Uncompleting the Gamma-function

Abstract: The taylor-series of the gamma-function has a very small range of convergence but shows a remarkable pattern of coefficients. It suggests to search for a separation into two powerseries where one has possibly a closed-form and the other (the residual) is better converging than the gamma-function itself. I discuss one reduction using a sum of zetas and one using a sum of geometric series. The second seems to be the better solution.

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Contents:

1. A Taylor-/powerseries expression for the *Γ*-("gamma")-function

The *Γ*-("gamma")-function as generalization of the factorial-function is often defined[a)] by the Euler-integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

A reference for the taylorseries is not easily obtained –for instance in the "handbook of mathematical function" [A&S] we get a lot of other representations and the taylorseries only in numerical approximation of a handful of coeffcients – and only for *1/ Γ(x)*

However some software like "Maxima" or Wolfram's mathematica site give exact expressions – but what a highly complicated structure of its coefficients do we encounter: (Laurent-expansion of the gamma-function)

Series(Gamma(1+x),x)=

$$1 - \gamma x + \frac{1}{12}\left(6\gamma^2 + \pi^2\right)x^2 + \frac{1}{6}x^3\left(-\gamma^3 - \frac{\gamma\pi^2}{2} + \psi^{(2)}(1)\right) +$$
$$\frac{1}{24}x^4\left(\gamma^4 + \gamma^2\pi^2 + \frac{3\pi^4}{20} - 4\gamma\psi^{(2)}(1)\right) +$$
$$\frac{x^5\left(-12\gamma^5 - 20\gamma^3\pi^2 - 9\gamma\pi^4 + 120\gamma^2\psi^{(2)}(1) + 20\pi^2\psi^{(2)}(1) + 12\psi^{(4)}(1)\right)}{1440}$$
$$+ O\left(x^6\right)$$

http://www.wolframalpha.com/input/?i=Series+Gamma%281%2Bx%29

The software Maxima gives the same in terms of the *zeta*-function instead of the derivatives of the *psi*; however, the *zeta* at even arguments are given a powers of π^2 and so I had to rework this below.

```
taylor(gamma(1+x),x,0,4);
```

```
1 - %gamma x

                 2      2   2
        (6 %gamma  + %pi ) x
      + ---------------------
                  12

                   3       2                         3
        (2 %gamma  + %pi  %gamma + 4 zeta(3)) x
      - ----------------------------------------
                            12

                    4          2        2                          4   4
        (20 %gamma  + 20 %pi  %gamma  + 160 zeta(3) %gamma + 3 %pi ) x
      + ---------------------------------------------------------------
                                      480
```

output by *software "Maxima" at sourceforge*,

[a)]See for instance [A&S], [NIST] or [WeissMW]

Here I needed guesses concerning the ambiguities when powers of π^2 are decomposed: for instance π^6 can arbitrarily be decomposed in terms of $\zeta(2)^3$, $\zeta(4)*\zeta(2)$ and $\zeta(6)$. I arrived at the following provisorical representation:

$$
taylor(\Gamma(1+x),x,0,5) = \begin{aligned}[t] & 1-\gamma x \\ & +\frac{\gamma^2+1\zeta(2)}{2!}x^2 \\ & -\frac{\gamma^3+3\zeta(2)\gamma+2*1\zeta(3)}{3!}x^3 \\ & +\frac{\gamma^4+6\zeta(2)\gamma^2+2*4\zeta(3)\gamma+6*\zeta(4)+3*\zeta(2)^2}{4!}x^4 \\ & -\frac{\gamma^5+10\zeta(2)\gamma^3+2*10\zeta(3)\gamma^2+6*5\zeta(4)\gamma+3*5*\zeta(2)^2\gamma+20\zeta(3)\zeta(2)+24\zeta(5)}{5!}x^5 \\ & +\dots \end{aligned}
$$

This still rather complicated expression can be refined to get a memorizable description. Let us define some concise notation for repeating patterns in the above:

$$
\begin{aligned} a) \qquad & z_1 = \gamma/1 \\ b) \qquad & z_k \underset{k>1}{=} \zeta(k)/k \\ c) \qquad & z_k^{\circ p} = z_k^p / p! \end{aligned}
$$

and let us introduce an *umbral*-like operation for multiplication of z_k with the same index k:

$$
d) \qquad z_k^{\circ p} \cdot z_k^{\circ q} = z_k^{\circ p+q} \qquad \textit{defined by} \quad \left(\frac{z_k^p}{p!}, \frac{z_k^q}{q!} \right) \Rightarrow \frac{z_k^{p+q}}{(p+q)!}
$$

Finally let us remove the alternating signs at powers of x by rewriting this for $\Gamma(1-x)$. Then we can express that powerseries for $\Gamma(1-x)$ as

$$
taylor(\Gamma(1-x),x,0) = \begin{aligned}[t] & 1 \\ & + x(z_1^{\circ 1}) \\ & + x^2(z_1^{\circ 2} + z_2) \\ & + x^3(z_1^{\circ 3} + z_2 z_1 + z_3) \\ & + x^4(z_1^{\circ 4} + z_2 z_1^{\circ 2} + z_3 z_1 + (z_4 + z_2^{\circ 2})) \\ & + x^5(z_1^{\circ 5} + z_2 z_1^{\circ 3} + z_3 z_1^{\circ 2} + (z_4 + z_2^{\circ 2})z_1 + (z_5 + z_3 z_2)) \\ & + x^6(z_1^{\circ 6} + z_2 z_1^{\circ 4} + z_3 z_1^{\circ 3} + (z_4 + z_2^{\circ 2})z_1^{\circ 2} + (z_5 + z_3 z_2)z_1 \\ & \qquad\qquad + (z_2 z_4 + z_6 + z_2^{\circ 3} + z_3^{\circ 2}) \\ & + \dots \end{aligned}
$$

This can even be made more compact with a recursive notation with formal coefficients $c_0, c_1, c_2, \dots$ to arrive at the usual powerseries-notation:

$$\Gamma(1-x) = \sum_{k=0}^{\text{inf}} c_k * x^k$$

where

$$\begin{aligned}
c_0 &= 1 \\
c_1 &= c_0 z_1 \\
c_2 &= c_1 z_1 + z_2 \\
c_3 &= c_2 z_1 + z_3 \\
c_4 &= c_3 z_1 + (z_4 + z_2^{\circ 2}) \\
c_5 &= c_4 z_1 + (z_5 + z_3 z_2) \\
c_6 &= c_5 z_1 + (z_6 + z_4 z_2 + z_3^{\circ 2} + z_2^{\circ 3}) \\
&\dots
\end{aligned}$$

The construction of the c_k-coefficients follows an additive partitioningscheme: the new term at each new index contains the references to zetas, its powers and division by factorials by the additive partitions of the index; where indexes are working additive and exponents with indexes are multiplicative.

Additional notes:

a) The reciprocal of the gamma can then be expressed by a simple pattern of change of signs:

$$\begin{aligned}
\frac{1}{\Gamma(1+x)} = 1 & \\
&+x(z_1^{\circ 1}) \\
&+x^2(z_1^{\circ 2} - z_2) \\
&+x^3(z_1^{\circ 3} - z_2 z_1 + z_3) \\
&+x^4(z_1^{\circ 4} - z_2 z_1^{\circ 2} + z_3 z_1 - (z_4 - z_2^{\circ 2})) \\
&+x^5(z_1^{\circ 5} - z_2 z_1^{\circ 3} + z_3 z_1^{\circ 2} - (z_4 - z_2^{\circ 2}) z_1 + (z_5 - z_3 z_2)) \\
&+x^6(z_1^{\circ 6} - z_2 z_1^{\circ 4} + z_3 z_1^{\circ 3} - (z_4 - z_2^{\circ 2}) z_1^{\circ 2} + (z_5 - z_3 z_2) z_1 \\
&\qquad - (z_6 - z_4 z_2 + z_2^{\circ 3} - z_3^{\circ 2}) \\
&-\dots
\end{aligned}$$

Here I do not yet have the recipe how to determine the signs in the parentheses

---

b) Reordering of summation $\Gamma(1-x)$ in

$$\begin{aligned}
\mathit{Series}\,\Gamma(1-x) = \; & 1 \\
&+ x(z_1^{\circ 1}) \\
&+ x^2(z_1^{\circ 2} + z_2) \\
&+ x^3(z_1^{\circ 3} + z_2 z_1 + z_3) \\
&+ x^4(z_1^{\circ 4} + z_2 z_1^{\circ 2} + z_3 z_1 + (z_4 + z_2^{\circ 2})) \\
&+ x^5(z_1^{\circ 5} + z_2 z_1^{\circ 3} + z_3 z_1^{\circ 2} + (z_4 + z_2^{\circ 2}) z_1 + (z_5 + z_3 z_2)) \\
&+ x^6(z_1^{\circ 6} + z_2 z_1^{\circ 4} + z_3 z_1^{\circ 3} + (z_4 + z_2^{\circ 2}) z_1^{\circ 2} + (z_5 + z_3 z_2) z_1 \\
&\qquad + (z_2 z_4 + z_6 + z_2^{\circ 3} + z_3^{\circ 2}) \\
&+ \dots
\end{aligned}$$

gives also

$$
Series\,\Gamma(1-x) = \exp(\gamma x) * (\\
1 + x^2 z_2 + x^3 z_3 \\
+x^4 (z_4 + z_2^{\circ 2}) \\
+x^5 (z_5 + z_3 z_2) \\
+x^6 (z_6 + z_4 z_2 + z_3^{\circ 2} + z_2^{\circ 3}) \\
+\ldots)
$$

c) The formal powerseries for *Γ(1+x) Γ(1–x)* has a special simple form

$$
\begin{aligned}
\Gamma(1+x)\Gamma(1-x) &= \frac{\pi\, x}{\sin(\pi\, x)} \\
&= 2\sum_{k=0}^{\text{inf}} \eta(2k)\, x^{2k}
\end{aligned}
$$

where *η(k)* is Euler's zeta *ζ(k)* with alternating signs. Numerically the coefficients tend very soon to *2* since *η(2k)->1* when *k->inf* . The connection to the sine-function was already shown by Euler. Many more different representations and functional relations are known and easily available in articles online in the internet. [a)]

[a)] For instance I like much the introductory article by P. Sebah/X. Gourdon "Introduction to the Gamma-function", 2002 (see [SeGou]), and also two nice historical reviews (see [Davis], [Gronau])

2. Decompositions of the Gamma based on numerical heuristics

2.1. Looking at the powerseries for $\Gamma(1+x)$ numerically

My initial intention with all this was to look at the powerseries for $\Gamma(1+x)$ using Pari/GP, a software which is primarily designed for numerical work. This gave the following numerical expression for the powerseries for $\Gamma(1+x)$:

```
gamma(1+x) = 1 - 0.57721566*x + 0.98905600*x^2 - 0.90747908*x^3
            + 0.98172809*x^4 - 0.98199507*x^5 + 0.99314911*x^6 - 0.99600176*x^7
            + 0.99810569*x^8 - 0.99902527*x^9 + 0.99951566*x^10 - 0.99975660*x^11
            + 0.99987827*x^12 - 0.99993906*x^13 + 0.99996952*x^14 - 0.99998475*x^15
            + …
            + 1.0000000*x^60 - 1.0000000*x^61 + 1.0000000*x^62 - 1.0000000*x^63
            + 1.0000000*x^64
            + …
```

It is much interesting, that the complicated exact definition leads to such a regular expression: the most surprising fact is that the absolute value of the coefficients approach the constant values *(+ and –) 1* very fast.

A first thought is then, that the range of convergence for this series is like for the geometric series or that for the (mercatorseries of the) logarithm $\log(1+x)$ is $|x|<1$ so we can analoguously compute $\Gamma(y)$ for $0<y<2$ by means of that powerseries.

Such small range of convergence limits the usefulness of a powerseries much, but because of the simple functional relation between the unit-intervals $\Gamma(1+x) = x\,\Gamma(x)$ that series can still be used as basis for the computation of the whole range of positive reals.

But while this characteristic limits the value of the powerseries-representation it triggers another question: what if we "reduce" the gamma-function by some other functions $g(x)$ whose power series shows a similar shape but whose values at x are known in closed form (or easily and well approximated) and if we look at the residual function $r(x) = \Gamma(1+x) - g(x)$ which is then defined by the subtraction of the two formal powerseries?

2.2. Example 1: reducing gamma by sum of zetas: $r_1(x) = \Gamma(1+x) - z(x)$ [a)]

For instance, the sequence of $\zeta(k)$ for $k=2..inf$ converges quickly to 1; so that these zetas can be used as coefficients of a power series $z(x)$ with alternating signs:

$$z(x) = \sum_{k=0}^{\text{inf}} (-1)^k \zeta(2+k) * x^k$$

Then the difference-function as formal difference of the two taylor-series-representations writing $\mathcal{T}(f(x))$ for the taylorseries-representation of a function $f(x)$

$$\mathcal{T}(r_1(x)) = \mathcal{T}(\Gamma(1+x)) - \mathcal{T}(z(x))$$

we get the residual taylorseries $r_1(x)$

```
r₁(x) = -0.64493407 + 0.62484124*x - 0.093267238*x^2 + 0.12944868*x^3
    - 0.035614975*x^4 + 0.026354208*x^5 - 0.010928242*x^6 + 0.0060066324*x^7
    - 0.0028888813*x^8 + 0.0014689210*x^9 - 0.00073043048*x^10
    + 0.00036611584*x^11 - 0.00018297682*x^12 + 0.000091524030*x^13
    …
    + 0.0000000013969831*x^29 - 0.00000000069849166*x^30
    + 0.00000000034924588*x^31
    + …
```

The above sum-of-zetas $z(x)$ gives rational values for positive integer x; we have

$z(x) = h_1(x)/x$

where $h_1(x)$ are the harmonic numbers of order 1:

$h_1(1) = 1$
$h_1(2) = 1+1/2 = 3/2$
$h_1(3) = h_1(2)+1/3 = 11/6$
...

Thus also $r_1(x)$ gives rational values at that positive integer parameters x:

$x=0 \rightarrow$	$r_1(x) = 0! - \zeta(2)$	
$x=1 \rightarrow$	$r_1(x) = 1! - 1$	$=1! - 1/2 - 1/2$
$x=2 \rightarrow$	$r_1(x) = 2! - 3/4$	$=2! - 1/2 - 1/4$
$x=3 \rightarrow$	$r_1(x) = 3! - 11/18$	$=3! - 1/2 - 1/9$
$x=4 \rightarrow$	$r_1(x) = 4! - 25/48$	$=4! - 1/2 - 1/48$

However, having the explicite symbolic description for the taylorseries of $gamma(1+x)$ as in chap 1 this is no more too interesting.

[a)] I've found this idea somewhere reading online-material from the internet. Unfortunately I do not remember the source. I'll add the source if I find it again

2.3. Reducing *Γ(1+x)* by a sum of geometric series: $r_2(1+x)= \Gamma(1+x)-g(1+x)$

A sharper approximation to the powerseries of the gamma-function gives a sum of geometric series of consecutive parameters. Let's begin with the taylor-series of *Γ(1+x)*:

```
taylor(gamma(1+x) ,x)

= 1             - 5.7722 E-1*x   + 9.8906 E-1*x^2 - 9.0748 E-1*x^3 + 9.8173 E-1*x^4
- 9.8200 E-1*x^5 + 9.9315 E-1*x^6 - 9.9600 E-1*x^7 + 9.9811 E-1*x^8 - 9.9903 E-1*x^9 + O(x^10)
```

Compare this with the powerseries –representation of *1/(1+x)*:

```
taylor( 1.0/(1+x),x)

= 1.0000 E0     - 1.0000 E0*x   + 1.0000 E0*x^2 - 1.0000 E0*x^3 + 1.0000 E0*x^4
- 1.0000 E0*x^5 + 1.0000 E0*x^6 - 1.0000 E0*x^7 + 1.0000 E0*x^8 - 1.0000 E0*x^9 + O(x^10)
```

After subtraction we get

```
taylor(gamma(1+x),x) – taylor(1.0/(1+x),x)

= 0.0             + 4.2278 E-1*x    - 1.0944 E-2*x^2  + 9.2521 E-2*x^3  - 1.8272 E-2*x^4
+ 1.8005 E-2*x^5  - 6.8509 E-3*x^6  + 3.9982 E-3*x^7  - 1.8943 E-3*x^8  + 9.7473 E-4*x^9
- 4.8434 E-4*x^10 + 2.4340 E-4*x^11 - 1.2173 E-4*x^12 + 6.0936 E-5*x^13 - 3.0482 E-5*x^14
+ 1.5247 E-5*x^15 - 7.6255 E-6*x^16 + 3.8134 E-6*x^17 - 1.9069 E-6*x^18 + 9.5353 E-7*x^19
+ O(x^20)
```

Here we observe, that we seem to get a converging sequence of coefficients, and also that the ratio of consecutive coefficients is roughly *–1/2* – and inspecting more terms indicates that we have indeed something very similar to the powerseries for *1/(1+x/2)/2*.

So we subtract that (with adapted sign):

```
taylor(gamma(1+x),x)  - taylor(1.0/(1+x),x) + taylor(1/(1+x/2)/2,x)

= 5.0000 E-1      + 1.7278 E-1*x    + 1.1406 E-1*x^2  + 3.0021 E-2* x^3  + 1.2978 E-2 *x^4
+ 2.3799 E-3*x^5  + 9.6161 E-4*x^6  + 9.1990 E-5*x^7  + 5.8819 E-5 *x^8  - 1.8301 E-6 *x^9
+ 3.9373 E-6*x^10 - 7.3813 E-7*x^11 + 3.4163 E-7*x^12 - 9.9363 E-8 *x^13 + 3.5342 E-8 *x^14
- 1.1488 E-8*x^15 + 3.8736 E-9*x^16 - 1.2867 E-9*x^17 + 4.2976 E-10*x^18 - 1.4323 E-10*x^19
+ O(x^20)
```

and arrive at a formal powerseries whose coefficients diminuish even sharper. This means, that the range of convergence for this series has increased.

But we find much more: now seemingly we have a ratio of roughly *–1/3* between consecutive coefficients; and consequently we try, whether we can repeat that type of subtraction, possibly up to infinity.

The function *g(1+x)*

Indeed it seems that we can proceed, and this allows to formulate the following ansatz:

write $\mathcal{T}(f(x))$ for the taylor-expansion of some function *f(x)* and define

$$\begin{aligned}\mathcal{T}(\mathrm{g}(1+x)) &= \mathcal{T}(\frac{1}{(1+x)})\frac{1}{0!} - \mathcal{T}(\frac{1}{1+(1+x)})\frac{1}{1!} + \mathcal{T}(\frac{1}{2+(1+x)})\frac{1}{2!} - \mathcal{T}(\frac{1}{(3+(1+x)})\frac{1}{3!}\dots \\ &= \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{k!}\mathcal{T}(\frac{1}{(k+(1+x)})\right) = \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{k!}\mathcal{T}(\frac{1}{k+(1+x)})\right)\end{aligned}$$

so that finally

$$\mathcal{T}(\mathrm{g}(1+x)) = \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{k!}\mathcal{T}(\frac{1}{k+(1+x)})\right)$$

Usually, when we look at formal powerseries, we say that we have no concern about convergence. But here we sum infinitely many formal powerseries, so the convergence for each resulting coefficient must be shown. However – here this is easy: the factorials in the denominators make sure, that the sums of the coefficients at like powers of x are convergent, and all coefficients of *g(1+x)* are thus well defined.

The *sumalt()*-procedure in *Pari/GP* gives the following result:

```
sumalt(k=0,(-1)^k*(taylor(1/(k+1+x)+O(x^20) ,x)/k!) \\ this defines taylor(g(1+x))

= 0.63212      - 0.79660*x    + 0.89121*x^2  - 0.94308*x^3  + 0.97066*x^4
- 0.98502*x^5  + 0.99241*x^6  - 0.99617*x^7  + 0.99807*x^8  - 0.99903*x^9
+ 0.99951*x^10 - 0.99976*x^11 + 0.99988*x^12 - 0.99994*x^13 + 0.99997*x^14
- 0.99998*x^15 + 0.99999*x^16 - 1.0000*x^17  + 1.0000*x^18  - 1.0000*x^19
+ O(x^20)
```

This powerseries has obviously convergence-radius smaller than *1*; but the interesting aspect is, that if we use the series of the closed-forms for evaluation, then the convergence-radius is infinite excluded the x from the set of negative integers. So for the numerical evaluation we always use:

$$\mathrm{g}(1+x) = \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{k!}\cdot\frac{1}{k+(1+x)}\right) \qquad x \notin (-1,-2,-3,\dots)$$

or some better converging expressions.

The function $r_2(1+x)$

The above coefficients are very near to that of the taylorexpansion of $\Gamma(1+x)$; if we now define the residual function $r_2(1+x)$ by subtracting the coefficients of the formal powerseries for $\Gamma(1+x)$ and $g(1+x)$:

$$\mathcal{T}(r_2(1+x)) = \mathcal{T}(\Gamma(1+x)) - \mathcal{T}(g(1+x))$$

we find a taylor-series whose coefficients decrease quickly to zero; the convergence is much faster than that in the series r_1 of the previous example.

Here is the beginning of the taylor-series for $r_2(1+x)$:

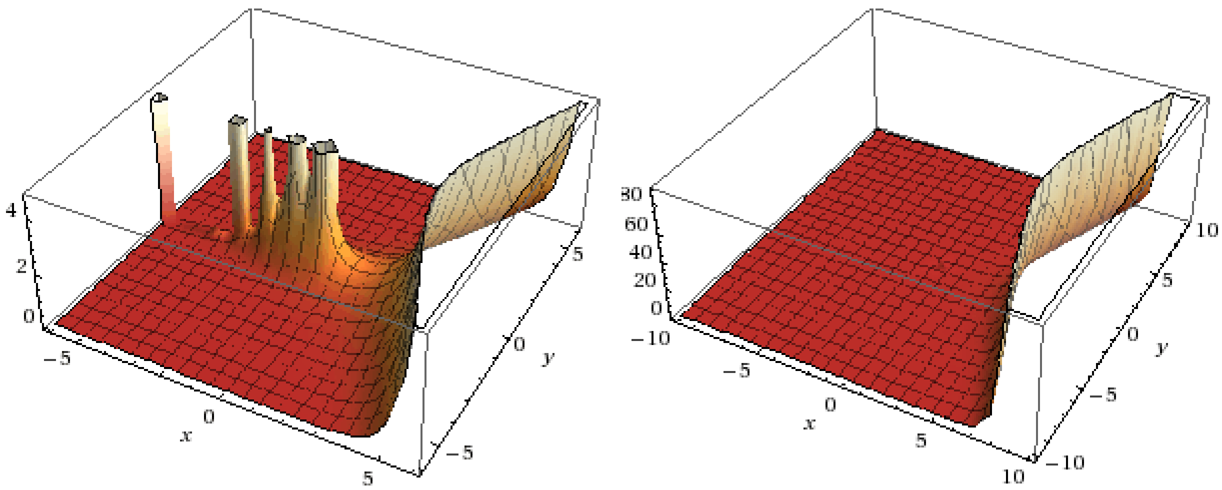
```
r₂(1+x) = 0.36787944 + 0.21938393*x + 0.097843197*x^2 + 0.035603492*x^3
        + 0.011070895*x^4 + 0.0030276112*x^5 + 0.00074265830*x^6
        + 0.00016575626*x^7 + 0.000034031395*x^8 + 0.0000064826098*x^9
        + 0.0000011537135*x^10 + 0.00000019293744*x^11 + 0.000000030464914*x^12
        + 0.0000000045607254*x^13 + 0.00000000064961995*x^14
        + …
        + 1.2448664 E-25*x^30 + 1.0172319 E-26*x^31 + O(x^32)
```

Heuristics suggest,

- that the coefficients decrease with some hypergeometric rate, thus this series seems to be entire (all poles of the gamma are captured by $g(1+x)$) and
- that the function has a zero only at $x=-infinity$.
- (obviously) the function has $r_2(1)= 1/e$
- that the function has a functional equation: $r_2(x+1) = x\cdot r_2(x) + 1/e$
- that the quotients of r_2 at consecutive x form a well known sequence $r_2(k)/r_2(k-1)$ begins $[1, 2, 5, 16, 65, 326, 1957,\ldots]$ [a)]

The following plot shows, that the poles of the gamma-function are captured by the subtraction of the $g(1+x)$-function: the function $r_2(1+x)$ seems to be very smooth in the shown area:

Plot 1.



$abs(\Gamma(1+x))$ $abs(1+r_2)$

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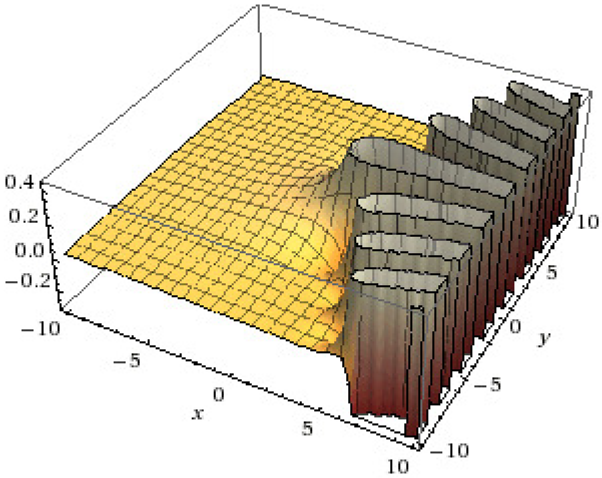
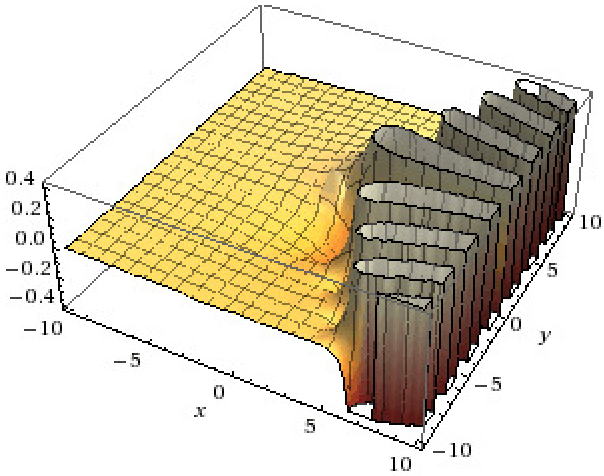
a) [OEIS] See more info in the OEIS: http://oeis.org/A000522 "*Total number of arrangements of a set with n elements: a(n) = Sum_{k=0..n} n!/k!*"

That this is not completely smooth shows the separation of the plot into that of the real and that of the imaginary part of *$r_2(1+z)$*:

Plot 2.



$Re(r_2)$

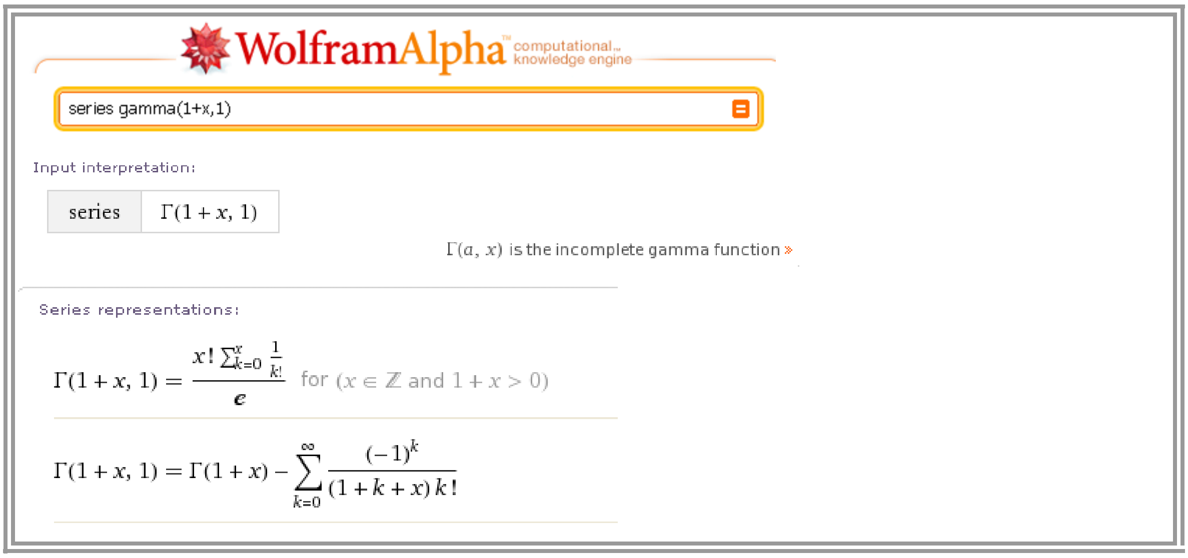


$Im(r_2)$

2.4. The function *r_2* is in fact the *"incomplete gamma"*

It was bit of surprise when I came across the wolframalpha-site and explored the whereabouts of *$\Gamma(1+x)$*, *$g(1+x)$* and *$r_2(1+x)$*. I was curious what the term "incomplete gamma function" would mean – and actually one of the series-representations for the (upper) incomplete gamma (*"gamma(1+x,1)"* in the notation of *Mathematica*) met the sum-expression which I used for *r_2*: (see Wolfram alpha "*Series(gamma(1+x,1))*1.0*")

(Incomplete gamma):



http://www.wolframalpha.com/input/?i=series+gamma%281%2Bx%2C1%29

and the last equation is identical to the definition by subtraction of the formal powerseries *$g(1+x)$* from *$\Gamma(1+x)$* as given above in the paragraph on *$g(1+x)$*:

$$r_2(1+x) \qquad = \Gamma(1+x) - g(1+x)$$

3. Further properties of the function $r_2(1+x)$ / incomplete Γ

a) approximation to $\Gamma(x)$ at positive real values

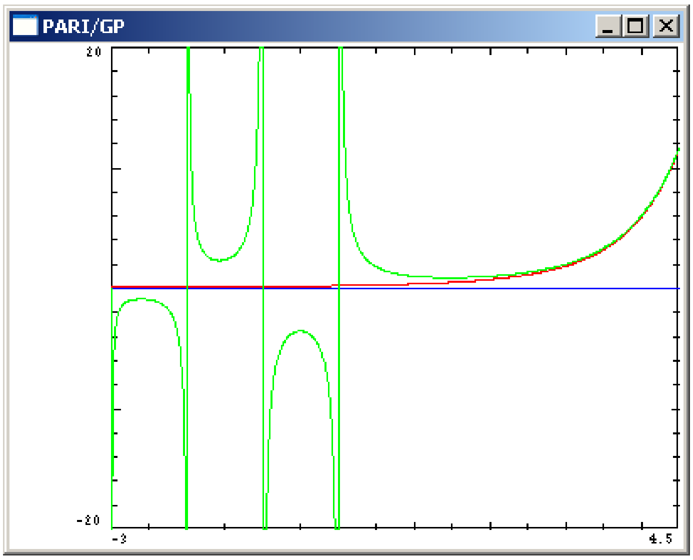
For positive larger values of x the value of the $g()$-function vanishes quickly and accordingly the new function $r_2(x)$ converges there quickly to the gamma-function.

A couple of values are (sequence of numerators see also [OEIS][1]):

$$\begin{aligned}
r(1) &= 1/e = 0*r(0) + 1/e \sim 0.37 \\
r(2) &= 2/e = 1*r(1) + 1/e \sim 0.74 \\
r(3) &= 5/e = 2*r(2) + 1/e \sim 1.84 \\
r(4) &= 16/e = 3*r(3) + 1/e \sim 5.89 \\
r(5) &= 65/e = 4*r(4) + 1/e \sim 23.91 \\
r(6) &= 326/e = 5*r(5) + 1/e \sim 119.93 \\
&\ldots
\end{aligned}$$

Here is an overlay; the green curve is $\Gamma(x)$, the red curve is $r_2(x)$

Plot 4:



green curve: gamma-function

red curve: r_2-function

b) functional equation

Surprisingly the function r_2 seems to have an interesting functional equation:

$$r_2(1+x) = x \cdot r_2(x) + 1/e$$

This is so far simply heuristic. Obviously at $x=0$ we have that $r_2(1+x) = 1/e$ as indicated by the constant term in the powerseries-expansion. (I have not analyzed the reason for this behaviour yet.)

If we formulate the family of such functional equations, distinguished only by the constant parameter, say

$$r_c(1+x) = x \cdot r_c(x) + 1/c$$
$$r_d(1+x) = x \cdot r_d(x) + 1/d$$

then all that functions are only scalings of each other:

$$c * r_c(x) = d * r_d(x)$$

All these functions have the property that they are entire and have no zero. It is remarkable, that the Euler-gamma-function is then the limit $r_m(x)$ where in the constant term $m\text{->}inf$ or $1/m=0$

[1] [OEIS] http://www.research.att.com/~njas/sequences/A000522

4. The log of *Γ* from the viewpoint of an *indefinite sum*... *(Draft-state)*

Meditating this way on the Gamma-function it comes to mind, what might come out if I develop the interpolation of the factorial the same way as I did it other functions, say iteration of polynomials and exponentiation – and see, whether this will lead to the gamma- or another function.

The iteration of the factorial is

$$(k+1)! = k! * (k+1)$$

and if we take the logarithm it converts to a sum-expression:

$$\ln((k+1)!) = \ln(1) + \ln(2) + \dots \ln(k) + \ln(k+1)$$

This looks like an expression designed for iteration... It reminds of the computation of sums of like powers which was solved by means of the bernoulli-/zeta-polynomials. So one could imagine a function

$$g(k) = \ln(k) + \ln(k+1) + \ln(k+2) + \dots$$

and then

$$\ln(\Gamma(k)) = g(1) - g(1+k)$$

Then to generalize this even to fractional k we needed some analogon to the closed form for the sum of integers, but such a closed form is not known for sums of logarithms.

Indefinite sum

However, summing with fractional bounds for the index is already a reasonably established exercise; part of that technique is the concept of "indefinite sum".

This requires an operator which performs the iteration by one step of the index. So we may ask for the operator for the indefinite sum $\ln(x)\to\ln(1+x)$, or said differently: with a unknown transfer function Φ we want:

$$\ln(1+x) = \Phi^{-1}(1+ \Phi(\ln(x)))$$

and one immediate solution for the function $\Phi(x)$ is $\Phi(x)=\exp(x)$.

So the associated operator has the composition

$$f: \ln \bullet \operatorname{add}(1) \bullet \exp$$

or $$f(x) = \ln(1+\exp(x))$$

Expressed as function $f(x)$ it has the taylorseries

$$f(x) \quad = \ln(1+\exp(x)) = \ln(2) + \frac{1}{2}\cdot\frac{x}{1!}+\frac{1}{4}\cdot\frac{x^2}{2!}-\frac{1}{8}\cdot\frac{x^4}{4!}+\frac{1}{4}\cdot\frac{x^6}{6!}-\dots+\dots$$

which seems to be

$$f(x)=\sum_{k=0}^{\inf}\frac{\eta(1-k)}{k!}x^k$$

where $\eta(n)$ is the alternating version of $\zeta(n)$.

With this we have

$$\ln(1+x) = f(\ln(x))$$

I work with Carlemanmatrices/matrixoperators which are assigned to some function; so this is the top-left aspect of ***M***, the matrix-operator for *f(x)*:

$$
M=\begin{bmatrix}
1 & 0.6931472 & 0.4804530 & 0.3330247 & 0.2308351 & 0.1600027 \\
0 & 1/2 & 0.6931472 & 0.7206795 & 0.6660493 & 0.5770877 \\
0 & 1/8 & 0.4232868 & 0.7000303 & 0.8871918 & 0.9768336 \\
0 & 0 & 1/8 & 0.3849302 & 0.7069134 & 1.016847 \\
0 & -1/192 & 0.008404717 & 0.1187342 & 0.3605346 & 0.7130570 \\
0 & 0 & -1/192 & 0.01260708 & 0.1124684 & 0.3431196 \\
0 & 1/2880 & -0.0008207311 & -0.004160259 & 0.01473128 & 0.1069713 \\
0 & 0 & 1/2880 & -0.001231097 & -0.003112185 & 0.01581641 \\
0 & -17/645120 & 0.00007740110 & 0.0002152093 & -0.001370526 & -0.002240473 \\
0 & 0 & -17/645120 & 0.0001161016 & 0.00008319646 & -0.001374487 \\
0 & 31/14515200 & -0.000007244122 & -0.00001145722 & 0.0001221699 & -0.00001946161 \\
0 & 0 & 31/14515200 & -0.00001086618 & 0.000003437238 & 0.0001119219
\end{bmatrix}
$$

If we extract the reciprocal factorials by leftmultiplication with ^dF = diagonal(0!,1!,2!,...) *we see the* η*-values at integer arguments in the second column:*

$\eta(1)=\ln(2)$, $\eta(0)=1/2$, $\eta(-1)=1/4$, $\eta(-2)=0$, ...

$$
{}^dF*M=\begin{bmatrix}
1 & 0.6931472 & 0.4804530 & 0.3330247 & 0.2308351 & 0.1600027 \\
0 & 1/2 & 0.6931472 & 0.7206795 & 0.6660493 & 0.5770877 \\
0 & 1/4 & 0.8465736 & 1.400061 & 1.774384 & 1.953667 \\
0 & 0 & 3/4 & 2.309581 & 4.241480 & 6.101082 \\
0 & -1/8 & 0.2017132 & 2.849621 & 8.652832 & 17.11337 \\
0 & 0 & -5/8 & 1.512849 & 13.49621 & 41.17436 \\
0 & 1/4 & -0.5909264 & -2.995387 & 10.60652 & 77.01936 \\
0 & 0 & 7/4 & -6.204727 & -15.68541 & 79.71473 \\
0 & -17/16 & 3.120812 & 8.677241 & -55.25961 & -90.33588 \\
0 & 0 & -153/16 & 42.13097 & 30.19033 & -498.7739 \\
0 & 31/4 & -26.28747 & -41.57597 & 443.3300 & -70.62229 \\
0 & 0 & 341/4 & -433.7432 & 137.2036 & 4467.566
\end{bmatrix}
$$

Indeed, if we use the matrixoperator ***M*** for a dotproduct, we get the expected results:

$$
\begin{aligned}
V(\ln(1))\sim * M &= [1, \ln(2), \ln(2)^2, \ln(2)^3, \ln(2)^4, ...]\sim \\
&= V(\ln(2))\sim \\[1ex]
V(\ln(2))\sim * M &= [1, \ln(3), \ln(3)^2, \ln(3)^3, \ln(3)^4, ...]\sim \\
&= V(\ln(3))\sim
\end{aligned}
$$

and in general

$$
\begin{aligned}
V(\ln(x))\sim * M &= [1, \ln(x+1), \ln(x+1)^2, \ln(x+1)^3, \ln(x+1)^4, ...]\sim \\
&= V(\ln(x+1))\sim
\end{aligned}
$$

A telescoping sum

Clearly, if we write

$V(\ln(1))$ $+ V(\ln(2))$ $+ V(\ln(3))$ $+ \ldots$ $+ V(\ln(x))$	$*(M - I) =$	$V(\ln(2))\sim - V(\ln(1))\sim$ $+ V(\ln(3))\sim - V(\ln(2))\sim$ $+ V(\ln(4))\sim - V(\ln(3))\sim$ $+ \ldots$ $+ V(\ln(x+1))\sim - V(\ln(x))\sim$	$= V(\ln(x+1))\sim - V(\ln(1))\sim$
	$=$	$[0, \ln(x+1), \ln(x+1)^2, \ldots]\sim$	

we get that

$$\left(\sum_{k=1}^{x} V(\ln(x))\right) * (M - I) = V(\ln(x+1)) - V(0)$$

$$= [0, \ln(x+1), \ln(x+1)^2, \ldots]$$

Here we ask, whether we can invert that computation:

Can we get

$$\big(V(\ln(x+1)) - V(0)\big) * S = \sum_{k=1}^{x} V(\ln(x))$$

*where **S** must somehow be the inverse of the non-invertible **(M-I)***

Unfortunately, the matrix ***(M-I)*** cannot be inverted since the first column is completely zero (and/or there is one eigenvalue zero).

Here we try one approach, which I became aware of by the *slog*-matrix of Andy Robbins. For the *slog*-matrix (for the inverse of the tetration-function) he proposes to discard simply the first column and try, whether the inverses of the remaining squared truncations "converge" with increasing size.

If we do this here, denoting the matrices where the first column is deleted and a trailing column is appended as ***M**** and ***I****, several times with increasing dimension, we seem to approximate to some definite matrix ***S***. (Since we cutted the leading zero-column in ***M-I*** we insert a leading zero-row in ***S***). The top-left segment of ***S*** approximates with increasing dimension to:

$$S=(I^*-M^*)^{-1}= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1.000000 & -0.5772157 & 0.1456317 & 0.02907109 & -0.008215338 & -0.01162685 \\ 0.5000000 & 0.5338592 & -0.6345700 & 0.1858111 & 0.06050253 & -0.003555469 \\ 0.1666667 & 0.3255788 & 0.3868589 & -0.6713043 & 0.2104926 & 0.09203596 \\ 0.04166667 & 0.1252741 & 0.2407114 & 0.3127569 & -0.6958103 & 0.2276073 \\ 0.008333333 & 0.03372565 & 0.09916203 & 0.1918968 & 0.2670350 & -0.7136514 \\ 0.001388889 & 0.006859354 & 0.02847304 & 0.08146653 & 0.1603119 & 0.2356665 \\ 0.0001984127 & 0.001172608 & 0.005923793 & 0.02452507 & 0.06891184 & 0.1381998 \\ 0.00002480159 & 0.0001831833 & 0.0009884023 & 0.005280337 & 0.02143180 & 0.05961816 \\ 0.000002755732 & 0.00002257650 & 0.0001620035 & 0.0008613236 & 0.004778981 & 0.01896129 \end{bmatrix}$$

We see two immediately interesting aspects here:

the first column (index *c=0*) gives just the coefficients for the function *exp(x)-1*

in the second column, the second entry *0.5772157* seems to be just the Euler-Gamma

In fact, we get by leftmultiplication with a vector *V(x)* (for *x* in a range of convergence)

$$* \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1.000000 & -0.5772157 & 0.1456317 & 0.02907109 & -0.008215338 \\ 0.5000000 & 0.5338592 & -0.6345700 & 0.1858111 & 0.06050253 \\ 0.1666667 & 0.3255788 & 0.3868589 & -0.6713043 & 0.2104926 \\ 0.04166667 & 0.1252741 & 0.2407114 & 0.3127569 & -0.6958103 \\ 0.008333333 & 0.03372565 & 0.09916203 & 0.1918968 & 0.2670350 \\ 0.001388889 & 0.006859354 & 0.02847304 & 0.08146653 & 0.1603119 \\ 0.0001984127 & 0.001172608 & 0.005923793 & 0.02452507 & 0.06891184 \\ 0.00002480159 & 0.0001831833 & 0.0009884023 & 0.005280337 & 0.02143180 \\ 0.000002755732 & 0.00002257650 & 0.0001620035 & 0.0008613236 & 0.004778981 \end{bmatrix}$$

$$V(\ln(3))\sim - V(\ln(2))\sim = [\; 1.000000 \quad 0.6931472 \quad 0.4804530 \quad 0.3330247 \quad 0.2308351 \;]$$

$$= [\; 1 \quad \ln(2) \quad \ln(2)\hat{}2 \quad \ln(2)\hat{}3 \quad \ldots \;]$$

	* *S*
V(ln(5))~-V(ln(1))~	= [4.000000 3.178054 3.609214 4.323191 5.380922]

4 = ln(1)^0	ln(1)	ln(1)^2	ln(1)^3	...
+ln(2)^0	+ln(2)	+ln(2)^2	+ln(2)^3	...
+ln(3)^0	+ln(3)	+ln(3)^2	+ln(3)^3	...
= +ln(4)^0	+ln(4)	+ln(4)^2	+ln(4)^3	...

Heuristically we find, that the coefficients in the *2nd* column (columnindex *c=1*) equal just that of the powerseries of

$g_1(x) = \ln(\Gamma(\exp(x)))$

as given by the Pari/GP-call of *lngamma(exp(x))*

Now to construct the analogy to the Hurwitz-zeta-differences, we assume a value Λ_p for the infinite sum of p'th powers of logarithms of consecutive bases (which we shall actually try to determine later):

$$\Lambda_p = \ln(1)^p + \ln(2)^p + \ln(3)^p + \dots = \sum_{k=1}^{\inf} \ln(k)^p$$

If we can define such values meaningfully, then we can also define the functions

$$T_p(n) = \Lambda_p - S_p(n) = \ln(n+1)^p + \ln(n+2)^p + \ln(n+3)^p + \dots$$

from where then

$$T_p(m) - T_p(n) = \ln(m+1)^p + \ln(m+2)^p + \dots + \ln(n)^p$$

and practically

$$T_p(0) - T_p(n) = \ln(1)^p + \ln(2)^p + \dots + \ln(n)^p$$
$$= S_p(n)$$

Because the functions $T_p(n)$ and $S_p(n)$ differ only by the constant and the sign, the power series for $T_p(n)$ has the same coefficients as we have already determined for $S_p(n)$, only the signs are inverted and the constant is appended. So we get the matrix ***T***:

T =

LAMB_0	LAMB_1	LAMB_2	LAMB_3	LAMB_4	LAMB_5
-1.000000	0.5772157	-0.1456317	-0.02907109	0.008215338	0.01162685
-0.5000000	-0.5338592	0.6345700	-0.1858111	-0.06050253	0.003555469
-0.1666667	-0.3255788	-0.3868589	0.6713043	-0.2104926	-0.09203596
-0.04166667	-0.1252741	-0.2407114	-0.3127569	0.6958103	-0.2276073
-0.008333333	-0.03372565	-0.09916203	-0.1918968	-0.2670350	0.7136514
-0.001388889	-0.006859354	-0.02847304	-0.08146653	-0.1603119	-0.2356665
-0.0001984127	-0.001172608	-0.005923793	-0.02452507	-0.06891184	-0.1381998

Now to have

$$T_1(0) = V(0)\sim * \boldsymbol{T}[,1]$$
$$= 1 * \Lambda_1 + 0*0.57721\dots + 0 * -0.5338592 + 0 * \dots.$$
$$= \ln(1) + \ln(2) + \ln(3) + \dots$$

we need that Λ_1 equals that sum of logarithms.

So: how can we assign some value to the Λ_p ?

There are at least two ways to achieve this.

- ***Divergent sumnmation/Ramanujan summation***

  According to the idea of Ramanujan-summation we use the power series of $ln(1+x)$ and replace each power of x^k by the according zeta-value $\zeta(-k)=1^k+2^k+3^k+\ldots$ as the representation of the sum of all $x \in N$. Also we need some integral. This reads then formally

$$\Lambda_1 = \int_1^0 \ln(t)dt + \sum_{k=1}^{\inf} \frac{(-1)^{k-1}\zeta(-k)}{k} \quad (P)$$

  The integral evaluates to $c_1=1$, and for the divergent sum (let's denote it by s_1) we must apply a summationmethod stronger than, for instance, Eulersummation. I've a specially taylored method (however unproven) according to Nörlund means which gives for the sum s_1 approximately -0.0810614667953. The value of this

$$\begin{aligned}\Lambda_1 &= c_1+s_1\\ &\sim 1 - -0.081061\\ &\sim 0.918938533205\end{aligned}$$

- ***Derivatives of the zeta***

There is another representation of the sums of logarithms, even of powers of logarithms and this are the derivatives of the zeta at zero-argument. The Ramanujan-sum agrees with the sum of the logarithms as defined by that derivatives; for $p=1$ we have even a closed form in terms of π and logarithm only:

$$\Lambda_1 = -\zeta'(0) = 0.5 * \ln(2\pi) \sim 0.91893853320467274178032973640 6$$

The closed forms for the higher derivatives become much complicated; but expressing the zeta as a power series/Laurent series with the help of the Stieltjes-constants allows a very simple computation of that derivatives:

$$\zeta(z) = \frac{1}{z-1} + \sum_{k=0}^{\infty} (-1)^k \frac{s_k}{k!}(z-1)^k$$

and the p'th derivatives are

$$\begin{aligned}\zeta^{(p)}(z) &= \left(\frac{1}{z-1}\right)^{(p)} + \sum_{k=0}^{\infty} (-1)^k\, p!\binom{k}{p}\frac{s_k}{k!}(z-1)^{k-p}\\ &= (-1)^p \frac{p!}{(z-1)^{p+1}} + \sum_{k=p}^{\infty}(-1)^k \frac{p!k!}{k!\,p!}\frac{s_k}{(k-p)!}(z-1)^{k-p}\\ &= -\frac{p!}{(1-z)^{p+1}} + (-1)^p \sum_{k=0}^{\infty} s_{p+k}\frac{(1-z)^k}{k!}\end{aligned}$$

The representation of the sums of powers of logarithms by derivatives of the zeta is

$$\zeta^{(p)}(z) = \sum_{k=1}^{\infty} \frac{(-\ln(k))^p}{k^z}$$

Setting $z=0$ we get the *"sum of like powers of logarithms"* as

$$\sum_{k=1}^{\infty} \ln(k)^p = (-1)^p \zeta^{(p)}(0)$$

With this the power series (which uses the Stieltjes-constants) reduces to

$$\zeta^{(p)}(0) = -\frac{p!}{(1-0)^{p+1}} + (-1)^p \sum_{k=0}^{\infty} s_{p+k} \frac{(1-0)^k}{k!}$$
$$= -p! + (-1)^p \sum_{k=0}^{\infty} \frac{s_{p+k}}{k!}$$

and

$$\Lambda_p = \sum_{k=1}^{\infty} \ln(k)^p$$
$$= (-1)^{p-1} p! + \sum_{k=0}^{\infty} \frac{s_{p+k}}{k!}$$

Here the sum converges very fast and allows many digits accuracy with few terms depending on the accuracy of the Stieltjes-numbers .

The first few values are:

$\Lambda_1 \sim \quad 1 - 0.0810614667 \quad \sim \quad 0.91893853321$
$\Lambda_2 \sim - \quad 2 - 0.0063564559 \quad \sim - \quad 2.00635645591$
$\Lambda_3 \sim \quad 6 + 0.0047111669 \quad \sim \quad 6.00471116686$
$\Lambda_4 \sim -24 + 0.0028968119 \quad \sim -23.9971031880$
...

Finally, it may be of interest to look at the sum of all Λ_1 Λ_2, Λ_3, ...

The sum of all c_p is

$$\sum_{p=1}^{\infty} c_p = \int_1^0 \ln(t)^1 + \ln(t)^2 + \ldots dt$$
$$= \int_1^0 \frac{\ln(t)}{1-\ln(t)} dt$$
$$\approx 0.403652637677\ldots$$
$$\approx 1 - 0.596347362323\ldots$$

where the *0.59634…* is the <u>*Gompertz*</u>-constant and represents the divergent series

0.59634… = 0! – 1! + 2! – 3! + … - …

The sum of the fractional parts seems to be divergent. With a Noerlund-summation I found an approximation at about

-0.080906… [2]

Gottfried Helms, 8'2012 (first version 08'2010)

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[2] In my previous version I documented the value to more precision -0.0809057233444163259461536334873... but my different summation routines might be not completely compatible with the number of estimated correct digits. I'll update the value later after external confirmations.

5. References/Links *Draft-state)*

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Projectindex

[Helms] "Mathematical Miniatures"
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