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Mathematical Miniatures

An infinite matrix *M* whose inverse is the Null-matrix?

Abstract: An inverse for a very simple infinite matrix is sought and it seems, the best answer is, that this would be the "null"-matrix.

*We consider the LDU-factors of **M** and describe the inverse **W=M^{-1}** as inner product of the inverses of that LDU-factors in the limit as the dimensions increase towards infinity.*

*The pattern of entries of the inverted **U**-factor exhibits the Stirling numbers first kind, when read along the diagonals in the matrix of that nubers. Also (and easier) that pattern suggests a concept of "recursive harmonic numbers". That concept can be found elsewhere, for instance in an article of D.Loeb [LOEB95] who discusses a generalization of the Stirling numbers first kind.*

The article is based on heuristics; the assumed (infinite) set of identities of sums based on Stirling numbers first kind, binomials and factorials, all converging zero, seems to be new in this generality and remains to be proven.

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Dec 2012, Sep 2012: typo corrections; Aug 2012 errata: formula on pg 9 was corrected
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Keywords: infinite matrix, Stirling numbers first kind, recursive harmonic numbers

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1 Problem, motivation and caveat

We consider the matrix ***M*** of infinite size with entries

$$m_{r,c} = \frac{1}{(r+1)^c} \qquad \text{for } r,c = 0 \ldots \infty$$

The top-left aspect of this matrix is

$$M_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1/2 & 1/4 & 1/8 & 1/16 & 1/32 \\ 1 & 1/3 & 1/9 & 1/27 & 1/81 & 1/243 \\ 1 & 1/4 & 1/16 & 1/64 & 1/256 & 1/1024 \\ 1 & 1/5 & 1/25 & 1/125 & 1/625 & 1/3125 \\ 1 & 1/6 & 1/36 & 1/216 & 1/1296 & 1/7776 \end{bmatrix}$$

The question, arising from an example in a discussion in MSE[1], is:

> *Is it possible to define a reciprocal for this matrix*
> *and if, what is its limiting expression?*

The initial idea was to check, whether the finite truncation M_n to size $n \times n$ has an inverse W_n and then whether the entries in W_n converge to some limit if n is increased without bounds.

The much surprising initial observation with this matrix was, that the top-left region of W_n tended completely to zero, and that that tendency and also the region of numbers increased when n was increased.

Top left 6x6 aspect of W_{32}

-1.21613E-34	8.09599E-24	-3.49293E-17	2.52097E-12	-0.0000000178187	0.0000274087
6.40898E-32	-4.25849E-21	1.83379E-14	-0.00000000132099	0.00000931918	-0.0143073
-1.61921E-29	1.07369E-18	-4.61406E-12	0.000000331699	-0.00233527	3.57793
2.61134E-27	-1.72773E-16	0.000000000740832	-0.0000531407	0.373310	-570.718
-3.01985E-25	1.99320E-14	-0.0000000852631	0.00610158	-42.7630	65224.8
2.66697E-23	-1.75570E-12	0.00000749097	-0.534704	3738.09	-5.68746E6

Top left 6x6 aspect of W_{dyn}

-1.64397E-63	-8.31335E-63	-1.14390E-62	1.15058E-62	3.00983E-62	2.22988E-62
8.87746E-64	1.82985E-62	-2.53710E-63	5.56516E-63	2.66908E-62	3.21911E-62
1.23599E-62	-2.09290E-62	6.59079E-63	-2.59867E-62	1.19654E-62	2.34612E-62
-2.44525E-63	1.65517E-62	-1.17786E-62	-4.35489E-63	3.61025E-63	1.14954E-62
-1.83684E-62	-1.01635E-62	1.62686E-62	1.07183E-62	-1.39427E-62	4.25526E-63
2.37686E-63	5.15990E-63	-1.85003E-62	-2.16536E-62	4.40998E-62	-1.94159E-62

The entries of W_{dyn} result from dotproducts of some infinite triangular matrix-factors (see below). That dot-products define series very similar to the exponential series which converge to a relative small number only after a (possibly) huge partial sum has been reached (the "hump"). W_{dyn} is computed with a dot-product-specific number of terms for each entry (until the "hump" was surpassed and the series begin to converge).
***Table**: needed number of terms of dotproducts to decrease (and then converge) below an epsilon $|\varepsilon|<1e-60$*

49	60	69	77	84	91
51	62	72	80	87	94
52	64	74	82	90	97
54	66	76	85	93	100
55	68	78	87	95	103

In short:

> ***Should the inverse of M_n indeed tend to the zero-matrix if n tends to infinity?***

[1] MSE: internet forum *math.stackexchange.org* , question in dec 2011
http://math.stackexchange.com/questions/89853

This bizarre impression was in fact the triggering observation for the following analysis. However with matrices of infinite size it is not trivial to define a meaningful inverse at all. In the following I try to present a sensical approach.

Caveat:

Matrices of infinite size may have multiple reciprocals, as for instance the matrix of the Stirling numbers second kind. If we define

$$A := a_{r,c} = s_{2\_r,c} \frac{c!}{r!}$$

where the s_2 denote the Stirling numbers 2nd kind, then this has not only the well known reciprocal

$$B := b_{r,c} = s_{1\_r,c} \frac{c!}{r!}$$

(s_1 denoting the signed Stirling numbers first kind) whose columns provide the coefficients for the formal power series for the logarithm $log(1+x)$ and all of its powers. But $\mathbf{A}$ has also the analogue matrices[2] $\mathbf{B}_k$ with the coefficients for $log(1+x)+k\cdot 2\pi\hat{\imath}$ as left-inverse/left-reciprocal. The only characteristic difference is, that $\mathbf{B}_0$ (the Carleman-matrix for the principal branch of the logarithm) is triangular and the other $\mathbf{B}_k$ are square. So we might distinguish the triangular $\mathbf{B}_0$ calling it the *"principal reciprocal"*.

So a multitude of inverses might be possible here as well, but in a reflection of the above we'll call the convergent, which is based on triangular inverse factors the *"principal reciprocal"* without further notification.

Brief aside: the determinant of truncations of M

The matrix-determinant of nxn truncations of the Vandermondematrix $\mathbf{VZ}$

$$VZ_n = matrix(n,n,r,c, (1+c)^r)$$

$$\begin{aligned} Det(VZ_n) &= (n-1)!! \\ &= (n-1)! \cdot (n-2)! \cdot \ldots \cdot 3! \cdot 2! \cdot 1! \cdot 0! \\ &= (n-1)^1 \cdot (n-2)^2 \cdot (n-3)^3 \cdot \ldots \cdot 3^{n-3} \cdot 2^{n-2} \cdot 1^{n-1} \end{aligned}$$

equals the <u>superfactorial</u>; here, with the matrix $\mathbf{M}$

$$M_n = matrix(n,n,r,c, (1+c)^{-r})$$

we find, that the nxn-truncation has the determinant

$$\begin{aligned} Det(M_n) &= (-1)^0 \cdot (-2)^{-1} \cdot (-3)^{-2} \cdot (-4)^{-3} \cdot \ldots (-n)^{1-n} \\ &= -1/n!^n / (n-1) \end{aligned}$$

[2] "*Analogue matrices*" refers here to the ***form*** of the matrix, which are known as "Carleman-matrices" or "Bell-matrices" or which I call in my collection of related articles as "matrix-operators".

2 Using the LDU-decomposition of *M* to determine the reciprocal/inverse $W=M^{-1}$

According to the initial idea, we could simply take the truncation of ***M*** to size of some *n x n* and compute the finite inverse W_n . Then we could try to determine the limit of this as *n* increases without bound and check whether it converges to fixed entries in $W_n \,|_{n\to oo}$. Empirically, with some small n<64 we find, that the top-left edge of Wn indeed tends to zero and that that area increases as n increases. So we have a first hint, that this problem might be really interesting...

But proceeding, trying to improve that observations and possibly arrive to a proof, we need a different technique.

Instead of that successive computation of inverses we can use the well known ***LDU***-factorization of ***M*** into triangular and diagonal factors, then invert that single factors - which is easy and has a unique "principal solution" because everything is triangular and even diagonal - and try, whether the dot-product of the inverses of the ***LDU***-factors converges to some fixed matrix W_{oo} . The advantage of this process is, that the entries of the ***LDU***-factors do not change when we increase the matrix size, and that this inherits also to the inverse triangular factors: the dot-products of the inverse triangular factors get simply new terms, and if that terms have a tractable analytic description, the dot-products making the entries of W_{oo} are represented by some series for which we might give a closed form and/or a finite value.

So we begin with the ***LDU***-decomposition of the matrix ***M***.

The ***LDU***- decomposition

$$M = L\cdot D \cdot U \qquad \text{// by LDU-decomposition L, D, U to be determined}$$

provides the following triangular factors ***L*** and ***U*** and a diagonal factor ***D*** (we see the top-left areas of the matrices ***L***,***D***,***U***):

$$L=\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 4/3 & 1 & . & . & . \\ 1 & 3/2 & 27/16 & 1 & . & . \\ 1 & 8/5 & 54/25 & 256/125 & 1 & . \\ 1 & 5/3 & 5/2 & 80/27 & 3125/1296 & 1 \end{bmatrix} \quad D=\begin{bmatrix} 1 \\ -1/2 \\ 1/9 \\ -1/64 \\ 1/625 \\ -1/7776 \end{bmatrix} \quad U=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 3/2 & 7/4 & 15/8 & 31/16 \\ . & . & 1 & 11/6 & 85/36 & 575/216 \\ . & . & . & 1 & 25/12 & 415/144 \\ . & . & . & . & 1 & 137/60 \\ . & . & . & . & . & 1 \end{bmatrix}$$

In the case of infinite matrices we can sometimes find multiple inverses, so for instance the matrix L_{oo} may have non-triangular left-inverses ***X*** such that

$$Y_{oo} \cdot L_{oo} = I_{oo}$$

but only one unique right-inverse X_{oo} such that

$$L_{oo} \cdot X_{oo} = I_{oo} = X_{oo} \cdot L_{oo}$$

because ***L*** is *rowfinite*. Thus here we want to use that unique inverses and shall call them *"principal inverses"* or *"principal reciprocals"*.

The triangularity means: for any selection of matrix-size we compute the same principal inverses of the ***LDU***-factors except for appending rows/columns.

Here is the top-left aspect of the matrices $\boldsymbol{K} = \boldsymbol{L}^{-1}$ and $\boldsymbol{C} = \boldsymbol{D}^{-1}$:

diag(C), K =

diag(C)						
1	1	.	.	.	.	.
-2	-1	1	.	.	.	.
9	1/3	-4/3	1	.	.	.
-64	-1/16	3/4	-27/16	1	.	.
625	1/125	-32/125	162/125	-256/125	1	.
-7776	-1/1296	5/81	-5/8	160/81	-3125/1296	1

Because the left-multiplication by $\boldsymbol{C}$ is just a row-scaling of $\boldsymbol{K}$ and also because it makes the symbolic description of the entries easier we introduce $\boldsymbol{G} = \boldsymbol{C}\cdot\boldsymbol{K}$ and get

G = diag(C)·K=

1	.	.	.	.	.
2	-2	.	.	.	.
3	-12	9	.	.	.
4	-48	108	-64	.	.
5	-160	810	-1280	625	.
6	-480	4860	-15360	18750	-7776

with which we shall work from now.

The third matrix $\boldsymbol{T} = \boldsymbol{U}^{-1}$ begins like

T =

1	-1	1/2	-1/6	1/24	-1/120
.	1	-3/2	1	-5/12	1/8
.	.	1	-11/6	35/24	-17/24
.	.	.	1	-25/12	15/8
.	.	.	.	1	-137/60
.	.	.	.	.	1

Now our ansatz should allow to find a meaningful approximation for all entries in $\boldsymbol{W}$ where

$$W = \lim_{n\to oo} T_n \cdot G_n = T_n\cdot (C_n\cdot K_n) = U_n^{-1} \cdot D_n^{-1} \cdot L_n^{-1}$$

consisting of the dot-products of rows in $\boldsymbol{T}$ and columns in $\boldsymbol{G}$.

If the entries in $\boldsymbol{T}$ and $\boldsymbol{G}$ have then some evaluable analytic description depending on their row- and column-indexes we can describe the vectorial dot-products analytically as infinite series

$$w_{r,c} = \sum_{k=0}^{oo} t_{r,k} \cdot g_{k,c}$$ *(where the small letters indicate entries of the resp. matrices*

Such sums might (or may not) be expressible in a closed form also for the infinite case. If ***all*** dot-products in $\boldsymbol{T} \cdot \boldsymbol{G}$ *(=W)* (of rows in $\boldsymbol{T}$ with columns in $\boldsymbol{G}$) define convergent or − in a bit more generality − at least Euler-/Abel summable[3] series, or if we even arrive at closed forms then we have also the possibility to determine finite values (or otherwise singularities, if this should happen) and thus the possibility to describe the "principal reciprocal/inverse" $\boldsymbol{W}$ of $\boldsymbol{M}$ also for the infinite case.

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[3] "*Euler-/Abel-summability*" refers here to the fact, that the terms of the dot-product might diverge in their absolute values but anyway might be summable with some common concept of divergent summation because their signs alternate.

3 Description of entries of the non-inverse LDU-factors

Although later we do not need the analytic description of the non-inverted matrices let's just take them for the record.

The patterns in the ***L*** nd ***D*** - matrices seem to be:

$$l_{r,c} = \binom{r}{c}\left(\frac{1+c}{1+r}\right)^c$$

$$d_{r,r} = \left(-\frac{1}{1+r}\right)^c$$

for instance the ***L***-matrix

	0	1	2	3	...
0	1	.	.	.	.
1	1	$1 \cdot 2^1/2^1$	.	.	.
2	1	$2 \cdot 2^1/3^1$	$1 \cdot 3^2/3^2$	.	.
3	1	$3 \cdot 2^1/4^1$	$3 \cdot 3^2/4^2$	1	.
4	1	$4 \cdot 2^1/5^1$	$6 \cdot 3^2/5^2$	$4 \cdot 4^3/5^3$	...
...	...	...	...	...	...

the ***D***-matrix

...
$1 = (-1)^0$
- $1/2 = (-2)^{-1}$
$1/9 = (-3)^{-2}$
- $1/64 = (-4)^{-3}$
$1/625 = (-5)^{-4}$...
...

The entries of the ***U*** are a bit hairy; at least, rescaled by powers of factorials, we seem to get integer numbers and that numbers are already listed in the [OEIS], described by W.D. Lang. However, I find the definition there a bit unhandy.

"recursive harmonic numbers"

A better description might be the following concept of a "recursive harmonic number".

Consider the sequence of harmonic numbers

$$h_1 = 1, \quad h_2 = 1+1/2, \quad h_3 = 1+1/2+1/3+,...$$

Then consider the "harmonic composition" of harmonic numbers:

$$h_{2,1} = 1, \quad h_{2,2} = 1h_1 + 1/2h_2, \quad h_{2,3} = 1h_1 + 1/2h_2 + 1/3h_3 + ,...$$

and the recursive generalization

$$h_{0,1} = 1, \; h_{0,\cdot} = 1$$
$$h_{r,1} = 1,$$
$$h_{r,2} = 1\, h_{r-1,1} + 1/2\, h_{r-1,2} \;,$$
$$h_{r,3} = 1\, h_{r-1,1} + 1/2\, h_{r-1,2} + 1/3\, h_{r-1,3}$$

or

$$h_{r,k} = h_{r,k-1} + h_{r-1,k}/k$$

```
\\ ========== Pari/GP ================
{h(rec,m) = local(s);
     if(rec==0,return(1));
     s = sum(k=0,m,h(rec-1,k)/(1+k));
   return(s); }
matrix(6,6,r,c,if(c>=r,h(r-1,c-r)))
```

where the first index r refers to the recursion depth.

Then we get the description of ***U***:

$$u_{r,c} = h_{r,1+c-r}$$
$$u_{r,c} = u_{r,c-1} + u_{r-1,c}/c$$

(here r means the row-index(!), c the column-index of the matrix)

*The top-left aspect of **U** in symbolic notation:*

	0	1	2	3	4	5	...
0	1	1	1	1	1	1	
1	.	1	$h_{1,2}$	$h_{1,3}$	$h_{1,4}$	$h_{1,5}$	
2	.		1	$h_{2,2}$	$h_{2,3}$	$h_{2,4}$	
3	.			1	$h_{3,2}$	$h_{3,3}$	
4	.				1	$h_{4,2}$	...
...	.					...	...

(More related information to that "recursive harmonic numbers" see [RHN] in the appendix).

4 Description of entries in the inverse L-D-U-factors K-C-T

Here is the top-left aspect of the matrices $C = D^{-1}$ and $K = L^{-1}$:

$$\mathrm{diag}(C),K = \begin{bmatrix} 1 \\ -2 \\ 9 \\ -64 \\ 625 \\ -7776 \end{bmatrix} \begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 1/3 & -4/3 & 1 & . & . & . \\ -1/16 & 3/4 & -27/16 & 1 & . & . \\ 1/125 & -32/125 & 162/125 & -256/125 & 1 & . \\ -1/1296 & 5/81 & -5/8 & 160/81 & -3125/1296 & 1 \end{bmatrix}$$

Because the left-multiplication by C is just a row-scaling of K and makes the description of the entries easier we write $G = C \cdot K$ and get

$$G = \mathrm{diag}(C) \cdot K = \begin{bmatrix} 1 & . & . & . & . & . \\ 2 & -2 & . & . & . & . \\ 3 & -12 & 9 & . & . & . \\ 4 & -48 & 108 & -64 & . & . \\ 5 & -160 & 810 & -1280 & 625 & . \\ 6 & -480 & 4860 & -15360 & 18750 & -7776 \end{bmatrix}$$

(Heuristically) the pattern in the G matrix is

$$g_{r,c} = (-1)^c (1+c)^r \binom{r+1}{c+1}$$

***G**-matrix (using a linear notation (n:m) for the binomials)*

	0	1	2	...
0	1 = (1:1)	.	.	.
1	2 = (2:1)	- 2 = -2 (2:2)		
2	3 = (3:1)	- 12 = -2^2 (3:2)	9 = 3^2 (3:3)	
3	4 = (4:1)	- 48 = -2^3 (4:2)	108 = 3^3 (4:3)	
4	5 = (5:1)	-160 = -2^4 (5:2)	810 = 3^4 (5:3)	
...	...	...	...	...

(Heuristically) the entries in T can be expressed by a analogy of the above introduced recursive harmonic numbers.

Consider the sequence of the "non-harmonic" numbers

$y_{1,1} = 1,$ $y_{1,2} = 1+2=3,$ $y_{1,3} = 1+2+3 =6$,...

Then consider the "non-harmonic composition" of "non-harmonic" numbers:

$y_{2,1} = 1y_{1,0}$,
$y_{2,2} = 1y_{1,0} + 2y_{1,1}$,
$y_{2,3} = 1y_{1,0} + 2y_{1,1} + 3y_{1,2}$,
$y_{2,4} = 1y_{1,0} + 2y_{1,1} + 3y_{1,2} + 4y_{1,3}$,...

and the recursive generalization

$y_{0,1} = 1,\ y_{0,\cdot}=1$
$y_{r,1} = 1\ y_{r-1,0}$
$y_{r,2} = 1\ y_{r-1,0} + 2\ y_{r-1,1}$,
$y_{r,3} = 1\ y_{r-1,0} + 2\ y_{r-1,1} + 3\ y_{r-1,2}$

or

$y_{r,1+k} = y_{r,k} + (1+k)\ y_{r-1,k}$

```
\\ ========== Pari/GP ================
{h(rec,m) = local(s);
     if(rec==0,return(1));
     s = sum(k=0,m,h(rec-1,k)/(1+k));
   return(s); }
matrix(6,6,r,c,if(c>=r,h(r-1,c-r)))
```

where the first index r refers to the recursion depth.

We get:

$$t_{r,c} = (-1)^c \cdot \frac{y(c+1, c-r+1)}{c!}$$

T-*matrix*

	0	1	2	3	
0	1	$-y_{0,1}$	$y_{0,2}$	$y_{0,3}$	...
1	.	$-y_{1,1}$	$y_{1,2}$	$y_{1,3}$	...
2	.	.	$y_{2,2}$	$y_{2,3}$	..·
3	.	.	.	$y_{3,3}$	...
4	.	.	.	.	...
...	.	.	.	.	...

and have also a recursive definition:

$$t_{r,c} = t_{r,c-1}/c + t_{r-1,c-1}$$

which allows to compute quickly sequences of consecutive t in one row.

(Heuristically) the entries in ***T*** can also be expressed by the (signed) Stirling numbers first kind:

$$t_{r,c} = (-1)^c \cdot \frac{s_1(1+c, 1+c-r)}{c!}$$

T-*matrix*

	0	1	2	3	
0	1	$-s_{1\_2,2}/1!$	$s_{1\_3,3}/2!$	$-s_{1\_4,4}/3!$	...
1	.	$-s_{1\_2,1}/1!$	$s_{1\_3,2}/2!$	$-s_{1\_4,3}/3!$	...
2	.		$s_{1\_3,1}/2!$	$-s_{1\_4,2}/3!$	..·
3	.	.	.	$-s_{1\_4,1}/3!$	...
4	.	.	.	.	...
...		.	.	.	...

Perhaps this definition helps for analytic results for the dot-products; at least MATHEMATICA at *Wolframalpha* can evaluate several of the dotproducts analytically to zero when they are given in terms of that Stirling numbers

5 The formulae for the entries in the resulting inverse matrix $W = M^{-1}$

By the previous we get the definition (based on described heuristic evidence):

(using the Stirling-numbers 1st kind)

$$w_{r,c} = \sum_{k=a}^{\inf} t_{r,k} \cdot g_{k,c} \qquad where \quad a = \max(r,c)$$

$$= \sum_{k=a}^{\inf} \left[(-1)^k \cdot \frac{s_1(1+k,1+k-r)}{k!} \right] \cdot \left[(-1)^c (1+c)^k \binom{k+1}{c+1} \right]$$

$$w_{r,c} = 0 \qquad (\text{for small } r,c \ \ \text{supposedly for all } r,c)$$

We can now extend the computation of approximations of ***W*** to as many terms we wish and can thus improve the numerical evidence. In fact for some dozen entries of the left top of the ***W***-matrix all numerical evidence indicated the convergence to zero -either by the blind ***LDU***-factor inversion based on the empirical matrices as well as by this explicite formula.

As already mentioned the above definition gives more. Feeding them into a symbolic algebra-system like Mathematica we can get the analytical result for all $w_{r,c}$ that we may ask to be zero[4]; however unfortunately a complete result keeping *r* and *c* both variable seems out of reach.

The formula that can be used for Mathematica at wolframalpha[5] is the following:

> one has to insert the wished row- and column-index (beginning at zero) for the capital letters *R* and *C*. The lower bound for the summation-index must then be *X* which is the greater value of the two indexes *X=max(R,C)*

$w_{R,C}=$ *sum (-1)^k · StirlingS1(k+1,1+k-R)/k! · (1+C)^k · binomial(k+1,C+1), for k=X to infty*

Mathematica[6] at wolframalpha returns then the exact/analytical evaluation to zero.

[4] the policy at wolframalpha has changed and the new granted computation time does no more suffice for the analytical result if we ask for row-/column indexes >2

[5] openly accessible via http://www.wolframalpha.com

[6] Code in Pari/GP using Stirling numbers 1st kind up to index $s_{1,512,512}$:

```
St1=makemat_St1(512)            \\ userdefined routine;
w(r,c,maxn=n)=local(a);a=max(r,c);sum(k=a,maxn-1,(-1)^k*1.0*St1[1+k,1+k-r]/k!*(-1)^c*(1+c)^k*binomial(k+1,c+1))}
w(0,0,200)
```

However, for the first two rows in ***T*** the description of entries is much simpler, and Mathematica can even determine the analytical sum for the whole rows of ***W*** (by letting the column-index indeterminate *c*)

row 0 *sum_{k=c}^{infinity} [((-1)^k/k!) · ((-1)^ c ·(1 + c)^k·binomial(k+1, c +1))]*

row 1 *sum_{k=c}^{infinity} [((-1)^k·binomial(k+1,2)/k!) · ((-1)^ c ·(1 + c)^k·binomial(k+1, c +1))]*

(a similar expression can be made for the row 2 in ***T****, but is more complicated)*

It seems, that this matrix-inverse defines a set of identities involving Stirling numbers first kind, which is not yet known (at least not in that generality)

Gottfried Helms, 5.Dec 2012 (previous version 15. Dec 2011)

6 A view through exponential generating functions

We can proceed by an approach using the concept of *exponential generating function* ("egf" or "$\mathcal{E}$") . We ask: what are the *egf* for the dot-products of the left matrix ***T*** and the right matrix ***G*** ?

To work with the *egf*-concept we need the coefficients without the running reciprocal factorials. For the sake of notational easiness we rewrite also the matrix-factors a bit:

$$T \cdot D \cdot G_2 = \begin{bmatrix} 1 & -1 & 1/2 & -1/6 & 1/24 \\ . & 1 & -3/2 & 1 & -5/12 \\ . & . & 1 & -11/6 & 35/24 \\ . & . & . & 1 & -25/12 \\ . & . & . & . & 1 \end{bmatrix}_T \cdot diag\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}_Z\right) \cdot \begin{bmatrix} 1 & . & . & . & . \\ 1 & -1 & . & . & . \\ 1 & -4 & 3 & . & . \\ 1 & -12 & 27 & -16 & . \\ 1 & -32 & 162 & -256 & 125 \end{bmatrix}_H$$

To see the coefficients of the *egf* we would now remove the columnwise reciprocals from ***T*** by postmultiplication with the diagonalmatrix of the factorials, and because they are constant with respect to the dot-product with the columns of ***H*** (which given then egfs $\mathcal{E}_c$ for each such dotproducts) we include also the diagonal matrix ***Z*** into the lhs, so we get

$$U = T \cdot {}^dV(-1) \cdot {}^dF \cdot Z$$
$$H = {}^dV(-1) \cdot G_2$$

and then

$$X = U \cdot H = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ . & -2 & -9 & -24 & -50 \\ . & . & 6 & 44 & 175 \\ . & . & . & -24 & -250 \\ . & . & . & . & 120 \end{bmatrix}_U \cdot \begin{bmatrix} 1 & . & . & . & . \\ -1 & 1 & . & . & . \\ 1 & -4 & 3 & . & . \\ -1 & 12 & -27 & 16 & . \\ 1 & -32 & 162 & -256 & 125 \end{bmatrix}_H$$

Now for the dot-product of the first row and first column we can write the powerseries in the variable k:

$$\mathcal{E}_{0,0}(t) = \sum_{k=0}^{oo} (1+k)\frac{(-t)^k}{k!} = \sum_{k=0}^{oo} 1\frac{(-t)^k}{k!} + \sum_{k=0}^{oo} k\frac{(-t)^k}{k!} = e^{-t} - te^{-t} = (1-t)e^{-t}$$

So if we set $t=1$ (which simply means summing the dot-product) we get

$$x_{0,0} = \mathcal{E}_{0,0}(1) = 0$$

For the second row this is a bit trickier, however, we can find an expression for the entries in the second row in terms of a polynomial in the column-index k:

$$u_{1,k} = -1/2(1k + 2k^2 + 1k^3)$$

and thus

$$\mathcal{E}_{1,0}(t) = -\frac{1}{2}\sum_{k=0}^{oo}(1k+2k^2+1k^3)\frac{(-t)^k}{k!} = -\frac{1}{2}(4(-t)+5(-t)^2+1(-t)^3)e^{-t}$$
$$= \frac{1}{2}(4t-5t^2+1t^3)e^{-t}$$

What we want to do is now to find a transfer function, which converts directly the coefficients of some polynomial in k (which occurs at the series terms) into coefficients of the polynomial in t (which describes the composition of the exponential function).

Surprisingly this is very simple: we use just the matrix ***S_2*** of Stirling numbers *2*'nd kind

$$S2 = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 1 & 1 & . & . & . \\ 0 & 1 & 3 & 1 & . & . \\ 0 & 1 & 7 & 6 & 1 & . \\ 0 & 1 & 15 & 25 & 10 & 1 \end{bmatrix}$$

as transfermatrix and get for the general form

$$\mathcal{E}_{<general>}(t) = \sum_{k=0}^{oo} (a + bk + ck^2 + dk^3 + \ldots) \frac{(-t)^k}{k!} = (A + Bt + Ct^2 + Dt^3 + \ldots)e^{-t}$$

the sought coefficients *[A,B,C,D,...]* simply by the matrixproduct

$$[a,b,c,d,\ldots] \cdot S_2 = [A, B, C, D, \ldots]$$

Let's now call the polynomials in k on the lhs at the series terms as $\mathcal{P}_{r,c}$, where r indicates the row of the matrix ***U*** and c indicates the column of the matrix ***H*** which we use, and let's call the polynomials in t, which describe the composition of exponentials as $Q_{r,c}$.

The polynomials $\mathcal{P}$ in k have the form:

$\mathcal{P}_{0,0}(k)=$	1	+1k								
$\mathcal{P}_{1,0}(k)=$	0	-1/2k	$-1k^2$	$-1/2k^3$						
$\mathcal{P}_{2,0}(k)=$	0	-1/12k	$-5/24k^2$	$-1/24k^3$	$+5/24k^4$	$+1/8k^5$				
$\mathcal{P}_{3,0}(k)=$	0	0	$-1/24k^2$	$-1/16k^3$	$+1/24k^4$	$+1/12k^5$	$+0k^6$	$-1/48k^7$		
$\mathcal{P}_{4,0}(k)=$	0	1/120k	$+17/1440k^2$	$-1/36k^3$	$-41/1152k^4$	$+167/5760k^5$	$+91/2880k^6$	$-7/576k^7$	$-1/128k^8$	$+1/384k^9$

We get the coefficients for Q by the matrix-product

$$Q_{\cdot,0} = \mathcal{P}_{\cdot,0} \cdot S_2$$

and explicitely:

$Q_{0,0}(t)=$	1	+1t								
$Q_{1,0}(t)=$		-2t	$-5t^2/2!$	$-3t^3/3!$						
$Q_{2,0}(t)=$			$6t^2/2!$	$+26t^3/3!$	$+35t^4/4!$	$+15t^5/5!$				
$Q_{3,0}(t)=$				$-24t^3/3!$	$-154t^4/4!$	$-340t^5/5!$	$-315t^6/6!$	$-105t^7/7!$		
$Q_{4,0}(t)=$					$120t^4/4!$	$+1044t^5/5!$	$+3304t^6/6!$	$+4900t^7/7!$	$+3465t^8/8!$	$+945t^9/9!$

We get for

$$\mathcal{E}_{r,0}(t) = Q_{r,0}(-t) \cdot e^{-t}$$

and the according coefficient of the matrix ***X***:

$$x_{r,0} = \mathcal{E}_{r,0}(1) = Q_{r,0}(-1) \cdot e^{-1}$$

Empirically we get the expected (now exact) result for all rows:

$$x_{r,0} = 0 \qquad \text{for all } r>=0$$

and thus the matrix-vector-product

$$T \cdot G[,0] = W[,0] = 0$$

gives exact zeros in the first column of the inverse ***W***, for at least 64 rows (as far as empirically tested).

It is interesting (and surely the key for this observation), that all polynomials $Q_{\cdot,0}(t)$ have the factor $(1+t)$, which show immediately, that $t=-1$ is a zero of that polynomials. For proving $x_{r,0} = 0$ it would suffice to prove this for all $Q_{\cdot,0}$-polynomials.

7 References / Links

[MSE] math.stackexchange.com
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[MSE1112] "Does the inverse of this matrix of size *n* times n go to the zero matrix?"
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[MSE1101] "Does M^{-1}_n converge for a series of growing matrices M_n ?"
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online: http://people.math.sfu.ca/~cbm/aands/page_824.htm

[MWStir1] Stirling numbers of first kind
Eric W. Weisstein
From MathWorld--A Wolfram Web Resource
(The index in the online definition is not explicitely given; to match the indexes here it has to be taken as beginning at 1 (not zero!))
http://mathworld.wolfram.com/StirlingNumberoftheFirstKind.html

======= related discussions =================

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[RHN11] Discussions related to that "recursive harmonic numbers"
Mike Spivey ; 12'2011 answering at "math.stackexchange" (MSE)
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[2)] Blackett Laboratory, Imperial College, London SW7 2BZ, UK

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