Mathematical Miniatures

# A dream of a (number-) sequence

## *A factorization pxp(x) = (1+ax) (1+bx²) (1+cx³)... for the exponential-series exp(x) exhibits a beautiful sequence*

*Abstract: Some musings with the exponential-series and an attempt to express the series in terms of polynomial factors led to investigations on the denominators of the coefficients a,b,c,.. . A sequence with a beautiful inner symmetry and an eccentric growthrate occurred.*

*The treatise refers also to the article in seqfan-mailinglist "a dream of a series" of 4.6.2008*

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# 1. Can the exponential-series be expressed by an infinite product?

## 1.1. The problem

Some years ago I fiddled already with this question: *can the exponential-series possibly be expressed by an infinite product and what are the properties and whereabouts, then*?

$$exp(x) = 1 + x + x^2/2! + x^3/3! + \dots$$
$$pxp(x) = (1 + ax)(1 + bx^2)(1 + cx^3)(1+dx^4)\dots$$

*q: can exp(x) be expressed by a suitable version of pxp(x) ?*

I didn't go deeply into it and left the subject – until recently when some correspondent stumbled onto the same question and asked me for a comment. This reminded me of that old material and I gave it a new consideration. And –with the background of years of experiences with the OEIS-sequence-databases[1] – there occured a beautiful sequence with an exciting symmetry and an eccentric behave when the denominators of the coefficients *a,b,c…* were analyzed separately.

It has some similarity with a multiplicative sequence[2], but also involves powers of primefactors in a very symmetric way.

$$p^q * q^p$$

For instance, at indexes $n=p*q$, where $p$ and $q$ are distinct primefactors, the denominator has the very nice symmetric exponential form $p^q * q^p$. It looks, as if $p$ and $q$ lovingly carry the weight of each other … :-) .

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***Context/further reading***

> The idea of conversion of a powerseries into an infinite product-representation is not new and not only a game in recreational mathematics; according to H. Gingold/A. Knopfmacher in "*Analytic Properties of Power Product Expansions*" (1995) this idea *"appears to have first been studied in the 1930's by Ritt [R] and Feld [F]. More recently, Knopfmacher and Lucht [KL] (...)"*. The Gingold/Knopfmacher-article provides also a short list of further literature which I provide here at the references-section for the interested reader.
>
> In the seqfan-mailing list (associated with the ***O***nline ***E***ncyclopedia of ***I***nteger ***S***equences (OEIS)) there was some exchange on this idea; looking through the archives I find my own first article on that subject ("*A dream of a Series*", Jun2008) and another one initiated by Neil Fernandez ("*polynomial-to-product transform*", Nov 2008), in which H. Gould pointed to the Gingold/Gould/Mays-article "*Power Product Expansions*" in Utilitas Mathematics (34, 1998, 143-161).
>
> I did not yet incorporate any of that literature in this article because I got aware of that research only these days when I browsed the seqfan-archive for possible earlier discussions of this. Also in this article the main focus are the properties of the specific sequence of coefficients as found in *pxp(x)*.

[1] see two sequences in [OEIS] which refer to the same notion, called *pxp(x)* here
[2] see for instance "multiplicative function" in wikipedia [wiki-m]

## 1.2. Two ways to find a solution *pxp(x)*

The conversion of *exp(x)* into *pxp(x)* is solvable by two simple ways.

a) iteratively dividing *exp(x)* by the obvious factors to get *pxp(x)*:

| | | |
|---|---|---|
| $E_0(x) = exp(x)$ | $= \underline{1 + x} + x^2/2! + x^3/3!$ | $F_1(x) = (1 + x)$ |
| $E_1(x) = E_0(x) / F_1(x)$ | $= \underline{1 + 1/2x^2} + ...$ | $F_2(x) = (1 + 1/2\, x^2 )$ |
| $E_2(x) = E_1(x) / F_2(x)$ | $= \underline{1 - 1/3x^3} + ...$ | $F_3(x) = (1 - 1/3\, x^3 )$ |
| $E_3(x) = E_2(x) / F_3(x)$ | $= \underline{1 + 3/8x^4} + ...$ | $F_4(x) = (1 + 3/8\, x^4 )$ |

...

$pxp(x) = F_1(x)*F_2(x)*F_3(x)* F_4(x)*...$

A ***Pari/GP***-program can be used as follows:

```
\ps 16
coeffs = vector(16);
E_tmp = exp(x);
{ for(k=1,#coeffs,
         C = polcoeff(E_tmp,k);
         coeffs[k] = C;
         E_tmp = E_tmp/(1 + C * x^k);
      ); }
print(coeffs);
pxp(x) = prod(k=1,#coeffs, 1+coeffs[k]*x^k)
```

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b) using the logarithm-series, expanding and comparing coefficients:

$log(exp(x)) = x$

$log(pxp(x)) = log(1+ax) + log(1+bx^2) + log(1+cx^3) + ...$

Expand the logarithm-series and collect coefficients of like powers of $x$:

$$
\begin{array}{lllllll}
 & = ax & - a^2x^2/2 & + a^3x^3/3 & - a^4x^4/4 & + a^5x^5/5 & - a^6x^6/6 + ... \\
 & & + b\,x^2 & & - b^2x^4/2 & & + b^3x^6/3 \\
 & & & + c\,x^3 & & & - c^2x^6/2 \\
 & ... & & & & & \\
 & \text{-----} & & & & & \\
=log(exp(x)) & = 1x + & 0 & + 0 & + 0 & + 0 & + 0
\end{array}
$$

and solve for the coefficients. Then immediately it follows

$$a = 1,\ b = 1/2,\ c = -1/3,\ d = 3/8,\ etc...$$

*(See a better table below.)*

A program for ***Pari/GP***

```
{ prodcoeffs(f,dim=64) = local(logf,logfcoeffs,prdfcoeffs,a,lc);
     logf = log(f)
     logfcoeffs = vectorv(dim,k,polcoeff(lf,k)) ;
     prdfcoeffs = vectorv(dim);
     for(c=1,dim-1,
          prdfcoeffs[1+c]= a =logfcoeffs[1+c];
          lc=0;forstep(k=c,dim-1,c,lc++;logfcoeffs[1+k]-= -(-a)^lc/lc );
     );
 return(prdfcoeffs);}
 print (prodcoeffs(exp(x)))
```

Both methods give the same sequence of coefficients *a,b,c,...*

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### 1.2.1. Inspection of the computation-scheme using the logarithmic representation

The computation-scheme using the logarithm gives immediate insight in the composition of the coefficients, their denominators and numerators.

***Table 1***: expand the logarithm-series and order coefficients of like powers of *x* in columns. The columnsums must agree with the coefficients in the first row. Note, that the *log(f(x)) = log(exp(x)) = 0 + 1x + 0x + …* so all entries except the first entry are zero in the first row.

| | *x | *$x^2$ | *$x^3$ | *$x^4$ | *$x^5$ | *$x^6$ | *$x^7$ | *$x^8$ | *$x^9$ | *$x^{10}$ | … |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **log(f(x))=** | **1** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | |
| log(1+ax) | a | $-a^2/2$ | $a^3/3$ | $-a^4/4$ | $a^5/5$ | $-a^6/6$ | $a^7/7$ | $-a^8/8$ | $a^9/9$ | $-a^{10}/10$ | … |
| +log(1+b$x^2$) | | b | | $-b^2/2$ | | $b^3/3$ | | $-b^4/4$ | | | … |
| +log(1+c$x^3$) | | | c | | | $-c^2/2$ | | | $c^3/3$ | | … |
| +log(1+d$x^4$) | | | | d | | | | $-d^2/2$ | | | … |
| +log(1+e$x^5$) | | | | | e | | | | | $-e^2/2$ | … |
| +log(1+f$x^6$) | | | | | | f | | | | | … |
| +log(1+g$x^7$) | | | | | | | g | | | | … |
| +log(1+h$x^8$) | | | | | | | | h | | | … |
| +log(1+i$x^9$) | | | | | | | | | i | | … |
| +log(1+j$x^{10}$) | | | | | | | | | | j | … |
| +… | | | | | | | | | | | |

The coefficients can be found by a simple recursive process.

We begin determining *a=1* and insert in the second row *1* for *a*. Then for *b* follows it must equal $b=a^2/2 =1/2$ to make the second columnsum equal zero. Then we can insert the powers of *b* in the $2^{th}$, $4^{th}$, $6^{th}$, $8^{th}$,…,$(2n)^{th}$ column of the second row.

For *c* we can simply do the same: $c=-a^3/3 = -1/3$ and fill out the row with powers of *c*.

Then *d* is dependent on *a* and *b*, so we get $d=a^4/4 + b^2/2 = 1/4 +1/8 = 3/8$ .

What we got so far is:

***Table 2:***

| | *x | *$x^2$ | *$x^3$ | *$x^4$ | *$x^5$ | *$x^6$ | *$x^7$ | *$x^8$ | *$x^9$ | *$x^{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| =log(f(x)) | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| +log(1+ax) | 1 | -1/2 | 1/3 | -1/4 | 1/5 | -1/6 | 1/7 | -1/8 | 1/9 | -1/10 |
| +log(1+b$x^2$) | | 1/2 | | $-1/2^2/2$ | | $1/2^3/3$ | | $-1/2^4/4$ | | |
| +log(1+c$x^3$) | | | -1/3 | | | $-(-1/3)^2/2$ | | | $-(-1/3)^3/3$ | |
| +log(1+d$x^4$) | | | | 3/8 | | | | $-(3/8)^2/2$ | | |
| + | | | | | | | | | | |

It is easy to see how this continues. Obviously the composition of a coefficient at one index *n* depends on the ***divisors of n*** and their (previously determined) values.

| | *x | *$x^2$ | *$x^3$ | *$x^4$ | *$x^5$ | *$x^6$ | *$x^7$ | *$x^8$ | *$x^9$ | *$x^{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| =log(f(x)) | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| +log(1+ax) | **1** | -1/2 | 1/3 | -1/4 | 1/5 | -1/6 | 1/7 | -1/8 | 1/9 | -1/10 |
| +log(1+b$x^2$) | | **1/2** | | -1/8 | | 1/24 | | -1/64 | | |
| +log(1+c$x^3$) | | | -1/3 | | | -1/18 | | | 1/81 | |
| +log(1+d$x^4$) | | | | **3/8** | | | | -9/128 | | |
| + | | | | | | | | | | |

For prime *n* we have an especially simple description, and for *n* being a power of a prime the description of the coefficient is still easily deduced.

# 2. Some numerical properties

## 2.1. coefficients at prime- and prime-power indexes

Let's have a look at the product-formula *pxp(x)* with known coefficients inserted:

$$\begin{aligned} pxp(x) = {} & (1 + x) \\ & *(1 + 1/2\, x^2) \\ & *(1 - 1/3\, x^3) \quad *(1 + 3/8\, x^4) \\ & *(1 - 1/5\, x^5) \quad *(1 + 13/72\, x^6) \\ & *(1 - 1/7\, x^7) \quad *(1 + 27/128\, x^8)*(1 - 8/81\, x^9)*(1 + 91/800\, x^{10}) \\ & *(1 - 1/11\, x^{11})*(1 + 1213/13824\, x^{12}) \\ & *(1 - 1/13\, x^{13})* \ldots \\ & \ldots \end{aligned}$$

An immediate observation: at odd prime-indexes $p$ the coefficient is just $-1/p$ .

Let's denote an index with $n$ and the coefficient at that index $a(n)$ as used in the ***O**nline **E**ncyclopedia of **I**nteger **S**equences* (OEIS), and some prime with the letter $p$, $q$ or $r$. Then

$$\begin{aligned} &\text{num } a(n)_{n=p>2} = -1 && \text{// numerator} \\ &\text{den } a(n)_{n=p>2} = p && \text{// denominator} \end{aligned}$$

Using table 1 this is obvious from the fact, that a prime index has no divisors except $1$ and itself.

A deeper look at the denominators exhibits more interesting details. For another instance: if $n$ is a prime-power $n=p^k$ then we find cyclotomic expressions in the exponents of $p$, or, with another notion, the so-called "$q$-analogues" of powers of $p$.

$$\text{den } a(p^k) = p^{p^{"k}}$$
$$\text{where } p^{"k} := (p^k - 1)/(p - 1) \text{ the q-analogue[3] of } p^k$$

So for $n=4=2^2$, $n=8=2^3$, $n=9=3^2$, $n=343=7^3$ we have

$$\begin{aligned} &\text{den } a(2^2) = 8 && = 2^{2^{"2}} = 2^{(2^2-1)/(2-1)} = 2^3 \\ &\text{den } a(2^3) = 128 && = 2^{2^{"3}} = 2^{(2^3-1)/(2-1)} = 2^7 \\ &\text{den } a(3^2) = 81 && = 3^{3^{"2}} = 3^{(3^2-1)/(3-1)} = 3^4 \\ &\text{den } a(7^3) = xxx && = 7^{7^{"3}} = 7^{(7^3-1)/(7-1)} = 7^{57} \\ & && = 1481113296616977741464105532513750734030421355207 \end{aligned}$$

[3] see "q-analogue" in wikipedia [wiki-q]

## 2.2. Symmetry and eccentric growth: coefficients at composite indexes

### 2.2.1. Index *n* is composite but squarefree

The property, which was initially the most astonishing for me, is the most beautiful and intriguing symmetry occuring at indexes of composite squarefree integers.

We have at an index *n* as product of two different primes $n=p*q$:

$$\text{den } a(pq) = p^q * q^p = (p^{1/p}*q^{1/q})^n$$

*Example:*

$$\text{den } a(14) = 2^7 * 7^2 = 128 * 49 = 6272$$

If we have three primefactors in *n* : $n=p*q*r$ then

$$\text{den } a(pqr) = p^{qr} * q^{pr} * r^{pq} = p^{n/p}\, q^{n/q}\, r^{n/r} = (p^{1/p}\, q^{1/q}\, r^{1/r})^n$$

*Example:*

$$\text{den } a(30) = 2^{3*5} * 3^{2*5} * 5^{2*3} = 30233088000000$$

The sequence strongly resembles multiplicativity, but one could say, it is some "overdriven" multiplicativity.

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To have a closer look at these coefficients I show a table. The first few coefficients are

| Index n | coefficient a(n) numerator / denominator | com fac | Index n | coefficient a(n) numerator / denominator | com fac |
|---|---|---|---|---|---|
| 1: | 1 | | 21: | -34943 / 750141 | |
| 2: | 1 / 2 | | 22: | 12277 / 247808 | |
| 3: | -1 / 3 | | 23: | -1 / 23 | |
| 4: | 3 / 8 | | 24: | 593806671 / 13759414272 | 9 |
| 5: | -1 / 5 | | 25: | -624 / 15625 | |
| 6: | 13 / 72 | | 26: | 57331 / 1384448 | |
| 7: | -1 / 7 | | 27: | -58528 / 1594323 | |
| 8: | 27 / 128 | | 28: | 195948483 / 5035261952 | |
| 9: | -8 / 81 | | 29: | -1 / 29 | |
| 10: | 91 / 800 | | 30: | 1052424027703 / 30233088000000 | |
| 11: | -1 / 11 | | 31: | -1 / 31 | |
| 12: | 3639 / 41472 | 3 | 32: | 77010795 / 2147483648 | |
| 13: | -1 / 13 | | 33: | -7085759 / 235782657 | |
| 14: | 505 / 6272 | | 34: | 1179631 / 37879808 | |
| 15: | -1919 / 30375 | | 35: | -37497599 / 1313046875 | |
| 16: | 2955 / 32768 | | 36: | 169147809135192 / 5777633090469888 | 24 |
| 17: | -1 / 17 | | 37: | -1 / 37 | |
| 18: | 196456 / 3359232 | 8 | 38: | 5242861 / 189267968 | |
| 19: | -1 / 19 | | 39: | -89281919 / 3502727631 | |
| 20: | 1136313 / 20480000 | | 40: | 355723139681937 / 13421772800000000 | |

However, since we have rational numbers and ***Pari/GP*** displays values in the most cancelled version, at indexes *n* of more complexity (where prime-powers and multiple primes are involved) numerators and denominators may have common factors, which are then cancelled in the display and the regularity seems to be broken. However, I found one definition in ***OEIS*** (see [**OEIS**]) for the composition of a sequence which agrees with the denominators here, except that it seems to contain (the) common factors. Such assumed common factors are red marked in the table above; ***Pari/GP*** gives the reduced fraction. For instance at $n=12$ we get the above expected coefficient *3639/41472* cancelled by *3* as *1213 / 13824* which is shown by ***Pari/GP***, when computed with the logarithmic scheme.

---

### 2.2.2. Index *n* is composite and not squarefree

The following expressions are based on heuristics only, not yet rigorously derived from the evaluation scheme using the logarithmic representation.

Denote as for the previous composite, but squarefree case, *n* by its primefactorization (use again *3* sample-primefactors *p,q,r* and now their exponents *a,b,c* here)

$$n = p^a q^b r^c \dots$$

Then, with some *A,B,C* we find heuristically

$$\text{den } a(n) = p^A * q^B * r^C$$
$$A = p^{"a} * q^b * r^c \qquad \text{// again } p^{"a} \text{ is the q-analogue of the } p^a\text{, see (1.1)}$$
$$B = p^a * q^{"b} * r^c$$
$$C = p^a * q^b * r^{"c}$$

This can be simplified to get:

$$\begin{aligned} A &= p^{"a}/p^a * n \\ &= (p^a - 1)/(p-1)/p^a * n \\ &= (1 - 1/p^a)/(p-1) * n \\ &= -(p^{-a} - 1)/(p-1) * n \\ &= -p^{"-a} * n \end{aligned}$$

*// analoguously for each of the primefactors*

$$\text{den } a(n) = p^{-p^{"-a} * n} q^{-q^{"-b} * n} r^{-r^{"-c} * n} = \left(\frac{1}{p^{p^{"-a}}} \frac{1}{q^{q^{"-b}}} \frac{1}{r^{r^{"-c}}}\right)^n$$

Here (after cancellations) the empirical denominators of the coefficients are at least divisors of the values, which we expect by the analytical description.

---

## 2.3. Example-computations for coefficients at simple-structured indexes

I didn't derive the full description for the denominators yet; especially the composition of the numerators seems to be too complex to have a closed form. But the following scheme may indicate, how the composition of coefficients can recursively be determined with a saving of effort.

Assume we want to compute the composition of a coefficient at an index *n* where *n* is a prime-power. Say, $n=3$, $n=3^2$, $n=3^3$, and so on. Then a recursive scheme, (shorter than the complete representation of the sums of logarithm series because only few positions are relevant) is

| | coefficient | denominators of sum terms | max of den. | numerator |
|---|---|---|---|---|
| $n=3^1$ | $a(3) = -(1^3/3)$ | $3^1$ | | |
| $n=3^2$ | $a(9) = -(1^9/9 + a(3)^3/3)$ | $3^2, 3^{3+1}$ | $3^4$ | $(3^2 - 1)$ |
| $n=3^3$ | $a(27) = -(1^{27}/27 + a(3)^9/9 + a(9)^3/3)$ | $3^3, 3^{9+2}, 3^{4*3+1}$ | $3^{13}$ | $3^{10} - (3^2 - 1)^3 \dots$ |
| $n=3^4$ | $a(81) = -(1^{81}/81 + a(3)^{27}/27 + a(9)^9/9 + a(27)^3/3)$ | $3^4, 3^{27+3}, 3^{4*9+2}, 3^{13*3+1}$ | $3^{40}$ | ... |
| ... | ... | ... | ... | ... |

In more generality for an index $n$, which is the $k$'th power of a prime $p$, the denominator is

| | coefficient $s_k = a(n)$ where $n=p^k$ | denominators of sum terms | max of den. |
|---|---|---|---|
| $n=p^1$ | $s_1=-(1/p)\ (1)$ | $p^1$ | $p^{p^{"1}}$ |
| $n=p^2$ | $s_2=-(1/p^2)(1 + p\, s_1^{\,p})$ | $p^2, p^{p+1}$ | $p^{p^{"2}}$ |
| $n=p^3$ | $s_3=-(1/p^3)(1 + p\, s_1^{\,p^2} + p^2 s_2^{\,p})$ | $p^3, p^{p^2+2}, p^{p^{"2}*p+1}$ | $p^{p^{"3}}$ |
| $n=p^4$ | $s_4=-(1/p^4)(1 + p\, s_1^{\,p^3} + p^2 s_2^{\,p^2} + p^3 s_3^{\,p})$ | $p^4, p^{p^3+3}, p^{p^{"2}*p^2+2}, p^{p^{"3}*p+1}$ | $p^{p^{"4}}$ |
| | ... | | |

*(numerators are too complicated and not displayed here)*
*(in matrix-notation: $S = -{}^dV(1/p) * [s_k] * V(p)$)*

For some higher composite indexes follow the primefactor-decomposition for the denominator.

| Index n | den a(n) formula | details for exponents |
|---|---|---|
| $180 = 2^2 * 3^2 * 5$ | $2^{135}*3^{80}*5^{36}$ | $135 = 2^{"2}*3^2*5 = 3*9*5$<br>$80 = 2^2*3^{"2}*5 = 4*4*5$<br>$36 = 2^2*3^2*5^{"1} = 4*9$ |
| $900 = 2^2 * 3^2 * 5^2$ | $2^{675} * 3^{400} * 5^{216}$<br>$2^7*3^4$ cancelled(*1) | $675 = 2^{"2}*3^2*5^2 = 3*225$<br>$400 = 2^2*3^{"2}*5^2 = 4*4*25$<br>$216 = 2^2*3^2*5^{"2} = 4*9*6$ |
| $1800 = 2^3 * 3^2 * 5^2$ | $2^{1575} * 3^{800} * 5^{432}$<br>$2^7*3^2$ cancelled(*1) | $1575 = 2^{"3}*3^2*5^2 = 7*225$<br>$800 = 2^3*3^{"2}*5^2 = 8*4*25$<br>$432 = 2^3*3^2*5^{"2} = 8*9*6$ |

(*1) *primefactors to indicated power empirically seem to be cancelled against numerator due to rational arithmetic in Pari/GP*

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## 2.4. An external definition-formula for the denominators? (1)

One definition can be found in [OEIS] and its implementation in ***Pari/GP***-code is

```
\\ OEIS(A067911) : a(n) = Product_{ d divides n } d^phi(n/d)
\\                  (Vladeta Jovovich, Mar 2004)
\\ Pari/GP
den_a(n)=local(res);
         res=1;        \\ fordiv: d running over all divisors of n
         fordiv(n, d, res = res * d^eulerphi (n/d));
         return(res)
```

This is at a first glance nearly identical to the procedure using the logarithmic scheme, except that I did not yet translate the Euler-phi-function appropriately. So except for the cancellation of common factors at high-composite indexes this procedure matches my result up to index $n=2000$. The empirical values, computed with one of the methods here, have smaller compositions than expected by the analytic formula for the denominator only if the index $n$ contains more than one prime-power. This is very likely due to cancellations with the numerator.

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(1) See two more descriptions, matching my approach here, in [OEIS]

# 3. Additional considerations

## 3.1. Usability/limitation of the function *pxp(x)*

The function *pxp(x)* is extremely bad suited for practical-use: apparently it has the very limited radius of convergence of $|x|<1$ and partial evaluation needs extremely many terms to get approximation to a handful of decimals of the known value by the *exp(x)*-function.

The coefficients themselves decrease very slowly (although on composite indexes *n* extremely high values occur in denominators compared to *n*). Here is a table of the real-arithmetic values of coefficients for *n=1..16* and *n=1985..2000*

| *n* | *coefficient* | *n* | *coefficient* |
|---|---|---|---|
| 1 | 1.00000000000 | 1985 | -0.000503778337511 |
| 2 | 0.500000000000 | 1986 | 0.000504030578240 |
| 3 | -0.333333333333 | 1987 | -0.000503271263211 |
| 4 | 0.375000000000 | 1988 | 0.000503526205670 |
| 5 | -0.200000000000 | 1989 | -0.000502764064949 |
| 6 | 0.180555555556 | 1990 | 0.000503017600544 |
| 7 | -0.142857142857 | 1991 | -0.000502260170768 |
| 8 | 0.210937500000 | 1992 | 0.000502512944452 |
| 9 | -0.0987654320988 | 1993 | -0.000501756146513 |
| 10 | 0.113750000000 | 1994 | 0.000502007527095 |
| 11 | -0.0909090909091 | 1995 | -0.000501251999332 |
| 12 | 0.0877459490741 | 1996 | 0.000501506028120 |
| 13 | -0.0769230769231 | 1997 | -0.000500751126690 |
| 14 | 0.0805165816327 | 1998 | 0.000501000354463 |
| 15 | -0.0631769547325 | 1999 | -0.000500250125063 |
| 16 | 0.0901794433594 | 2000 | 0.000500502022436 |
| ... | ... | ... | ... |

## 3.2. Factorizing – but *exp(x)* has no zeros?!

The product-representation *pxp(x)* suggests, that we have zeros of *exp(x)* at $x_n=-1/a(n)^{1/n}$. However *exp(x)* has ***no*** zero, so what does this mean here? Can the remainder be said to be divergent at that "zeros"? I don't have an answer for this yet.

## 3.3. Modifications of the *exp(x)-pxp(x)* function

In other contexts I fiddled with the complementary function which occurs, if the coefficients in the product-representation change their signs. We have then

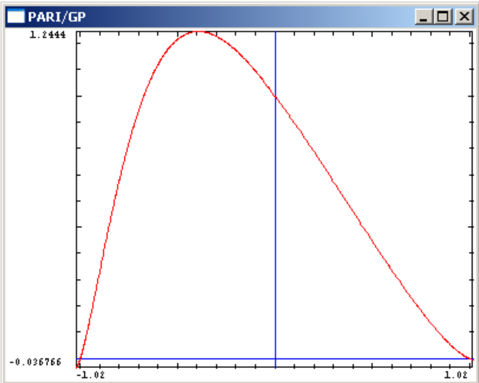
$$\begin{aligned} qxp(x) = (1 - x) & \\ & *(1 - 1/2\, x^2) \\ & *(1 + 1/3\, x^3) \quad *(1 - 3/8\, x^4) \\ & *(1 + 1/5\, x^5) \quad *(1 - 13/72\, x^6) \\ & *(1 + 1/7\, x^7) \quad *(1 - 27/128\, x^8)*(1 + 8/81\, x^9)*(1 - 91/800\, x^{10}) \\ & *(1 + 1/11\, x^{11})*(1 - 1213/13824\, x^{12}) \\ & *(1 + 1/13\, x^{13})* \ldots \\ & \ldots \end{aligned}$$

Rewritten as powerseries we get

$$qxp(x) = 1 - x - 1/2\, x^2 + 5/6\, x^3 - 17/4!\, x^4 + 49/5!\, x^5 - 19/6!\, x^6 - 449/7!\, x^7 + O(x^8)$$
$$= 1 - x - 0.5\, x^2 + 0.83333\, x^3 - 0.70833\, x^4 + 0.40833\, x^5 - 0.026389\, x^6 - 0.089087\, x^7 + 0.028150\, x^8 + O(x^9)$$

*// where the absolute values of the coefficients seem to decrease slowly.*

The range of convergence is surprisingly small compared with the original function *exp(x)*, whose range of convergence is infinity. A plot of *qxp(x)* for the interval $-1<x<1$ shows the following graph:

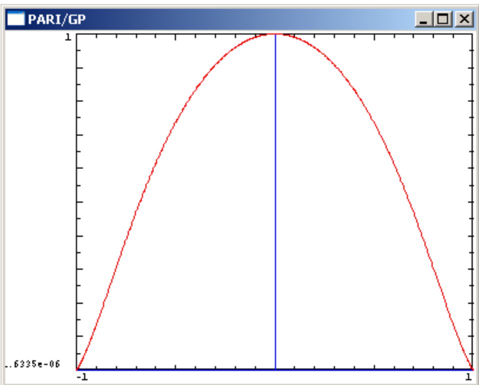



already a bit outside of this range the partial sums of the series cannot be summed even if *512* terms are taken into account.

If we use the product we get

$$qxp(x)*exp(x) = (1-a^2x^2)(1-b^2x^4)(1-c^2x^6)(1-d^2x^8)\ldots$$
$$= (1-x^2)(1-1/4x^4)(1-1/9x^6)(1-9/64x^8)(1-1/25x^{10})\ldots$$
$$= 1 - 1\, x^2 - 0.25\, x^4 + 0.13889\, x^6 - 0.029514\, x^8 + 0.12840\, x^{10} + 0.014778\, x^{12} + 0.0026609\, x^{14} + O(x^{16})$$

and the plot for the convergent range




The product can also be understood by two other complementary functions:

$$qxp(x)*exp(x) = (1-a^2x^2)(1-b^2x^4)(1-c^2x^6)(1-d^2x^8)\ldots$$
$$=poxp(x)*nexp(x)$$

where all coefficients of the product-series are positive for *poxp(x)* and all are negative for *nexp(x)* (for *pxp* and *qxp* the signs alternate in a chaotic way). We get then for *poxp(x)* and *nexp(x)*, expanded as powerseries:

$$poxp(x) = 1 + x + 0.5\, x^2 + 0.83333\, x^3 + 0.70833\, x^4 + 0.74167\, x^5 + 0.73472\, x^6 + 0.73591\, x^7 + 0.73574\, x^8 + 0.73576\, x^9 + O(x^{10})$$
$$nexp(x) = 1 - x - 0.5\, x^2 + 0.16667\, x^3 - 0.041667\, x^4 + 0.34167\, x^5 + 0.040278\, x^6 + 0.075198\, x^7 - 0.13614\, x^8 + 0.099341\, x^9 + O(x^{10})$$

### 3.4. A step aside to the Goldbach-problem

When contemplating the conversion from the productseries representation to that of the usual powerseries representation, we find some flavour of the Goldbach-problem: *is each even natural number the sum of two primes?*

If we consider the expansion of the product-formula into the powerseries expression, then for the coefficient at, for instance, $x^8$ of the powerseries of *exp(x)* the following partial products of the *pxp(x)*-function are involved:

$1/8!x^8$ determined by

$a(8)x^8$, $(1+a(1)x)*(1+a(7)x^7)$, $(1+a(2)x^2)*(1+a(6)x^6)$,
$(1+a(3)x^3)*(1+a(5)x^5)$
$(1+a(1)x^1)* (1+a(2)x^2)*(1+a(5)x^5)$
$(1+a(1)x^1)* (1+a(3)x^3)*(1+a(4)x^4)$

More precisely

$$1/8!x^8 = (1*a(8) + a(1)*a(7) + a(2)*a(6) + a(3)*a(5) + a(1)*a(2)*a(5) + a(1)*a(3)*a(4))x^8$$
$$=(27/128 + 1 *(-1/7) + 1/2*13/72 + 1/3*1/5 - 1*1/2*1/5 - 1*1/3*3/8)\, x^8$$

So for the coefficient at index 2k of the powerseries, products of coefficients a(n) of the productseries are involved, whose indexes sum up to 2k. The Goldbach-conjecture says then, that for each coefficient 2k of the powerseries (at least) one pair of prime-indexes p,q with p+q=2k are involved – and moreover, that 1/(2k)! is composed involving at least one of such products.

The bell which is ringing here is, that the denominators of *a(n)* at composites indexes (and thus of their product) are complicated and have high value, and the denominators of *a(n)* at prime indexes (and thus of their product) is small.

So for the index *2k=8* in the powerseries expansion we have the product of the coefficients of the productseries $a(3)*a(5) = -1/3*-1/5$ involved.

I didn't investigate this deeper, for the reader it may serve as a interesting detail.

### 3.5. Conclusion

The properties of this infinite product-series are far from being exhaustingly discussed. Maybe I'll find another spare time to proceed. The interested reader is cordially invited to send comments and/or extensions.

Besides of that, I'll enjoy (and hope, you'll do too) the discovered (and also the still uncovered) rhythms and the symmetries of this beautiful "dream of a sequence".

Gottfried Helms

# 4. Entries in OEIS

*Slightly shortened and formatted descriptions. For original source see links at [OEIS]*

Sequence of denominators (den(n)) equals **A067911** except that at indexes of highly-composite n some prime-powers are cancelled against the numerator.

```
A067911  Product of GCD(k,n) for 1 <= k <= n.
         1, 2, 3, 8,
         5, 72,
         7, 128, 81, 800,
         11, 41472,
         13, 6272, 30375, 32768,
         17, 3359232,
         19, 20480000, 750141, 247808,
         23, 13759414272, 15625, 1384448, 1594323, 5035261952,
LINKS    T. D. Noe, Table of n, a(n) for n=1..500
FORMUL   a(n) = Product_{ d divides n } d^phi(n/d).
         Vladeta Jovovic (vladeta(AT)Eunet.yu), Mar 08 2004
CROSSR   Cf. A018804, where product is replaced by sum.
         Product of terms in n-th row of A050873.
AUTHOR   Sharon Sela (sharonsela(AT)hotmail.com), Mar 10 2002
EXTENS   Extended and edited by John W. Layman (layman(AT)math.vt.edu) Mar 14 2002
```

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The sequence of coefficients factorially scaled (a(n) = num(n)/den(n)*n!)

```
A137852  G.f.: Product_{n>=1} (1 + a(n)*x^n/n!) = exp(x).
         1, 1, -2, 9, -24, 130, -720, 8505, -35840, 412776, -3628800, 42030450, -479001600,
         7019298000, -82614884352, 1886805545625, -20922789888000, 374426276224000, …
COMMENT  Equals signed A006973 (except for initial term), where A006973 lists the dimensions
         of representations by Witt vectors.
FORMUL   a(n) = (n-1)!*[(-1)^n + Sum_{d divides n, 1<d<n} d*( -a(d)/d! )^(n/d) ] for n>1
         with a(1)=1.
EXAMPLE  exp(x) = (1+x)*(1+x^2/2!)*(1-2*x^3/3!)*(1+9*x^4/4!)*(1-
         24*x^5/5!)*(1+130*x^6/6!)*...*(1+a(n)*x^n/n!)*...
PROGRAM  (PARI)
         {a(n)=if(n<1, 0,
                   if(n==1, 1
                          , (n-1)!*((-1)^n + sumdiv(n, d,
                                                    if(d<n&d>1, d*(-a(d)/d!)^(n/d))
                                                  )
                                    )
                      )
                 )}
CROSSR   Cf. A006973.
AUTHOR   Paul D. Hanna (pauldhanna(AT)juno.com), Feb 14 2008
```

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The same, unsigned additional leading zero, was introduced even earlier:

```
A006973  Dimensions of representations by Witt vectors.
         0, 1, 2, 9, 24, 130, 720, 8505, 35840, 412776, 3628800, 42030450, 479001600,
         7019298000, 82614884352, 1886805545625, 20922789888000, 374426276224000,…
REFERE   Borwein, Jonathan; Lou, Shi Tuo, Asymptotics of a sequence of Witt vectors. J.
         Approx. Theory 69 (1992), no. 3, 326-337. Math. Rev. 93f:05007
         Reutenauer, Christophe; Sur des fonctions symetriques reliees aux vecteurs de Witt.
         [ On symmetric functions related to Witt vectors ] C. R. Acad. Sci. Paris Ser. I
         Math. 312 (1991), no. 7, 487-490.
         Reutenauer, Christophe; Sur des fonctions symetriques liees aux vecteurs de Witt et
         a l'algebre de Lie libre, Report 177, Dept. Mathematiques et d'Informatique, Univ.
         Quebec a Montreal, 26 March 1992.
CROSSR   Cf. A137852.
AUTHOR   Simon Plouffe (simon.plouffe(AT)gmail.com)
EXTENS   More terms from Michael Somos, Oct 07, 2001
         More terms from Paul D. Hanna (pauldhanna(AT)juno.com), Feb 14 2008
```

## 5. References/Links

[OEIS] Online encyclopedia of integer sequences
N. J. A. Sloane
http://www.research.att.com/~njas/sequences/A067911
http://www.research.att.com/~njas/sequences/A137852
http://www.research.att.com/~njas/sequences/A006973

[wiki-q] "Q-analogues" in "wikipedia"
http://en.wikipedia.org/wiki/Q-analogue

[wiki-m] "multiplicative function" in "wikipedia"
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Further readings

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*Reciprocal Sums Over Partitions and Compositions,*
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Linear Algebra and its Applications
Volume 430, Issues 11-12, 1 June 2009, Pages 2835-2858
http://dx.doi.org/10.1016/j.laa.2008.12.019

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My own project pages:

[Helms] Main index for math-pages
http://go.helms-net.de/math

Gottfried Helms, Kassel, 01.01.2010
(first version: 14.02.2009)