An exercise in MSE

Problem: Find the generating function $g(x)$ for a sequence $\{a_k\}_{k=0..oo}$ where

$$a_0 = 1$$
$$a_k = k\, a_0 + (k-1)\, a_1 + \dots 1\, a_{k-1}$$

Remark: A "*generating function*" is $g(x)$ if we write

$$g(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

(as far as the radius of convergence of $g(x)$ is nonzero).

Possibly we have even some "closed form" for $g(x)$.

Solution:

We use a matrix-based notation.

a) Assume, the coefficients a_k are in a column-vector ***A***:

$$A = column(a_0,a_1,a_2,\dots)$$

b) Assume a type of vector

$$V(x) = row(1, x, x^2, x^3, x^4,\dots)$$

Then we can formally write

c) $g(x) = V(x) * A$

where ***A*** contains so-far unknown coefficients.

By the recursive definition of ***A*** we have (with the given $a_0=1$)

$$\begin{pmatrix} 1 & . & . & . & . \\ 1 & . & . & . & . \\ 2 & 1 & . & . & . \\ 3 & 2 & 1 & . & . \\ 4 & 3 & 2 & 1 & . \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} * \begin{pmatrix} 1 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \dots \end{pmatrix} = \begin{pmatrix} 1 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \dots \end{pmatrix}$$

where we note, that if we extract the left-top element, then the lhs has a complete systematic structure when the columns are considered.

Let's call the matrix with the first element subtracted ***M*** and that matrix with only the first element as ***U*** :

$$M = \begin{pmatrix} . & . & . & . & . \\ 1 & . & . & . & . \\ 2 & 1 & . & . & . \\ 3 & 2 & 1 & . & . \\ 4 & 3 & 2 & 1 & \\ \dots & & & & \end{pmatrix} \qquad U = \begin{pmatrix} 1 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & \\ \dots & & & & \end{pmatrix}$$

So we have by the given definition for ***A***

$$A = (M + U) * A$$

which we must solve for ***A***. *(Obviously this is also an eigenvalue-problem, and thus could be solved by finding the nontrivial eigenvector of **M+U**, but we leave this aside here).*

Instead of the eigenvector-ansatz by another application of the concept of generating functions we can write:

$$V(x) * A = g(x)$$

and $$V(x) * (M+U) * A = g(x)$$

Now we consider the left part of the lhs and use associativity:

$$\begin{aligned} V(x) * (M + U) &= V(x)*M + V(x)*U \\ &= V(x)*M + V(0) \end{aligned}$$

Now the first column of $\boldsymbol{M}$ seen as coefficients have another generating function, let's call it $f(x)$. At the moment it is not yet important, what that $f(x)$ is, we can proceed completely formal.

If we look at the next columns, these are just the same, only shifted by one factor x. So by the dot-product $\boldsymbol{V}(x) * \boldsymbol{M}$ we get a resulting vector:

$$Y = V(x) * M = [f(x) , x*f(x) , x^2 *f(x) , ...]$$

where we can extract the scalar $f(x)$:

$$\begin{aligned} Y &= V(x) * M \\ &= f(x) * [1, x, x^2 , ...] \\ &= f(x) * V(x) \end{aligned}$$

which is then just a scalar multiple of $V(x)$.

Now we put that together:

$$g(x) = V(x) * A$$

and

$$\begin{aligned} g(x) &= V(x) * (M + U)* A \\ &= (V(0) + f(x)*V(x))*A \\ &= V(0)*A + f(x)*V(x)*A \\ &= a_0 + f(x)* g(x) \end{aligned}$$

Then we have

$$\begin{aligned} g(x) &= a_0 + f(x)* g(x) \\ g(x) * (1 - f(x)) &= a_0 \\ g(x) &= a_0 /(1 - f(x)) \end{aligned}$$

(which is a general solution even for a whole class of similar problems!).

It remains to determine a closed form for $f(x)$; and this is just

$$f(x) = x/(1-x)^2$$

so we get the closed form for $g(x)$ in our current problem, where also $a_0=1$ as

$$\begin{aligned} g(x) &= a_0 /(1- x/(1-x)^2) \\ &= (1-x)^2 / ((1-x)^2 - x) \\ &= (1-2x+x^2)/(1-3x+x^2) \\ &=1+x/(1-3x+x^2) \end{aligned}$$

The coefficients $\{a_k\}$ are

$$\{a_k\} = [1,1,3,8,21,55,144,...]$$

Appendix: We can also observe, that for $k>2$ we have a simple recursion

$$a_k = 3*a_{k-1} - a_{k-2}$$

but which holds **only for $k>2$** and for $k=2$ has an inconsistency if a_0 is set to $a_0<>0$.

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