Stirlingnumbers 2'nd kind expressed in cyclotomic functions and binomial-coefficients

St2(n,k)

	1	$\frac{1*(2^n-1^n)}{0!}$	$\frac{1*(3^n-2^n)/1 \; -1*(3^n-1^n)/2}{1!}$	$\frac{1*(4^n-3^n)/1 \; -2*(4^n-2^n)/2 \; +1*(4^n-1^n)/3}{2!}$	$\frac{1*(5^n-4^n)/1 \; -3*(5^n-3^n)/2 \; +3*(5^n-2^n)/3 \; -1*(5^n-1^n)/4}{3!}$
n					
0	1	0	0	0	
1	1	1	0	0	
2	1	3	1	0	
3	1	7	6	1	
4	1	15	25	10	1

result of the decomposition of the recurrence definition, see the both last table at the end

Bernoullinumbers expressed in Stirlingnumbers 2nd and first kind , and reciprocals of natural numbers

ß0	1*0!				
ß1	1*0!	- 1*1!/2			
ß2	1*0!	- 3*1!/2	+ 1*2!/3		
ß3	1*0!	- 7*1!/2	+ 6*2!/3	-1 *3!/4	
ß4	1*0!	-15*1!/2	+25*2!/3	-10*3!/4	+ 1*4!/5

(S2 = St-numbers 2'nd kind, S1=St-numbers 1'st kind

$$ß_n = \sum_{k=0}^{n} \left(S2_{n,k} * \frac{S1_{n,0}}{k+1} \right)$$

according to the matrix equation, where the bernoulli-numbers occur in 0'th column of $\mathbf{G_p}$

$$\mathbf{G_p}_{[*,0]} = \mathbf{S2} * \mathbf{N}^{-1} * \mathbf{S1}_{[*,0]}$$

Bernoullinumbers expressed in cyclotomical expressions, binomial numbers, and reciprocals of natural numbers with best cancelling of constants

	1	-	$\frac{1*(2^n-1^n)}{2}$	+	$\frac{2*(3^n-2^n) \; -1*(3^n-1^n)}{3}$	-	$\frac{3*(4^n-3^n) \; -3*(4^n-2^n) \; +1*(4^n-1^n)}{4}$	+	$\frac{4*(5^n-4^n) \; -6*(5^n-3^n) \; +4*(5^n-2^n) \; -1*(5^n-1^n)}{5}$
n									
ß0	1								
ß1	1	+	$\frac{-1*(2^1-1^1)}{2}$						
ß2	1	+	$\frac{-1*(2^2-1^2)}{2}$	+	$\frac{+2*(3^2-2^2) \; -1*(3^2-1^2)}{3}$				
ß3	1	+	$\frac{-1*(2^3-1^3)}{2}$	+	$\frac{+2*(3^3-2^3) \; -1*(3^3-1^3)}{3}$	+	$\frac{-3*(4^3-3^3) \; +3*(4^3-2^3) \; -1*(4^3-1^3)}{4}$		
ß4	1	+	$\frac{-1*(2^4-1^4)}{2}$	+	$\frac{+2*(3^4-2^4) \; -1*(3^4-1^4)}{3}$	+	$\frac{-3*(4^4-3^4) \; +3*(4^4-2^4) \; -1*(4^4-1^4)}{4}$	+	$\frac{+4*(5^4-4^4) \; -6*(5^4-3^4) \; +4*(5^4-2^4) \; -1*(5^4-1^4)}{5}$

$$ß_n = 1 + \sum_{k=2}^{n+1} \left(\frac{1}{k} \sum_{j=1}^{k-1} \left((-1)^j \binom{k-1}{j-1} \left(k^n - j^n\right) \right) \right)$$

Bernoullinumbers in terms of powers of consecutive natural numbers and reciprocals

	1	$\frac{-1*(2^n-1^n)}{2}$	$\frac{2*(3^n-2^n) -1*(3^n-1^n)}{3}$	$\frac{-3*(4^n-3^n) +3*(4^n-2^n) -1*(4^n-1^n)}{4}$	$\frac{4*(5^n-4^n) -6*(5^n-3^n) +4*(5^n-2^n) -1*(5^n-1^n)}{5}$
n					
	equals				
	$\frac{1*1^n}{1}$	$\frac{+1*1^n -1*2^n}{2}$	$\frac{+1*1^n -2*2^n +1*3^n}{3}$	$\frac{+1*1^n -3*2^n +3*3^n -1*4^n}{4}$	$\frac{+1*1^n -4*2^n +6*3^n -4*4^n +1*5^n}{5}$
ß0	1				
ß1	$1^n(1+1/2)$ - 2^n (1/2)				
ß2	$1^n(1+1/2+1/3)$ - 2^n (1/2+2/3) + $3^n(1/3)$				
ß3	$1^n(1+1/2+1/3+1/4)$ - 2^n (1/2+2/3+3/4) + $3^n(1/3+3/4)$ - $4^n(1/4)$				
ß4	$1^n(1+1/2+1/3+1/4+1/5)$ - 2^n (1/2+2/3+3/4+4/5) + $3^n(1/3+3/4+6/5)$ - $4^n(1/4+4/5)$ + 5^n (1/5)				

$$ß_n = \sum_{k=1}^{n+1} \left((-1)^k k^n * \sum_{i=k}^{n+1} \frac{c(i-1,k-1)}{i} \right)$$

(appears to be like eq 29 in mathworld)

Matrixnotation

$\mathbf{1} = col(1,1,1,1)$ = columnvector of ones
$\mathbf{I} = diag(1,1,1,1,...)$ = unit-matrix
$\mathbf{J} = diag(1,-1,1,-1,...)$ = unit-matrix with alternating signs
$\mathbf{N} = diag(1,2,3,4,...)$,
$\mathbf{P}$ = = lower triangular binomialmatrix, row index assumed at 0
$sum(\mathbf{M}) = \mathbf{1' M 1}$ = summing all entries of a matrix into a scalar

then

$$\beta_n = sum\ (\mathbf{N}^{-1}\ \mathbf{P}\ \mathbf{J}\ \mathbf{N}^n)$$

and, surprisingly

$$\beta_{n+1} = sum\ ((\mathbf{N}+\mathbf{I})^{-1}\ \mathbf{P}\ \mathbf{J}\ \mathbf{N}^n)$$

Unfortunately, constructing a function:

$$f(n,k) := \beta_{n+k} = sum\ ((\mathbf{N}+k*\mathbf{I})^{-1}\ \mathbf{P}\ \mathbf{J}\ \mathbf{N}^n)$$

gives no good interpolation; the derivative-function $f'(n,k)$ is not continuous (while the interpolated values may have some significance anyway).

$$\beta_3 = 1^n(2+1/12) - 2^n (2-1/12) + 3^n(1+1/12) - 4^n(1/4)$$

$$\beta_4 = 1^n(2+17/60) - 2^n (3-17/60) + 3^n(2 + 17/60) - 4^n(1+3/60) + 5^n (1/5)$$

$$\beta_n = 1^n \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n+1}\right)$$

$$-2^n \left(\frac{0}{1} + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \ldots \right)$$

$$+3^n \left(\frac{0}{1} + \frac{0}{2} + \frac{1}{3} + \frac{3}{4} + \frac{6}{5} + \ldots \right)$$

$$-4^n \left(\frac{0}{1} + \frac{0}{2} + \frac{0}{3} + \frac{1}{4} + \frac{4}{5} + \frac{10}{6} + \ldots \right)$$

Differences of Bernoulli-Numbers

```
ß_4 - ß_3 =  1^4 1/5 + (1^4 - 1^3)(1/1.+ 1/2 + 1/3 +.1/4) =  1^4 1/5 + 1^3  0 (1/1.+ 1/2 + 1/3 +.1/4)
            -2^4 4/5 - (2^4 - 2^3)(      1/2 + 2/3 + 3/4)   -2^4 4/5 - 2^3  1 (      1/2 + 2/3 + 3/4)
            +3^4 6/5 + (3^4 - 3^3)(            1/3.+.3/4)   +3^4 6/5 + 3^3  2 (            1/3.+.3/4)
            -4^4 4/5 - (4^4 - 4^3)(                  1/4)   -4^4 4/5 - 4^3  3 (                  1/4)
            +5^4 1/5 + (5^4 - 5^3)(                  0 )   +5^4 1/5 + 5^3  4 (                  0 )

ß_4 - ß_3 =   1^4 1/5 + 1^3  0
             -2^4 4/5 - 2^3  1 (      1/2 + 2/3 + 3/4)
             +3^4 6/5 + 3^3  2 (            1/3.+.3/4)
             -4^4 4/5 - 4^3  3 (                  1/4)
             +5^4 1/5 +      0


ß_4 + ß_3 =   1^4 1/5 + 1^3  2 (1/1.+ 1/2 + 1/3 +.1/4)
             -2^4 4/5 - 2^3  3 (      1/2 + 2/3 + 3/4)
             +3^4 6/5 + 3^3  4 (            1/3.+.3/4)
             -4^4 4/5 - 4^3  5 (                  1/4)
             +5^4 1/5 +      0
```

Since odd-indexed ß-numbers = 0

$$b_{2k} - b_{2k+-1} = b_{2k} \qquad (2k>2\ )$$

```
ß_4  =   1^4 1/5 + 1 1^3 1/4  + 1^2 1^2  1/3  + 1^3 1  1/2   + 1^4 1  1/1
        -2^4 4/5 - 2 2^3 3/4  - 2^2 2^2  2/3  - 2^3 2  1/2
        +3^4 6/5 + 3 3^3 3/4  + 3^2 3^2  1/3.
        -4^4 4/5 - 4 4^3 1/4
        +5^4 1/5

ß_3  =           +   1^3 1/4  + 1  1^2  1/3  + 1^2 1  1/2   + 1^3 1  1/1
                 -   2^3 3/4  - 2  2^2  2/3  - 2^2 2  1/2
                 +   3^3 3/4  + 3  3^2  1/3.
                 -   4^3 1/4


ß_2  =                        +   1^2  1/3  + 1  1  1/2   + 1^2 1  1/1
                              -   2^2  2/3  - 2  2  1/2
                              +   3^2  1/3.


ß_1  =                                    +    1  1/2   + 1 1  1/1
                                          -    2  1/2
```

Stirling-numbers decomposition according to their recursive definition

Example, column 4 (indexing starts at 0), entries of columns (0,1,2,3,4) are called (e,d,c,b,a):

```
a = 0
a' = b + 4*a
a'' = b' + 4*a'
a''' = b'' + 4*a''
a'''' = b''' + 4*a'''
...
```

$a^{(k+1)} = b^{(k)} + 4\ a^{(k)}$ a(0)=0
$b^{(k+1)} = c^{(k)} + 3\ b^{(k)}$ b(0)=0
$c^{(k+1)} = d^{(k)} + 2\ c^{(k)}$ c(0)=0
$d^{(k+1)} = e^{(k)} + 1\ d^{(k)}$ d(0)=0
$e^{(k+1)} = 1$ e(0)=1

Legend: 4443 = 4^3*3; column k+1 is the decomposition of column k

decomp	b+4a	c+3b	d+2c	e+1d	
a''''	b''''	c'''	d''	0 1d'	 1e + 1d
			2c''	2d' 22c'	2e + 2d 22d + 222c
		3b'''	3c''	3d' 32c'	3e + 3d 32d + 322c
			33b''	33c' 333b'	33d + 332c 333c + 3333b
	4a''''	4b'''	4c''	4d' 42c'	4e + 4d 42d + 422c
			43b''	43c' 433b'	43d + 432c 433c + 4333b
		44a'''	44b''	44c' 443b'	44d + 442c 443c + 4433b
			444a''	444b' 4444a'	444c + 4443b 4444b + 44444a
		a''' 4^2	a'' 4^3	a' 4^4	
		b''' $(4^2-3^2)/(4-3)$	b'' $(4^3-3^3)/(4-3)$	b' $(4^4-3^4)/(4-3)$	
		c''' 1	c'' 2+3+4	c' $(3^3-2^3)/(3-2)$ c' $(4^2-3^2)/(4-3)$ c' 3c'	
			d'' 1	d' 1+2+3+4	
				e' 1	
	a'''' 4	a''' 44	a'' 444	a' 4444	a 44444
	b'''' 1	b''' 4 + 3	b'' 44 + 43 33	b' 444 + 443 + 433 + 333	b 3333+4333+4433+4443+4444
		c''' 1	c'' 4 + 3 + 2	c' 44 + 43 + 42 33 + 32 22	c 444 + 443 + 442 433 + 432 + 422 +333 + 332 322 + 222
			d'' 1	d' 4 + 3 + 2 + 1	d 44 + 43 + 42 + 41 33 + 32+ 31 + 22 + 21 + 11
				e' 1	e 4 + 3 + 2 + 1

1				
1	1			
11	11 + 21	11		
111	111+ 211 + 221	111+121 311	111	
1111	1111+ 2111+2211+2221	1111+1211+1221 3111+3121+3311	1111+1121+1311 4111	
1				
1	1			
1	1+2	1		
1	1+2(1+2)	1+2+3 $=(3^3-1)/2-(2^3-1)$ $=(3^3-2^3-1^3)/2 - (2^2-1)$	1	
1	1+2(1+2(1+2))	1(1) 2(1+2) + 3(1+2+3) $=(3^4-1)/2- (2^4-1)$ 25	1+2+3+4 $=(3^3-2^3-1^3)/2 - (2^2-1)$ $+ (3^2-1)/2$ $=(3^3-1)/2-(2^3-1)$ $+ (3^2-1)/2$ 10 = 6 + 4	1
1	1+2+4+8+16	1+2(1+2(1+2)) +3(1+2(1+2)) $+3^2(1+2+3)$ 90	1(1) 2(1+2) 3(1+2+3) 4(1+2+3+4) $=(3^4-1)/2- (2^4-1)$ $+ (3^4-1)/2$ 65 = 25 + 40	1+2+3+4+5