Integer relations between the *sinh/cosh/asinh/cosh:* some accidental observations

*Accidentally I came across observations about integer ratios of the asinh() and acosh() - functions at integer arguments, which surprised me much. The **OEIS** gives helpful background informations, especially concerning relations to the Pell-equation-problem.*

Gottfried Helms, 15.3.2013

1. About the *sinh( )/asinh( )*-function

Consider the *asinh(n)* at integer *n*. In general, that numbers are transcendental. It is then somehow surprising, that some *asinh(n)* are integer multiples of *asinh(1)*.

Or, say,

$$\text{asinh}(a(n)) / \text{asinh}(1) = b(n) \qquad \text{where } b(n) \text{ are integers}$$

We can even find sequences of *a(n)* for natural indexes *n* which have all such integer ratios with the *asinh(1)*. We get - the presumably infinite - sequence

\n	1	2	3	4	5	...
a(n)	1	7	41	239	1393	...
b(n)	1	3	5	7	11	...

Interestingly, we find a similar table also for integer multiples of *asinh(2)*:

$$\text{asinh}(a_2(n))/\text{asinh}(2) = b_2(n)$$

We get

\n	1	2	3	4	5	...
$a_2(n)$	2	38	682	12238	219602	...
$b_2(n)$	1	3	5	7	11	...

and because of that immediate success we produce the complete(able) table by the formula

$$a_r(n) = \sinh(\ b(n) \cdot \text{asinh}(r)\ )$$

Indeed we get empirically always integers and this gives table ***T***:

r \ b(n) \ $a_r(n)$	1	3	5	7	9	11	...
1	1	7	41	239	1393	8119	...
2	2	38	682	12238	219602	3940598	...
3	3	117	4443	168717	6406803	243289797	...
4	4	268	17684	1166876	76996132	5080577836	...
5	5	515	52525	5357035	546365045	55723877555	...
6	6	882	128766	18798954	2744518518	400680904674	...
...	...	...	...	...	...	...	...

Reading the formula in reverse, the table gives us

$$\text{asinh}(a_r(n))/\text{asinh}(r) = b(n)$$

example:

$$\text{asinh}(4443)/\text{asinh}(3) = 5$$

or

$$\text{asinh}(515)/\text{asinh}(5) = 3$$

The rows have common factors; we can cancel this factor and get T_1:

\ b(n) r \ $a_r(n)$	1	3	5	7	9	11	...
1	1	7	41	239	1393	8119	...
2	1	19	341	6119	109801	1970299	...
3	1	39	1481	56239	2135601	81096599	...
4	1	67	4421	291719	19249033	1270144459	...
5	1	103	10505	1071407	109273009	11144775511	...
6	1	147	21461	3133159	457419753	66780150779	...
...	...	...	...	...	...	...	...

The sequences along the columns have a polynomial formula in its index r; we get

b(n)	Formula	
1	1	$=a_r(1)$
3	$4r^2+3$	$=a_r(3)$
5	$16r^4+20r^2+5$	$=a_r(5)$
7	$64r^6+112r^4+56r^2+7$	$=a_r(7)$
9	$256r^8+576r^6+432r^4+120r^2+9$	$=a_r(9)$
11	$1024r^{10}+2816r^8+2816r^6+1232r^4+220r^2+11$	$=a_r(11)$
...	...	...

We could also express them by binomial weighting of the coefficients in the following table, which occurs after $P^{-1} \cdot T_1$ (where P is the lower triangular Pascal-matrix)

\ b(n) r \ $a_r(n)$	1	3	5	7	9	11	...
1	1	7	41	239	1393	8119	...
2		12	300	5880	108408	1962180	...
3		8	840	44240	1917392	77164120	...
4			960	141120	13170240	1032757440	...
5			384	217728	44652672	6542904192	...
6				161280	82736640	22956595200	...
7				46080	85570560	48115077120	...
8					46448640	61823139840	...
9					10321920	47800811520	...
10						20437401600	...
11						3715891200	...
...	...	...	...	...	...	...	...

Where now, for instance to determine the 4'th row in column 2 is

$$a_4(2)=(1\cdot 7+2\cdot 12+1\cdot 8)\cdot 4=4\cdot 39$$
$$a_5(2)=(1\cdot 7+3\cdot 12+3\cdot 8+1\cdot 0)\cdot 5=5\cdot 67$$

and so on. From the rows can still common factors be cancelled; we can rescale by the reciprocal factorials, powers of 2 and then each second row by $r\cdot(r-1)$ but this does not show immediately some more interesting pattern.

Sequences along rows

The rows define sequences with a simple recursive formula. We get

$$a_r(n) = - a_r(n-2) + (2+4r^2) \cdot a_r(n-1)$$

OEIS

The found numbers $a_r(n)$ have some more interesting properties, due to remarks in the OEIS, where we find the first few sequences, when read along the rows.

We have, for the row *1*, that $a_1(n)$ are numbers, whose transformes are also squares, by the formula

$$d_1(n)^2 = 2\cdot(a_1(n)^2 + 1)$$

the sequence

$$d(n) = [2, 10, 58, 338, 1970, 11482]$$

OEIS : http://oeis.org/A075870 : [1,7,41,239,1393,8119,...]
2*n^2 - 4 is a square

Accordingly, we find the analogues for the other rows. This gives then table ***U***:

r \ b(n) \ $a_r(n)$	1	3	5		Formula	sequence $d_r(n)$
1	1	7	41	...	$d_1(n)^2 = 2\cdot(a_1(n)^2 + 1)$	2, 10, 58, 338, 1970, 11482
2	2	38	682		$d_2(n)^2 = 5\cdot(a_2(n)^2 + 1)$	5, 85, 1525, 27365, 491045,
3	3	117	4443		$d_3(n)^2 = 10\cdot(a_3(n)^2 + 1)$	10, 370, 14050, 533530, 20260090,
4	4	268	17684		$d_4(n)^2 = 17\cdot(a_4(n)^2 + 1)$	17, 1105, 72913, 4811153, 317463185,
5	5	515	52525		$d_5(n)^2 = 26\cdot(a_5(n)^2 + 1)$	26, 2626, 267826, 27315626, 2785926026,
6	6	882	128766		$d_6(n)^2 = 37\cdot(a_6(n)^2 + 1)$	37, 5365, 783253, 114349573, 16694254405
...	...	...	...		...	...
r	...	...	...		$d_r(n)^2 = (1+r^2)(a_r(n)^2+1)$	

where the coefficients at the parenthese are $(1+r^2)$

See a collection of relations to other formulae in the excerpt from the OEIS-comments in sect. 3.

2. What about the *cosh()*-function?

If we introduce also the *cosh( )/acosh( )*-function, we get also for the even indexes in the columnheaders integer multiples:

r \ b(n) \ $c_r(n)$	0	2	4	6	8	10	12
1	1	3	17	99	577	3363	19601
2	1	9	161	2889	51841	930249	16692641
3	1	19	721	27379	1039681	39480499	1499219281
4	1	33	2177	143649	9478657	625447713	41270070401
5	1	51	5201	530451	54100801	5517751251	562756526801
6	1	73	10657	1555849	227143297	33161365513	4841332221601
7	1	99	19601	3880899	768398401	152139002499	30122754096401
8	1	129	33281	8586369	2215249921	571525893249	147451465208321
9	1	163	53137	17322499	5647081537	1840931258563	600137943210001
10	1	201	80801	32481801	13057603201	5249124005001	2110134792407201
11	1	243	118097	57394899	27893802817	13556330774163	6588348862440401
12	1	289	167041	96549409	55805391361	32255419657249	18643576756498561

By that table we get

$$acosh(c_r(n))/asinh(r) = b(n)$$

example:

$$acosh(19)/asinh(3) = 2$$

or

$$acosh(5201)/asinh(5) = 4$$

In terms of polynomials we find now:

b(n)	Formula	
0	1	$=c_r(1)$
2	$2 r^2 + 1$	$=c_r(2)$
4	$8 r^4 + 8 r^2 + 1$	$=c_r(3)$
6	$32 r^6 + 48 r^4 + 18 r^2 + 1$	$=c_r(4)$
8	$128 r^8 + 256 r^6 + 160 r^4 + 32 r^2 + 1$	$=c_r(5)$
10	$512 r^{10} + 1280 r^8 + 1120 r^6 + 400 r^4 + 50 r^2 + 1$	$=c_r(6)$
		...

As binomially rescaled $P^{-1} \cdot T$ version we get

r \ b(n) \ $c_r(n)$	0	2	5	6	8	10	...
1	1	3	17	99	577	3363	...
2		6	144	2790	51264	926886	...
3		4	416	21700	936576	37623364	...
4			480	70080	6514560	509793600	...
5			192	108672	22217472	3249125760	...
6				80640	41287680	11436929280	...
7				23040	42762240	24014753280	...
8					23224320	30888345600	...
9					5160960	23895244800	...
10						10218700800	...
11						1857945600	...
...						...	...

where we can cancel common factors in the rows .

OEIS

Similarly as with the *sinh( )/asinh( )* we have, for the row *1*, that $c_1(n)$ are numbers, whose transformes are also squares, by the formula

$$d_1(n)^2 = 2 \cdot (c_1(n)^2 - 1)$$

giving the sequence

$$d(n) = [\, 0, 4, 24, 140, 816, 4756, \dots \,]$$

OEIS : http://oeis.org/A001541 : [1,3,17,99,...]
2*n^2 - 2 is a square
http://oeis.org/A005319 : [0, 4, 24, 140, 816, 4756, ...]

Again accordingly, we find the analogues for the other rows. This gives then table ***U***:

\ b(n) r \ $c_r(n)$	0	2	4		Formula	sequence $d_r(n)$
1	1	3	17	...	$d_1(n)^2 = 2\cdot(c_1(n)^2 - 1)$	0, 4, 24, 140, 816, 4756, ...
2	1	9	161		$d_2(n)^2 = 5\cdot(c_2(n)^2 - 1)$	0, 20, 360, 6460, 115920, 2080100, ...
3	1	19	721		$d_3(n)^2 = 10\cdot(c_3(n)^2 - 1)$	0, 60, 2280, 86580, 3287760,...
4	1	33	2177		$d_4(n)^2 = 17\cdot(c_4(n)^2 - 1)$	0, 136, 8976, 592280, 39081504,...
5	1	51	5201		$d_5(n)^2 = 26\cdot(c_5(n)^2 - 1)$	0, 260, 26520, 2704780, 275861040,...
6	1	73	10657		$d_6(n)^2 = 37\cdot(c_6(n)^2 - 1)$	0, 444, 64824, 9463860, 1381658736,...
...	...	...	...		...	...
r	...	...	...		$d_r(n)^2 = (1+r^2)(c_r(n)^2-1)$	

Gottfried Helms, 15. 3. 2013

3. References into OEIS:

3.1. The *sinh( )/asinh( )* - sequences of integer ratios:

r	sequence OEIS	"Comment" in OEIS
1	1,7,41,239,1393,8119 http://oeis.org/A002315	NSW numbers: a(n) = 6*a(n-1) - a(n-2); Named after the Newman-Shanks-Williams reference. Also numbers n such that A125650(3*n^2) is an odd perfect square. Such numbers 3*n^2 form a bisection of A125651. - Alexander Adamchuk, Nov 30 2006 For positive n, a(n) corresponds to the sum of legs of near-isosceles primitive Pythagorean triangles (with consecutive legs). - Lekraj Beedassy, Feb 06 2007 Also numbers n such that n^2 is a centered 16-gonal number; or a number of the form 8k(k+1)+1, where k = A053141(n) = {0, 2, 14, 84, 492, 2870, ...}. - Alexander Adamchuk, Apr 21 2007 A002315(n)=A001333(2*n+1) [From Ctibor O. ZIZKA (ctibor.zizka(AT)seznam.cz), Aug 13 2008] The lower principal convergents to 2^(1/2), beginning with 1/1, 7/5, 41/29, 239/169, comprise a strictly increasing sequence; numerators=A002315 and denominators=A001653. - Clark Kimberling, Aug 27 2008 The upper intermediate convergents to 2^(1/2) beginning with 10/7, 58/41, 338/239, 1970/1393 form a strictly decreasing sequence; essentially, numerators=A075870, denominators=A002315. - Clark Kimberling, Aug 27 2008 General recurrence is a(n)=(a(1)-1)*a(n-1)-a(n-2), a(1)>=4, lim n->infinity a(n)= x*(k*x+1)^n, k =(a(1)-3), x=(1+sqrt((a(1)+1)/(a(1)-3)))/2. Examples in OEIS: a(1)=4 gives A002878, primes in it A121534. a(1)=5 gives A001834, primes in it A086386. a(1)=6 gives A030221, primes in it not in OEIS {29,139,3191,...}. a(1)=7 gives A002315, primes in it A088165. a(1)=8 gives A033890, primes in it not in OEIS (does there exist any ?). a(1)=9 gives A057080, primes in it not in OEIS {71,34649,16908641,...}. a(1)=10 gives A057081, primes in it not in OEIS {389806471,192097408520951,...}. [From Ctibor O. Zizka, Sep 02 2008] Numbers n such that (ceiling(sqrt(n*n/2)))^2 = (1+n*n)/2 [From Ctibor O. Zizka, Nov 09 2009] A001109(n)/a(n) converges to cos^2(Pi/8) = 1/2 + 2^(1/2)/4 [From Gary Detlefs, Nov 25 2009] a(n) represents all positive integers K for which 2(K^2+1) is a perfect square. [From Neelesh Bodas (neelesh.bodas(AT)gmail.com), Aug 13 2010] For positive n, a(n) equals the permanent of the (2n)X(2n) tridiagonal matrix with sqrt(8)'s along the main diagonal, and i's along the superdiagonal and subdiagonal (i is the imaginary unit). [From John M. Campbell, Jul 08 2011] Integers n such that A000217(n-2) + A000217(n-1) + A000217(n) + A000217(n+1) is a square (cf. A202391). [From Max Alekseyev, (maxale(AT)gmail.com), Dec 19 2011]
2	2,38,682,12238,219602 http://oeis.org/A075796	Numbers k such that 5*k^2 + 5 is a square. Lim. n-> Inf. a(n)/a(n-1) = 8*phi + 1 = 9 + 4*Sqrt(5).
2'	1,19,341,6119,109801, 1970299 http://oeis.org/A049629	a(n)=(F(6n+5)-F(6n+1))/4=(F(6n+4)+F(6n+2))/4, where F=A000045 (the Fibonacci sequence) a(n) ~ 1/4*(sqrt(5) + 2)^(2*n+1) - Joe Keane (jgk(AT)jgk.org), May 15 2002 For all members x of the sequence, 20*x^2 + 5 is a square. Lim. n -> Inf. a(n)/a(n-1) = 9 + 2*Sqrt(20) = 9 + 4*Sqrt(5). The 20 can be seen to derive from the equation "20*x^2 + 5 is a square". - Gregory V. Richardson, Oct 12 2002 a(n) = [[(9 + 4*Sqrt(5))^N - (9 - 4*Sqrt(5))^N] + [(9 + 4*Sqrt(5))^(N-1) - (9 - 4*Sqrt(5))^(N-1)] / (8*Sqrt(5)) - Gregory V. Richardson, Oct 12 2002 G.f.: (1+x)/(1-18x+x^2). a(n)=A049660(n)+A049660(n+1). [From R. J. Mathar, Nov 04 2008] a(n)=18*a(n-1)-a(n-2) for n>1; a(0)=1, a(1)=19. [From Philippe DELEHAM, Nov 17 2008]
3	3,117,4443,168717, http://oeis.org/A173127	sinh((2n-1)*arcsinh(3)). Numbers n such that ((n^2+1)/10) is square. - Vincenzo Librandi, Jan 02 2012
3'	1,39,1481,56239, 2135601 http://oeis.org/A097314	Pell equation solutions (3*a(n))^2 - 10*b(n)^2 = -1 with b(n):=A097315(n), n>=0
4	4,268,17684,1166876, 76996132,5080577836 ---	---(not existent, see sequence *4' = $a_4(n)/4$* instead)
4'	1,67,4421,291719, 19249033,1270144459 http://oeis.org/A078989	Chebyshev sequence with Diophantine property. One fourth of bisection (even part) of A041024. (4*a(n))^2 - 17*A078988(n)^2= -1 (Pell -1 equation, see A077232-3).
5		---(not existent, see sequence *5' = $a_5(n)/5$* instead)
5'	1,103,10505,1071407, 109273009 http://oeis.org/A097726	Pell equation solutions (5*a(n))^2 - 26*b(n)^2 = -1 with b(n):=A097727(n), n>=0

3.2. The *cosh( )/acosh( )* - sequences of integer ratios:

r	sequence OEIS	"Comment" in OEIS
1	1,3,17,99,577,3363, http://oeis.org/A001541	Chebyshev polynomials of the first kind evaluated at 3. a(n) solves for x in x^2 - 8*y^2 = 1, the corresponding y being A001109(n). For n>0, the ratios a(n)/A001090(n) may be obtained as convergents to sqrt(8): either successive convergents of [3; -6] or odd convergents of [2; 1, 4]. - Lekraj Beedassy, Sep 09 2003 Formula: ((-1+sqrt(2))^n+(1+sqrt(2))^n+(1-sqrt(2))^n+(-1-sqrt(2))^n)/4 (with interpolated zeros) E.g.f. cosh(x)cosh(sqrt(2)x) (with interpolated zeros). - Paul Barry, Sep 18 2003 Also gives solutions to the equation x^2-1=floor(x*r*floor(x/r)) where r=sqrt(8) - Benoit Cloitre, Feb 14 2004 Appears to give all solutions >1 to the equation : x^2=ceiling(x*r*floor(x/r)) where r=sqrt(2). - Benoit Cloitre, Feb 24, 2004 a(n+1) - A001542(n+1) = A090390(n+1) - A046729(n) = A001653(n); a(n+1) - 4*A079291(n+1) = (-1)^(n+1). Formula generated by the floretion - .5'i + .5'j - .5i' + .5j' - 'ii' + 'jj' - 2'kk' + 'ij' + .5'ik' + 'ji' + .5'jk' + .5'ki' + .5'kj' + e - Creighton Dement (creighton.k.dement(AT)uni-oldenburg.de), Nov 16 2004 This sequence give numbers n such that (n-1)*(n+1)/2 is a perfect square. Remark : (i-1)*(i+1)/2 = (i^2-1)/2 = -1 = i^2 with i=sqrt(-1) so i is also in the sequence. - Pierre CAMI, Apr 20 2005 a(n) is prime for n = {1, 2, 4, 8}. Prime a(n) are {3, 17, 577, 665857}, which belong to A001601(n). a(2k-1) is divisible by a(1) = 3. a(4k-2) is divisible by a(2) = 17. a(8k-4) is divisible by a(4) = 577. a(16k-8) is divisible by a(8) = 665857. - Alexander Adamchuk, Nov 24 2006 a(n)=A001333(2*n) [From Ctibor O. Zizka, Aug 13 2008] The upper principal convergents to 2^(1/2), beginning with 3/2, 17/12, 99/70, 577/408, comprise a strictly decreasing sequence; essentially, numerators=A001541 and denominators=A001542. - Clark Kimberling, Aug 26 2008 Also index of sequence A082532 for which A082532=1 [From Carmine Suriano, Sep 07 2010] Numbers n such that sigma(n-1) and sigma(n+1) are both odd numbers. [From Juri-Stepan Gerasimov, Mar 28 2011] Also, numbers such that floor[a(n)^2/2] is a square: base 2 analog of A031149, A204502, A204514, A204516, A204518, A204520, A004275, A001075. - M. F. Hasler, Jan 15 2012
2	1,9,161,2889,51841, http://oeis.org/A023039	The primitive Heronian triangle 3*a(n) +/- 2, 4*a(n) has the latter side cut into 2*a(n) +/- 3 by the corresponding altitude and has area 10*a(n)*A060645(n). - Lekraj Beedassy, Jun 25 2002 Chebyshev's polynomials T(n,x) evaluated at x=9. The a(n) give all (unsigned, integer) solutions of Pell equation a(n)^2 - 80*b(n)^2 = +1 with b(n)=A049660(n), n>=0. Also gives solutions to the equation x^2-1=floor(x*r*floor(x/r)) where r=sqrt(5) - Benoit Cloitre, Feb 14 2004 Appears to give all solutions >1 to the equation : x^2=ceiling(x*r*floor(x/r)) where r=sqrt(5). - Benoit Cloitre, Feb 24, 2004 For all members x of the sequence, 5*x^2 - 5 is a square, A004292(n)^2. The a(n) are the y-values in the integer solutions of x^2-5y^2=1, see the comment in A060645. - Sture Sjöstedt, Nov 29 2011
3	1,19,721,27379, 1039681, http://oeis.org/A078986	Chebyshev T(n,19) polynomial. a(n+1)^2 -10*(6*A078987(n))^2 = 1, n>=0 (Pell equation +1, see A033313 and A033317). Also gives solutions to the equation x^2-1=floor(x*r*floor(x/r)) where r=sqrt(10) - Benoit Cloitre, Feb 14 2004 Numbers n such that 10*(n^2-1) is a square. [From Vincenzo Librandi, Aug 08 2010]
4	1,33,2177,143649, 9478657 http://oeis.org/A099370	Chebyshev's polynomial of the first kind, T(n,x), evaluated at x=33 Used in A099369. Solutions of the Pell equation x^2 - 17y^2 = 1 (x values). After initial term this sequence bisects A041024. See A121470 for corresponding y values. a(n+1)/a(n) apparently converges to (4+sqrt(17))^2. (See related comments in A088317, which this sequence also bisects.). - Rick L. Shepherd, Jul 31 2006