Toying with the odd perfect numbers

The question of the existence of odd perfect numbers is unsolved. Perfect numbers are natural numbers N with the property, that the sum of its divisors $\sigma(N)$ equals just the double of N, or in short

If

$\sigma(N) = 2 \cdot N$

then we call N a "perfect number".

That idea of the "perfect number" stems from the ancient, where it was observed, that $\sigma(6) = 1+2+3+6 = (1+2)\cdot(1+3) = 2\cdot 6$ and $\sigma(28) = 1+2+4+7+14+28 = (1+2+4)\cdot(1+7) = 2\cdot 28$. In the *18*'th century L. Euler has then proved the general formula for even perfect numbers:

If 2^p-1 *is prime for some p then for*

$N = 2^{p-1}\cdot(2^p-1)$

we have that

$\sigma(N) = 2\cdot N$

and N is a perfect number.

All that numbers are obviously even; the problem is not yet solved, whether there are also odd perfect numbers. However, there are very strong restrictions known and it is thus widely believed, that no odd perfect numbers exist.

To get some better intuition about that problem let's assume first, that there is some N where N is the product of two different (odd) prime-factors only and where also the primefactors can occur to some powers. So we begin with the assumption

$N = p^a\cdot q^b$

For such a number N the sum of the divisors can easily be formalized, it is

(1.1) $$\begin{aligned}\sigma(N) = {} & 1+p+p^2+p^3+\ldots+p^a \\ & + q + pq + p^2q+p^3q+\ldots+p^aq \\ & + q^2 + pq^2 + p^2q^2+p^3q^2+\ldots+p^aq^2 \\ & + \ldots \\ & + q^b + pq^b + p^2q^b+p^3q^b+\ldots+p^aq^b\end{aligned}$$

which can be shortened to

(1.2) $$\sigma(N) = (1+p+p^2+p^3+\ldots+p^a)\cdot(1+q+q^2+q^3+\ldots+q^b) = \frac{p^{a+1}-1}{p-1}\cdot\frac{q^{b+1}-1}{q-1}$$

After that we can restate our question as follows

$\sigma(N) =?= 2\cdot N$

(1.3) $\sigma(N)/N =?= 2$

Using the above formula this means

$$\frac{p^{a+1}-1}{p-1}\cdot\frac{q^{b+1}-1}{q-1} \stackrel{?}{=} 2\cdot N = 2\cdot p^a q^b$$

and

(2) $$\frac{p^{a+1}-1}{p-1}\frac{1}{p^a}\cdot\frac{q^{b+1}-1}{q-1}\frac{1}{q^b} \stackrel{?}{=} 2 \qquad (2)$$

(Note, that although we have assumed that N has only two different primefactors that the lhs of this equation can easily be generalized to as many primefactors as desired)

This formula can be seen with two different focuses.

The first focus is that on the primefactor decomposition of cyclotomic polynomials. Each of the bigger fractions in (2) is a cyclotomic polynomial and it is obvious, that neither p^a can be a factor of $p^{a+1}-1$ nor q^b can be a factor of $q^{b+1}-1$, so p^a must be the only factor of $q^{b+1}-1$ and also q^b must be the only factor of $p^{a+1}-1$ – except of exactly one additional primefactor *2*. So we should rotate the denominators and write

$$\frac{p^{a+1}-1}{(p-1)q^b}\cdot\frac{q^{b+1}-1}{(q-1)p^a}\overset{?}{=}2 \tag{3.1}$$

Now we can discuss divisibility properties of that cyclotomic polynomials depending on suitable parameters *p, q, a* and *b*.

The second focus is that of the ratios of *p* and *p-1* resp *q* and *q-1* and to check, whether that ratios can be adjusted sufficiently (by selection of the primes) to allow to arrive at 2 (or at least in the near of *2*). Thus we rewrite the formula into the following form:

$$\frac{p}{p-1}\frac{q}{q-1}\cdot\left(\frac{p^{a+1}-1}{p^{a+1}}\right)\cdot\left(\frac{q^{b+1}-1}{q^{b+1}}\right)\overset{?}{=}2$$

$$\frac{p}{p-1}\frac{q}{q-1}\cdot\left(1-\frac{1}{p^{a+1}}\right)\cdot\left(1-\frac{1}{q^{b+1}}\right)\overset{?}{=}2 \tag{3.2}$$

This second focus is a bit less sharp than that on the divisibility in *(3.1)*, but it is without change immediately generalizable for arbitrarily many primefactors, so we begin our discussion with this focus.

All factors in *(3.2)* are near to *1*, where the two left factors are greater and the right two factors are smaller than *1*. So we separate them into

$$L=\frac{p}{p-1}\frac{q}{q-1} \qquad\qquad R=\left(1-\frac{1}{p^{a+1}}\right)\cdot\left(1-\frac{1}{q^{b+1}}\right) \tag{3.3}$$

and we note the first observation

R<1 *for any finite exponents a and b*
R increases with increasing p,q, and a and/or b

Thus, the first requirement from that observation is, that

L >2 *because R<1*

The second observation is, that

L>1 *for any finite p and q*
L decreases for increasing p and q

That means that for *p,q -> oo* we have the limit *L=1, R=1, L·R=1 =/=2* and we shall have no solutions for *p,q* above some upper bound, thus *p* and *q* must be small.

We try the smallest possible odd primefactors *p=3, q=5* and get

L = 3/2 · 5/4 = 15/8

and get, that *L<2* . Thus, because *R* is always smaller than *1*, we have that

L·R < 2 =/= 2

This shows, that there is no solution for *(3.2)* for a pair of odd primefactors *p,q*, and we conclude easily:

> No solution in *(2)* for *N*, where *N* is composed by two different primefactors. This is also valid if that two primefactors occur in arbitrary multiplicity, or:

- ***There are no odd perfect number N which contains only two different odd primefactors. This is valid also for N with arbitrary multiplicity of its factors.***

We can extend this argument quickly to the question, whether there can an odd perfect number exist which has three different primefactors. The smallest ones are *p=3, q=5, r=7*. We proceed as before

$$L=\frac{p}{p-1}\frac{q}{q-1}\frac{r}{r-1} \qquad R=\left(1-\frac{1}{p^{a+1}}\right)\cdot\left(1-\frac{1}{q^{b+1}}\right)\cdot\left(1-\frac{1}{r^{c+1}}\right) \tag{3.2.2}$$

$$L=\frac{3}{2}\frac{5}{4}\frac{7}{6} \qquad R=\left(1-\frac{1}{3^{a+1}}\right)\cdot\left(1-\frac{1}{5^{b+1}}\right)\cdot\left(1-\frac{1}{7^{c+1}}\right) \tag{3.2.3}$$

Now we get *L=35/16 = 2+3/16* which is bigger than *2* and for some high *a,b,c* the value for *R* can be made sufficiently near to *1* such that we can arrive at smaller as well at bigger values . Exactly *R* must have the value

> *R =1 - 3/35*

We find this by rearranging

$$L\cdot R=2$$

$$L\cdot\left(1-\frac{1}{3^{a+1}}\right)\cdot\left(1-\frac{1}{5^{b+1}}\right)\cdot\left(1-\frac{1}{7^{c+1}}\right)=2$$

$$1-\frac{1}{7^{c+1}}=\frac{2}{L\cdot\left(1-\frac{1}{3^{a+1}}\right)\cdot\left(1-\frac{1}{5^{b+1}}\right)}$$

$$\frac{1}{7^{c+1}}=1-\frac{2}{L\cdot\left(1-\frac{1}{3^{a+1}}\right)\cdot\left(1-\frac{1}{5^{b+1}}\right)}$$

$$c+1=-\log_7\left(1-2\frac{3^a5^b(3-1)(5-1)}{\left(3^{a+1}-1\right)\cdot\left(5^{b+1}-1\right)}\cdot\left(1-\frac{1}{7}\right)\right)$$

Required (exact) exponent at primefactor 7 for combinations of a and b in the exponents of the prime-factors 3 and 5:

a\b	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	0	0
2	1.318	0.618	0.550	0.537	0.535	0.534	0.534	0.534
3	0.712	0.390	0.346	0.338	0.336	0.336	0.336	0.336
4	0.609	0.334	0.295	0.288	0.286	0.286	0.286	0.286
5	0.579	0.317	0.279	0.272	0.270	0.270	0.270	0.270
6	0.569	0.312	0.274	0.267	0.265	0.265	0.265	0.265
7	0.566	0.310	0.272	0.265	0.264	0.263	0.263	0.263
8	0.565	0.309	0.272	0.265	0.263	0.263	0.263	0.263

We see, that there is no exact integer value existent; and moreover, the table has a simple structure in that along the rows and along the columns we have monotonuous decrease, such that in effect any possible solution is only on the top-left where the entries switch from one unit-interval to the next.

The red marked entries in the first row say that the resulting *L* is smaller than *2* and thus there is apriori no solution possible (exponents at primfactor *7* had to be complex to allow an exact solution)

If we choose the primefactor *r=17* then *L=3/2 · 5/4 · 17/16 = 255/128 < 2* and so no solution shall be possible.

But if *r=13* we have *L = 15/8 · 13/12 = 65/32 = 2 + 1/32 > 2* and so some solution with high exponents might be possible. We get the following table

Required (exact) exponent at primefactor 13 for combinations of a and b in the exponents of the primefactors a=3 and b=5:

a\b	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	0	1.540	1.298	1.263	1.257	1.255	1.255
4	0	1.223	0.8060	0.7583	0.7494	0.7477	0.7473	0.7472
5	0	0.9894	0.7099	0.6722	0.6651	0.6637	0.6634	0.6634
6	0	0.9350	0.6826	0.6474	0.6407	0.6394	0.6391	0.6391
7	0	0.9185	0.6739	0.6394	0.6329	0.6316	0.6314	0.6313
8	0	0.9131	0.6711	0.6368	0.6303	0.6291	0.6288	0.6288

*65/32 * (1-1/3^x)(1-1/5^3)(1-1/13^2)*
(3^x-1)/3^x(11/25)(7/13)=2
*1 –1/3^(1+a) = 2*13*25/7/11*
1 – 650/77 = 1/3^(1+a)

We could now proceed with other combinations of primefactors and get the following table

p	q	r	L	further check needed
3	5	7	2.188	true
3	5	11	2.063	true
3	5	13	2.031	true
3	*5*	*17*	*1.992*	
3	*5*	*..*	*...*	
3	*7*	*11*	*1.925*	
3	*...*	*...*	*...*	
5	*7*	*11*	*1.604*	
...	...	...	...	

which shows, that we need at most three tests, whether an odd perfect number with 3 different primefactors can exist.