Tetration

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update 01.12-4

Mathematical Miniatures

Observations about a possible inconsistency, when using the Bell-matrix in an infinite series

The possibility to use the coefficient-array according to the exponential-series in the Bell-form as a matrix (with all the helpful rules in the matrix-algebra) provides a promising tool for application to tetration, but must be proved to be consistent with other results.

This is interesting because we can then use the established matrix-rules:

- for linear compositions of the parameters,
- for finding an inverse/a reciprocal to implement the inverse tetration,
- for use of fractional and even complex powers with the help of the matrix-logarithm/matrix-exponential and/or eigensystem-decomposition
- and thus for implementing the whole set of operations for continuous tetration

- all this in a completely natural way.

In the context of infinite matrices some of these properties are not always well defined, and/or seemingly obvious properties may be misleading.

As a basic pattern, for example, consider the row-sums and column-sums in the ***B***-array; we would like that the order of summing were insignificant when summed to a total sum. But I a naïve approach this is not the case.

In this article I'll point to a possible subtle inconsistency, or put it in less dramatic words: to a possible problem.

Discussion along the heuristic exploration of the Bell-matrix *B* (or *Bs*)

Consider the following matrix-multiplication-scheme, where the row-sums and column-sums are also appended (the reciprocal factorial common denominators of each row are extracted to the right of the Bell-matrix for better display):

Example (1):

```
                     | 1   1    1     1     1 ...| /0!     ζ(0)/0!
                     | 1   2    3     4     5 ...| /1!     ζ(-1)/1!
                     | 1   4    9    16    25 ...| /2!     ζ(-2)/2!
                     | 1   8   27    64   125 ...| /3!     ζ(-3)/3!
                     | 1  16   81   256   625 ...| /4!     ζ(-4)/4!
                     | 1  32  243  1024  3125 ...| /5!     ζ(-5)/5!

[1  1  1  1  1 ...]  [ e  e^2  e^3  e^4   e^5  ...]
```

a) $\sum = \frac{e}{1-e} = \text{-}1.581...$ *by geometric-series formula*

b) $\sum = \frac{1}{1-e} = \text{-}0.581$ *by num. evaluation of zeta-sum*

Here a curious discrepancy occurs.

a) The formula for the sum of powers of *e* is just that of the geometric series.

b) The formula for the sum of zetas can be derived from the formula for the sum of bernoulli-numbers, or numerically approximated already by few terms.

So, where is the missing *1*?

Approach to cure, by "obscure" method A)

The problem can be cured if we introduce the first column of powers of zero, to also include the element $e^0=1$ and tacitly assume that the row again equals $\zeta(0)$:

Example (2):

```
                     | 1  1   1    1     1      1 ...| /0!     ζ(0)/0!
                     | 0  1   2    3     4      5 ...| /1!     ζ(-1)/1!
                     | 0  1   4    9    16     25 ...| /2!     ζ(-2)/2!
                     | 0  1   8   27    64    125 ...| /3!     ζ(-3)/3!
                     | 0  1  16   81   256    625 ...| /4!     ζ(-4)/4!
                     | 0  1  32  243  1024   3125 ...| /5!     ζ(-5)/5!

[1  1  1   1  1 ...] [ 1  e   e^2  e^3   e^4    e^5   ...]
```

$$\sum \quad = \frac{1}{1-e} = -0.581... \qquad\qquad = \frac{1}{1-e} = -0.581$$

The unelegant situation, that $\zeta(0)$ begins already one column earlier than the following zetas, may be accepted.

But changing the $[1,1,1,...]$ vector into a general powerseries reintroduces the problem:

Example (3):

```
                     | 1  1   1    1     1      1 ...| /0!     1 *ζ(0)/0!
                     | 0  1   2    3     4      5 ...| /1!     x *ζ(-1)/1!
                     | 0  1   4    9    16     25 ...| /2!     x^2*ζ(-2)/2!
                     | 0  1   8   27    64    125 ...| /3!     x^3*ζ(-3)/3!
                     | 0  1  16   81   256    625 ...| /4!     x^4*ζ(-4)/4!
                     | 0  1  32  243  1024   3125 ...| /5!     x^5*ζ(-5)/5!

[1  x  x^2  x^3  x^4 ...] [ 1 e^x  e^2x  e^3x   e^4x   e^5x   ...]
```

$$\sum \quad = \frac{1}{1-e^x} = \frac{e^x}{1-e^x} + 1 \qquad\qquad = \frac{e^x}{1-e^x} + \frac{1}{x}$$

Let's check this numerically with some x:

```
X      Sums of e^x    Sums of zetas   difference
1      -0.581...      -0.581            0  = -1 + 1
1/2    -1.541...      -0.541...         1  = -1 + 1/(1/2)
1/3    -2.527...      -0.527...         2  = -1 + 1/(1/3)
1/4    -3.520...      -0.520...         3  = -1 + 1/(1/4)
2      -0.156...      -0.656...       -1/2 = -1 + 1/2
..
```

We see, that the addition of the first column didn't resolve the problem of differing totals, when the summation-order is changed, if a free parameter $x=/=1$ and the according powerseries is involved.

Approach to cure, by "obscure" method B)

Now we consider another way to cure the discrepancy. I remember, that I had to solve a related discrepancy concerning summing to zeta, and a cure (which simply required to pose the problem in more generality) was, to consider also the series for $\zeta(1)$. *[He-SOLP]*

Example (4):

```
                           | 1 1/2  1/3   1/4   1/5 ...| /(-1)!    ζ(1)/(-1)!=-1
                           | 1   1    1     1     1 ...| /0!       ζ( 0)/0!
                           | 1   2    3     4     5 ...| /1!       ζ(-1)/1!
                           | 1   4    9    16    25 ...| /2!       ζ(-2)/2!
                           | 1   8   27    64   125 ...| /3!       ζ(-3)/3!
                           | 1  16   81   256   625 ...| /4!       ζ(-4)/4!
                           | 1  32  243  1024  3125 ...| /5!       ζ(-5)/5!

  [1  1  1  1  1 ...]      [ e   e^2  e^3   e^4   e^5  ...]
```

$$\sum \quad = \frac{e}{1-e} = -1.581... \qquad\qquad = \frac{1}{1-e} - 1 = -1.581$$

The red-marked fractions actually evaluate to zero individually and in the column-sum, since their denominator is $(-1)!$, which is an infinite expression. For the summing to a power of e the change is also infinitesimal; but infinitely many terms in the first row gives a non-neglectable term of -1 for $\zeta(1)/(-1)!$

Now let's look, what happens, if we use again a variable x in the leftmost vector (let's ignore the infinitesimal for the exponential-series to e):

Example (5):

```
                       | 1 1/2  1/3   1/4   1/5 ...| /(-1)!    1/x*ζ(1)/(-1)!=-1/x
                       | 1   1    1     1     1 ...| /0!       1   *ζ( 0)/0!
                       | 1   2    3     4     5 ...| /1!       x   *ζ(-1)/1!
                       | 1   4    9    16    25 ...| /2!       x^2 *ζ(-2)/2!
                       | 1   8   27    64   125 ...| /3!       x^3 *ζ(-3)/3!
                       | 1  16   81   256   625 ...| /4!       x^4 *ζ(-4)/4!
                       | 1  32  243  1024  3125 ...| /5!       x^5 *ζ(-5)/5!

  [1/x  1  x  x^2  x^3 ...]  [ e^x  e^2x  e^3x  e^4x  e^5x  ...]
```

$$\sum \quad = \frac{e^x}{1-e^x} \qquad\qquad = \frac{e^x}{1-e^x} + \frac{1}{x} - \frac{1}{x} = \frac{e^x}{1-e^x}$$

This way we have formal equality for all x.

Approach to cure, by "obscure" method combining A) and B)

The last step is then, to add the leading column of powers of zero to the matrix again, to see, how it could be made compatible.

Example (6):

```
                      | ?  1 1/2  1/3   1/4   1/5 ...| /(-1)!   1/x*ζ(1)/(-1)!=-1/x + ?/x
                      | 1  1   1    1     1     1 ...| /0!      1  *ζ( 0)/0!        + 1
                      | 0  1   2    3     4     5 ...| /1!      x  *ζ(-1)/1!
                      | 0  1   4    9    16    25 ...| /2!      x^2 *ζ(-2)/2!
                      | 0  1   8   27    64   125 ...| /3!      x^3 *ζ(-3)/3!
                      | 0  1  16   81   256   625 ...| /4!      x^4 *ζ(-4)/4!
                      | 0  1  32  243  1024  3125 ...| /5!      x^5 *ζ(-5)/5!

 [1/x  1  x  x^2  x^3 ...]  [e^0  e^x  e^2x  e^3x  e^4x  e^5x  ...]
```

$$\sum = \frac{1}{1-e^x} + \frac{??}{x} \qquad = \frac{e^x}{1-e^x} + 1 + \frac{??}{x}$$

The value at the position of the question-mark is formally *1/0/(-1)! = 1/(0*(-1)!)* and one thinkable evaluation would be *1/0! = 1*. But it seems, it should be assumed to be zero, see next paragraph.

Comparision of the "heuristical" formulae with the analytical Bernoulli-identities

Actually, the valid formula, derived from the Bernoulli-identities[1]

$$(1) \qquad \sum_{n=0}^{oo} \frac{B_n}{n!} = \frac{x}{e^x - 1} \qquad\qquad (2) \qquad B_n = (-1)^{n+1} n \zeta(1-n) \qquad\qquad B_1 = -\frac{1}{2}$$

can be stated as

$$(3) \qquad \frac{1}{1-e^x} = -\frac{1}{x} + 1 + \sum_{n=0}^{oo} \frac{\zeta(-n)}{n!} x^n$$

which agrees with the above scheme "example(6)" as noted in the right-most column.

It can be derived (see appendix), that not only $\zeta(0)/0!$ can be included in this formula but even by setting $\zeta(1)/(-1)! = -1$ the leading $-1/x$ with $\zeta(1)/(-1)!$ can be included into the sum-term

$$\frac{1}{1-e^x} = \sum_{n=-1}^{oo} \frac{(-1)^{n+1} \zeta(-n)}{n!} x^n$$

to make it extremely smooth (and looking much like using the formula (2) of bernoulli-number/zeta-conversion).

Moreover, since the sign-change is only effective at even n, but except of $\zeta(0)$ all $\zeta(2n)=0$ I return to formula (3), where only $-\zeta(0) = 1+\zeta(0)$ is used and the additional 1 positioned before the sum.

This makes the final formula:

$$(4) \qquad \boxed{\frac{1}{1-e^x} = 1 + \sum_{n=-1}^{oo} \frac{\zeta(-n)}{n!} x^n}$$

which gives an exact reflection of scheme "example (6)", where the *?*-marked position is assumed to be zero. (The complete derivation of this formula is given in appendix.)

[1] http://mathworld.wolfram.com/BernoulliNumber.html

Consideration of the Eta-function instead of the Zeta-function

Using the eta-function things usually come out easier. So we insert alternating signs to make the row-sums eta-function-looking:

Example (7):

```
                       | ? -1 1/2 -1/3   1/4  -1/5 ...| /(-1)!    ?+x^-1 *-η( 1)/(-1)!
                       | 1 -1   1   -1     1    -1 ...| /0!       1+1   *-η( 0)/0!
                       | 0 -1   2   -3     4    -5 ...| /1!        x    *-η(-1)/1!
                       | 0 -1   4   -9    16   -25 ...| /2!        x^2  *-η(-2)/2!
                       | 0 -1   8  -27    64  -125 ...| /3!        x^3  *-η(-3)/3!
                       | 0 -1  16  -81   256  -625 ...| /4!        x^4  *-η(-4)/4!
                       | 0 -1  32 -243  1024 -3125 ...| /5!        x^5  *-η(-5)/5!

  [1/x  1  x  x^2  x^3 ...]  [e^0  -e^x  e^2x  -e^3x  e^4x  -e^5x  ...]
```

$$\sum \quad = \frac{1}{1+e^x} + \frac{??}{x} \qquad\qquad = 1 - \frac{e^x}{1+e^x} + \frac{??}{x}$$

The sum of the red marked first line (neglecting the question-mark) is zero now, (*=log(2)/(-1)!*), and the area, covered by the η-function with parameters *(0,-1,-2,...)* is exactly the square array starting at second row, second column, and the etas have all inverted sign. The results of either way of summation (serial or matrix-based) match perfectly[2], good to know.

But this does now not help for the decision, whether the red line should be included, since its value is just infinitesimal *(log(2)/(-1)!)* .

Now, if we would add the red line (assuming *?/x=0*) infinitely often, scaled by the reciprocals of the consecutive naturals, this may produce a significant value and could thus be helpful to decide, whether this row is to be included or not (then: also for the "infinitesimal case")

log(2)/(-1)! + log(2)/(-1)! /2 + log(2)/(-1)! /3 +...
= log(2) (1 + 1/2 + 1/3 +...)/(-1)!
*= log(2) * (ζ(1) / (-1)!)) = log(2)* (-1)*
= – log(2)

If again we set *ζ(1)/(-1)! = -1* we get now a non-neglectable value, and this helps for the decision for the precise description of a total sum.

[2] and suggest numerically *?/x=0*

Sum of infinitely many e^x's alternating geometric series

So let's compute an infinite sum of totals over all $x=1..inf$.

For the alternating geometric series of e^x this means summing $1/(1+e^x)$ for all $x=1..inf$. Explicitely this is

$$s_e = 1/(1+e) + 1/(1+e^2) + 1/(1+e^3) +...$$

This converges numerically well to the value

$$s_e = 0.464163...$$

by direct summing.

If we apply this accordingly to the row-sums (taken as η-values) we get, if we neglect the first row containing $\eta(1)$ in example(7)

$$\begin{aligned} s_{test} &= (1+1+1+1+...)(1 - \eta(0)) \quad /0! \\ &\quad + (1+2+3+4+...) \ -\eta(-1) \quad /1! \\ &\quad + ... \\ s_{test} &= \zeta(0) * \quad 1 \quad /0! \\ &\quad - \zeta(0) * \quad \eta(0) \quad /0! \\ &\quad - \zeta(-1) * \quad \eta(-1) \quad /1! \\ &\quad - \zeta(-2) * \quad \eta(-2) \quad /2! \\ &\quad - .... \end{aligned}$$

Numerically, by a rough Euler-sum[3], this is about

$$S_{test}= -0.228983... ,$$

which does not match s_e. But indeed, the difference is just

$$s_e - s_{test}= 0.69314... ,$$

which seems to be the value of $\eta(1)=log(2)$ if everything is precisely approximated – suggestively too good for an assumtion, that it may be a purely incidental value.

Then the sum of all totals, computed over ***all*** involved rows/etas (including $\eta(1)$ for the first row) is

$$s_\eta = 0.464157...$$

which then matches considerably well the evaluation along the sums of the geometric series of e^x

This appears to me as an reasonable indicator, that the red-marked line should be included to achieve full consistency – at least for such esoteric applications as infinite series of powertowers...

[3] This value was taken from a Euler-sum "before it begins to diverge". It needs another verification!

Provisorial conclusion:

So these seem to be two arguments, that for full consistency of the matrixoperator ***B*** we might reconsider the Bell-matrix also for its application in tetration, and to check whether the current definition is precise enough to be sufficient also for infinite sums/ for infinitely many applications/iterations.

The last example suggests also another aspect: the open question, what to do with the top-left entry, marked by the question-mark. In the last computation there was nothing missing to equal the summing via the exponentials or via the η-functions. The same can be assumed in the above paragraph, regarding the zeta-summing, where it should be assumed to be an infinitesimal or zero. It seems, the best assumption is, that the position of the question-mark should be zero.

So our improved Bell-matrix should possibly look like

Example (8):

```
| 0  1 1/2  1/3   1/4   1/5 ...| /(-1)!
| 1  1   1    1     1     1 ...| /0!
| 0  1   2    3     4     5 ...| /1!
| 0  1   4    9    16    25 ...| /2!
| 0  1   8   27    64   125 ...| /3!
| 0  1  16   81   256   625 ...| /4!
| 0  1  32  243  1024  3125 ...| /5!
```

[1/x 1 x x^2 x^3 ...]

if we want to use it for infinite sums of powertowers, or infinitely many iterations.

Well, this discussion, focuses a problem, which may be seen as artificially and only motivated by the suggestive number-pattern in the given examples. A zeta-series is not the same as a powerseries. To migrate zeta-series into the context of powerseries (more exactly to laurent-series) we have to introduce constants like Stieltjes-constants. So some discrepancies or curious effects should always be expected, if such migrations are involved as used here by the suggestive display of the number-array.

Now, in the context of tetration we don't explicitely deal with zeta-series, so all the above fiddling might be inappropriate here. On the other hand, the use of powers of the Bell-matrix or especially any approach to deal with its reciprocal (even if reduced to the problem of its triangular factors) may introduce some of those effects; and then to deal with infinite series of its reciprocal obviously needs extremely more caution.

I've come across a very similar looking (but easier) problem earlier, and could resolve it along a related manner, as already mentioned. That was also the reason, that in my conjectures I always used the ***alternating*** series of powertowers: to prevent to run into this problem inadvertedly. In my current conjecture *[He-GS1]* which involves series of the inverse of the Bell-matrix ***B*** or $\boldsymbol{B}_s$ this alternating-sign safety-measure is possibly somehow compromised...

Gottfried Helms, 28.11.2007 / 1.12.2007 /12.4.2011

[He-SOLP] *G. Helms 2007;*
sums of like powers (Bernoulli/zeta-polynomials)
http://go.helms-net.de/math/binomial_new/04_3_SummingOfLikePowers.pdf

[He-GS1] *G. Helms 2007*
http://go.helms-net.de/math/tetdocs/Tetration_GS_short.pdf

Appendix

We want to convert the formula

$$\sum_{n=0}^{oo} \frac{B_n}{n!} x^n = \frac{x}{e^x - 1} \qquad \textit{using} \quad B_n = (-1)^{n+1} n \zeta(1-n) \qquad B_1 = -\frac{1}{2}$$

into an expression which contains the zeta-function only (instead of the bernoulli-numbers), to reflect the example-schemes above. Here is the derivation.

First we separate the first two bernoulli-numbers from the sum, since they fall a bit out of the simple order when converted into zeta-values.

$$\sum_{n=0}^{oo} \frac{B_n}{n!} x^n \qquad = 1 - \frac{1}{2} x + \sum_{n=2}^{oo} \frac{B_n}{n!} x^n$$

Now we write this in terms of zetas

$$\sum_{n=0}^{oo} \frac{B_n}{n!} x^n \qquad = 1 - \frac{1}{2} x + \sum_{n=2}^{oo} \frac{(-1)^{n+1} n \zeta(-(n-1))}{n!} x^n$$

and cancel n, extract x and convert $(-1)^{n+1}$ into $(-1)^{n-1}$ to adapt coefficients and exponents to the same index:

$$\sum_{n=0}^{oo} \frac{B_n}{n!} x^n \qquad = 1 - \frac{1}{2} x + x \sum_{n=2}^{oo} \frac{(-1)^{n-1} \zeta(-(n-1))}{(n-1)!} x^{n-1}$$

We reduce the index and factor the one cofactor x out of the complete expression:

$$\sum_{n=0}^{oo} \frac{B_n}{n!} x^n \qquad = x(\frac{1}{x} - \frac{1}{2} + \sum_{n=1}^{oo} \frac{(-1)^n \zeta(-n)}{n!} x^n)$$

Now each $\zeta(-2n)$ is zero, for instance $\zeta(-2)$, $\zeta(-4)$, which coincides with $n=2,4,6...$. For $n=1,3,5$ $\zeta(-1)$, $\zeta(-3)$, $\zeta(-5)$ the sign (-1) remains significant only and we may extract this sign-factor to the front of the sum:

$$\sum_{n=0}^{oo} \frac{B_n}{n!} x^n \qquad = x(\frac{1}{x} - \frac{1}{2} - \sum_{n=1}^{oo} \frac{\zeta(-n)}{n!} x^n)$$

We can adapt this formula to zetas only.

First we replace $-1/2 = -1 + 1/2$, and then $+1/2 = -\zeta(0)/0!$.

$$\sum_{n=0}^{oo} \frac{B_n}{n!} x^n \qquad = x(\frac{1}{x} - 1 - \frac{\zeta(0)}{0!} - \sum_{n=1}^{oo} \frac{\zeta(-n)}{n!} x^n)$$

If we assume $\zeta(1)/(-1)! = -1$, then we may also expand the $1/x$-term

$$\sum_{n=0}^{oo} \frac{B_n}{n!} x^n \qquad = x(-\frac{\zeta(1)}{(-1)!} x^{-1} - 1 - \frac{\zeta(0)}{0!} - \sum_{n=1}^{oo} \frac{\zeta(-n)}{n!} x^n)$$

pull the minus out of the expression and collect all zeta-terms into the sum to get:

$$\sum_{n=0}^{oo} \frac{B_n}{n!} x^n \qquad = -x(1 + \sum_{n=-1}^{oo} \frac{\zeta(-n)}{n!} x^n)$$

Now, since $\sum_{n=0}^{oo} \frac{B_n}{n!} x^n = \frac{x}{e^x - 1}$ it is also

$$\frac{x}{e^x - 1} = -x(1 + \sum_{n=-1}^{oo} \frac{\zeta(-n)}{n!} x^n)$$

and to make this the expression for the geometric series, we change sign and also cancel x to obtain

$$\frac{1}{1 - e^x} = 1 + \sum_{n=-1}^{oo} \frac{\zeta(-n)}{n!} x^n$$

which is the expression, used in example (6), where the single 1 can be seen as the value in the column for e^0: if we remove it on both sides of the expression, we get

$$\frac{e^x}{1 - e^x} = \sum_{n=-1}^{oo} \frac{\zeta(-n)}{n!} x^n$$

the exact representation of the example, where the first column is deleted, and the geometric series of e^x starts at $(e^x)^1$ instead of $(e^x)^0$.