Binomial-("Pascal")-Matrix: a beautiful identity

see also: http://mathworld.wolfram.com/PascalMatrix.html
(corrected and extended version 3 of 31.8.2005)

I found for the Binomial (or Pascal) triangular matrix the following beautiful identity and connections to other matrices by some interesting properties.

Assume a matrix ***L*** with ascending natural numbers starting at 1 in the subdiagonal:

•	•	•	•	•
1	•	•	•	•
•	2	•	•	•
•	•	3	•	•
•	•	•	4	•

Take the matrix exponential ***E = exp(L)***

1	•	•	•	•
1	1	•	•	•
1	2	1	•	•
1	3	3	1	•
1	4	6	4	1

and note a beautiful simple formula for the generation of Pascal's binomial-matrix.

This identity can easily be shown by expanding the euler-series for the exponential-function

$$exp(L) = I + L^1/1! + L^2/2! + L^3/3! + \dots$$

and multiplying out. Read the steps of summation top down:

	L^x	/x!	$I + L^1/1! + L^2/2! + L^3/3! \dots$
			1 • • • • • 1 • • • • • 1 • • • • • 1 • • • • • 1
L^1	• • • • • 1 • • • • • 2 • • • • • 3 • • • • • 4 •	/1!	1 • • • • 1 1 • • • • 2 1 • • • • 3 1 • • • • 4 1
L^2	• • • • • • • • • • 2 • • • • • 6 • • • • • 12 • •	/2!	1 • • • • 1 1 • • • 1 2 1 • • • 3 3 1 • • • 6 4 1
L^3	• • • • • • • • • • • • • • • 6 • • • • • 24 • • •	/3!	1 • • • • 1 1 • • • 1 2 1 • • 1 3 3 1 • • 4 6 4 1
L^4	• 24 • • • •	/4!	1 • • • • 1 1 • • • 1 2 1 • • 1 3 3 1 • 1 4 6 4 1
L^5	• •	/5!	1 • • • • 1 1 • • • 1 2 1 • • 1 3 3 1 • 1 4 6 4 1

We have the identity:

exp (	• • • • • 1 • • • • • 2 • • • • • 3 • • • • • 4 •	) =	1 • • • • 1 1 • • • 1 2 1 • • 1 3 3 1 • 1 4 6 4 1	This is the binomial matrix

Connection to the Delannoy-Numbers
see: http://mathworld.wolfram.com/DelannoyNumber.html

Let *c(a,b)* be the choose-function and let

$$\mathbf{P} = diag(2^{c(n,2)})$$

and

$$\mathbf{B} = \textit{the binomial-matrix}$$

and

$$\mathbf{D} = \mathbf{B} * \mathbf{P} * \mathbf{B}'$$

then $\mathbf{D}$ is the Delannoy-matrix, and $\mathbf{B}*\mathbf{P}^{1/2}$ is the cholesky-decomposition of $\mathbf{D}$. The diagonal of $\mathbf{D}$ are the *central Delannoy-numbers*

$$\mathbf{D}_{a,b} = \mathbf{D}_{a-1,b} + \mathbf{D}_{a,b-1} + \mathbf{D}_{a-1,b-1}$$

	B		D	Comment
	1 0 0 0 0 1 1 0 0 0 1 2 1 0 0 1 3 3 1 0 1 4 6 4 1		1 1 1 1 1 1 3 5 7 9 1 5 13 25 41 1 7 25 63 129 1 9 41 129 321	Delannoy-numbers http://www.research.att.com/projects/OEIS?Anum=A001850

Different starting-conditions in the logarithmic matrix ***L*** result in other known matrices / numberarrays.
The result is a function of the matrix-exponential ***E**:=exp(**L**)* and its first column:

L:

0	0	0	0	0
1	0	0	0	0
0	3	0	0	0
0	0	6	0	0
0	0	0	10	0

***E**:=exp(**L**)*:

1	0	0	0	0
1	1	0	0	0
1.5	3	1	0	0
3	9	6	1	0
7.5	30	30	10	1

Then the *diagonals* have to be normalized, so that their leading element (in column *1*) is *1*; thus the diagonals and subdiagonals must be divided by their leading elements in column *1*:

$$\mathbf{M} := \mathbf{M}_{r,\,r+c-1} = \mathbf{E}_{r,\,r+c-1} / \mathbf{E}_{r,\,1} \qquad \textit{(where r,c = cover rows and cols)}$$

use	L		M	Comment
c(n,0):	. . . . . 1 . . . . . 1 . . . . . 1 . . . . . 1 .		1 . . . . 1 1 . . . 1 1 1 . . 1 1 1 1 . 1 1 1 1 1	
c(n,1):	. . . . . 1 . . . . . 2 . . . . . 3 . . . . . 4 .		1 . . . . 1 1 . . . 1 2 1 . . 1 3 3 1 . 1 4 6 4 1	Pascal/Binomial- triangle
c(n,2):	. . . . . 1 . . . . . 3 . . . . . 6 . . . . . 10 .		1 . . . . 1 1 . . . 1 3 1 . . 1 6 6 1 . 1 10 20 10 1	Narayana-triangle http://www.research.att.com/projects/OEIS?Anum=A001263 Number of antichains (or order ideals) in the poset 2*m*n or plane partitions with rows <= m, columns <= n, and entries <= 2
c(n,3):	. . . . . 1 . . . . . 4 . . . . . 10 . . . . . 20 .		1 . . . . 1 1 . . . 1 4 1 . . 1 10 10 1 . 1 20 50 20 1	http://www.research.att.com/projects/OEIS?Anum=A056939 Number of antichains (or order ideals) in the poset 3*m*n or plane partitions with rows <= m, columns <= n, and entries <= 3
c(n,4):	. . . . . 1 . . . . . 5 . . . . . 15 . . . . . 35 .		1 . . . . 1 1 . . . 1 5 1 . . 1 15 15 1 . 1 35 10 35 1	http://www.research.att.com/projects/OEIS?Anum=A056940 Number of antichains (or order ideals) in the poset 4*m*n or plane partitions with rows <= m, columns <= n, and entries <= 4 n1,ord = 5,5

Another way to compute ***M*** is to use a modification *F(**X**)* of the exponential-function: consider the first three terms of the power-series expansion:

$$exp(\mathbf{L}) = \mathbf{I} + \frac{\mathbf{L}}{1} + \frac{\mathbf{L}^2}{1*2} + \frac{\mathbf{L}^3}{1*2*3} + \ldots$$

By the structure of the given subdiagonal-matrix ***L*** the diagonal and subdiagonals of the resulting exponential-matrix correspond perfectly to the above terms of different powers of ***L*** as shown in the table on page *1:* we find in the principal diagonal just the entries of the unit-matrix ***I***; in the first subdiagonal the entries of L_1, in the second subdiagonal the entries of L_2. The entries of all subdiagonals are just the products of the appropriate elements of the original matrix ***L***; with symbols for that entries we have for the intermediate matrix ***TMP*** :

	L						TMP					M				
	•	•	•	•	•		1	•	•	•	•	1	•	•	•	•
	a	•	•	•	•		a	1	•	•	•	1	1	•	•	•
	•	b	•	•	•		ab	b	1	•	•	1	b/a	1	•	•
	•	•	c	•	•		abc	bc	c	1	•	1	c/a	c/a	1	•
	•	•	•	d	•		abcd	bcd	cd	d	1	1	d/a	cd/ab	d/a	1

$$\mathbf{TMP} = \mathbf{I} + \mathbf{L} + \mathbf{L}^2 + \mathbf{L}^3 \ldots$$

*where the summands fill separately the subdiagonals of **TMP***

To normalize this matrix according to their subdiagonals, we have to divide the subdiagonals ***TMP*** by this leading value in column *1*. This suggests the modified powerseries ***F()***:

$$F(\mathbf{L}) := \mathbf{I} + \frac{\mathbf{L}}{a} + \frac{\mathbf{L}^2}{a*b} + \frac{\mathbf{L}^3}{a*b*c} + \ldots = \mathbf{M}$$

For the ***L***-matrix of first order (L_1) the entries *a,b,c,d* are just consecutive numbers starting at 1, and reproduce perfectly the matrix-exponential by ***F()***. For the ***L***-matrices of higher order (L_2,L_3,...) this modified powerseries ***F()*** is to be preferred, or the result of the *exp()*-function must be adapted accordingly as indicated in the previous page. For L_2 , which gives the *Narayana*-triangle, the entries of its subdiagonal are *(1,3,6,10,...) = choose(n,2)*. These values can be plugged into the symbols *(a,b,c,d,...)* and a powerseries-representation for that matrix can derived this way.

The concept can be extended to other types of entries in the ***L***-matrices for other questions; for instance, if they are different powers of the same base we find tricky sums in the exponents of the entries of the resulting matrix ***M***, after it is computed with the properly adapted function ***F()***.

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