- oder stronger Noerlundsummation procedures (implemented again by matrix-formulae).

Vectors are row-vectors by default; if they are used as diagonal-matrices then this is denoted by a small super-prefix *d*, example: $^dV(x)$, the index begins at zero (so for the practical computation one must compensate for the Pari/GP convention with *1*-based indexes). Also I use the Pari/GP-symbol for transposition : *V(x)~* means a transposed vector/matrix. Reference to a row as vector in a matrix by the first index: $\mathbf{M}_r$, reference to a column c as vector $\mathbf{M}_{,c}$. If M has some type-index, so is for instance the matrix $\mathbf{M}_1$, then the row- and column-indexes are separated by a colon: $\mathbf{M}_{1:r,c}$.

Vectors:

The ***Vandermonde-vector*** *V(x)* contains the consecutive powers of its argument *x:*

$V(x) = [1, x, x^2, x^3, ...]$ *// this is a row-vector*

The dot-product of *V(x)* with some column vector *A* containing coefficients defines a ***(formal) power series*** in *x* ; in the practical computation with finitely truncated vectors these are only polynomials of order *n-1* if size is *n* .

The ***"factorial" vector*** *F* :

$F = [\ 0!\ ,\ 1!\ ,\ 2!\ ,\ 3!\ , ...]$

The vector of powers of the ***reciprocals of natural numbers*** *Z(s):*

$Z(s) = [\ 1\ , 1/2^s\ ,\ 1/3^s\ ,\ 1/4^s\ , ...\ ]$

Matrices

A generic ***Carleman-matrix***[14] ***C*** is defined to allow a mapping from one Vandermonde-vector to another and thus allows composition and iteration of functions by the simple notation of matrix-equations:

$[1, x, x^2, x^3, ...\ ] \cdot \mathbf{C} = [1, f(x), f(x)^2, f(x)^3, f(x)^4, ...]$

or

$V(x) \cdot C = V(\ f(x)\ )$
$V(x) \cdot C^2 = V(\ f(f(x))\ )$

It means, that in the second column it contains the coefficients of some (formal) power series *f(x)*, and in the following consecutive colums the coefficients of the consecutive powers of the power series *f(x)*. The first column is always empty except the unit in the first row. This notation of functional composition (and especially iteration) in terms of Carleman-matrices is due to notational easiness.

[14] see [WP:Carlemanmatrix]. Note that Eri Jabotinsky introduced the extension of Carleman-matrices to negative indexes, so to handle Laurent series in the same framework. But this extension is never done here.

The upper triangular ***Pascal-matrix P***, which is the Carleman-matrix for the transfer *x -> x+1* (by operating on the formal power series):

P=	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 2 & 3 & 4 & 5 \\ . & . & 1 & 3 & 6 & 10 \\ . & . & . & 1 & 4 & 10 \\ . & . & . & . & 1 & 5 \\ . & . & . & . & . & 1 \end{bmatrix}$	*Carleman operation:* $V(x) \cdot P = V(x+1)$ $V(x) \cdot P^m = V(x+m)$

The matrices of ***Stirling numbers*** *2nd* and *1st* kind:

S2=	$\begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 1 & 1 & . & . & . \\ 0 & 1 & 3 & 1 & . & . \\ 0 & 1 & 7 & 6 & 1 & . \\ 0 & 1 & 15 & 25 & 10 & 1 \end{bmatrix}$
S1=	$\begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & -1 & 1 & . & . & . \\ 0 & 2 & -3 & 1 & . & . \\ 0 & -6 & 11 & -6 & 1 & . \\ 0 & 24 & -50 & 35 & -10 & 1 \end{bmatrix}$

The factorially similarity-scalings ***fS2F*** and ***fS1F***, which are the Carleman-matrices for the transfers *x -> exp(x)-1* and *x -> log(1+x)* on their formal power series, respectively:

$fS2F={}^dF^{-1} \cdot S2 \cdot {}^dF=$	$\begin{bmatrix} 1 & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & 1/2 & 1 & . & . & . \\ . & 1/6 & 1 & 1 & . & . \\ . & 1/24 & 7/12 & 3/2 & 1 & . \\ . & 1/120 & 1/4 & 5/4 & 2 & 1 \end{bmatrix}$	*Carleman operation:* $V(x) \cdot fS2F = V(exp(x)-1)$
$fS1F={}^dF^{-1} \cdot S2 \cdot {}^dF=$	$\begin{bmatrix} 1 & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & -1/2 & 1 & . & . & . \\ . & 1/3 & -1 & 1 & . & . \\ . & -1/4 & 11/12 & -3/2 & 1 & . \\ . & 1/5 & -5/6 & 7/4 & -2 & 1 \end{bmatrix}$	*Carleman operation:* $V(x) \cdot fS1F = V(log(1+x))$

The ***ZETA***-matrix as representation of the Neumann-series of the matrix ***P*** which contain the coefficients for the integrals of the Bernoulli-polynomials column-wise with the factorially scaled ζ-values at zero and negative arguments in the first row:

ZETA=	$\begin{bmatrix} -1/2 & -1/12 & . & 1/120 & . & -1/252 \\ -1 & -1/2 & -1/6 & . & 1/30 & . \\ . & -1/2 & -1/2 & -1/4 & . & 1/12 \\ . & . & -1/3 & -1/2 & -1/3 & . \\ . & . & . & -1/4 & -1/2 & -5/12 \\ . & . & . & . & -1/5 & -1/2 \end{bmatrix}$	$(V(a)-V(a+n)) \cdot ZETA =$ $[\ n\ ,$ $(a+1)+(a+2)+...+(a+n)\ ,$ $(a+1)^2+(a+2)^2+...+(a+n)^2\ ,$ $...\ ]$
	for r≤c $ZETA[r,c] = (c:r)\zeta(r-c)$ *for r=c+1* $ZETA[r,c] = -1/r$ *for r>c+1* $ZETA[r,c] = 0$	*where r,c denote the row/col-indices, beginning at zero, and (a:b) denotes the binomial-coefficient*

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