# Infinite alternating sums of powertowers of like height
# a conjecture

## Note:

The following is a part of not yet finished manuscript, which derives a (new?) general identity concerning infinite series of powertowers of like height. It is meant to allow criticism and correction of errors before compiling the final article with a possible serious error. Comments are welcome!

I'm stating the notation first and then the conjectured identity. In the following is the derivation which reflects the more extensive introduction of the general aspects of the matrix-method as given in my first article:

[PowTowSum] An infinite alternating sum of powertowers of increasing height (the initial heuristic)
*http://go.helms-net.de/math/binomial_new/10_4_Powertower.pdf*

A shorter version of the main conjecture is at
*http://go.helms-net.de/math/tetdocs/Tetration_GS_short.pdf*

Relevant references will be supplied in the coming article.

Comments are invited to *mailto:helms@uni-kassel.de* , subject: tetration-conjecture 0708

Gottfried Helms, 15.10.2007 / 6.8.2007 (only reference added)

# 1. Tetration (Powertowers) :a conjectured general series-identity

## 1.1. Notation for scalar expressions

For the scalar expression I'll use the notation

(1.1.1.) $\{s.x\}\text{^^}\, y = s^{s^{\cdot^{\cdot^{s^x}}}}$ *with y-fold repetition of the base-parameter s*

where such a general expression for the scalars $s$ and $x$ will be necessary. For the simple case $x=1$ it may be sufficient to write in shorter form:

(1.1.2.) $s\text{^^}\, y = \{s.1\}\text{^^}\, y = s^{s^{\cdot^{\cdot^{s}}}}$ *with y-fold repetition of the base-parameter s*

## 1.2. The conjecture

The result here is a theorem concerning alternating series of powertowers, which seem new and may be crosschecked. It is described in chap 2. and the final paragraph 2.4. It states the following identities

***Theorem:***

(1.2.1.)
$$\sum_{k=0}^{oo}(-1)^k\{s.k\}\text{^^}m + \sum_{k=0}^{oo}(-1)^k\{s.-k\}\text{^^}m$$
$$= \{s.0\}\text{^^}m$$
$$= s\text{^^}(m-1)$$

***Examples:***

*note, that for the display I reordered the summation terms of the above general formula without checking here, whether this is in fact admissible*

(1.2.2.) 1)
$$CS_1(s) = \sum_{k=0}^{inf}(-1)^k * (s^k + s^{-k}) = \sum_{k=0}^{inf}(-1)^k * (s^k + \frac{1}{s^k}) = s\text{^^}0$$
$$(s^0 + \frac{1}{s^0}) - (s^1 + \frac{1}{s^1}) + (s^2 + \frac{1}{s^2}) - (s^3 + \frac{1}{s^3}) + \ldots - \qquad = 1$$

(1.2.3.) 2)
$$CS_2(s) = \sum_{k=0}^{inf}(-1)^k * (s^{s^k} + s^{s^{-k}}) = \sum_{k=0}^{inf}(-1)^k * (s^{s^k} + \sqrt[s^k]{s}) = s\text{^^}1$$
$$(s^{s^0} + \sqrt[s^0]{s}) - (s^{s^1} + \sqrt[s^1]{s}) + (s^{s^2} + \sqrt[s^2]{s}) - (s^{s^3} + \sqrt[s^3]{s}) + \ldots - \ldots \qquad = s$$

(1.2.4.) 3)
$$CS_3(s) = \sum_{k=0}^{inf}(-1)^k * (s^{s^{s^k}} + s^{s^{s^{-k}}}) = \sum_{k=0}^{inf}(-1)^k * (s^{s^{s^k}} + s^{\sqrt[s^k]{s}}) = s\text{^^}2$$
$$(s^{s^{s^0}} + s^{\sqrt[s^0]{s}}) - (s^{s^{s^1}} + s^{\sqrt[s^1]{s}}) + (s^{s^{s^2}} + s^{\sqrt[s^2]{s}}) - (s^{s^{s^3}} + s^{\sqrt[s^3]{s}}) + \ldots - .. \qquad = s^s$$

### 1.3. Tetration by a matrix-operator

For the analysis of tetration I introduce some matrices of theoretically infinite dimension, and the derivation of the final theorem is analytical, though depending on some assumptions.

For numerical confirmation in praxi the theoretical model must be approximated by finite matrices of manageable size (in terms of memory, float-precision and of cpu-usage in iterative routines like eigensystem analysis). Dimension 64 and float precision of 80 digits seemed to be sufficient for most problems, for eigensystem-analysis I had to go down even to dimension 32 and float precision of 200 or even 800 digits apparently due to non-optimal software implementation. But still that low dimension shows good confirmation for the general underlying idea.

The following matrices are defined:

*(1.3.1.)* *Let*

*$^dF$ be the diagonalmatrix of reciprocal factorials*

*V(x) be the columnvector of the powers of x, V(x)~ its transpose, $^dV(x)$ its diagonalmatrix*
*V(x) also be called a "vandermonde-vector of x"*

*VZ the above vandermondematrix with ascending powers along the colums and ascending bases along the rows, both beginning at zero*

*the symbol for transposition be the tilde ~ so VZ ~ = transpose(VZ)*

*a constant rowvector using brackets $[a_0, a_1, a_2, a_3,\ldots]$*

*The r'th component of a vector is denoted by indexing in brackets [r], where the index begins at zero; so, for instance*
*s = V(s) [1],*
*VZ[,0] = V(0) = [1,0,0,...]~ is the first column in VZ and*
*VZ[0,] = V(1)~ =[1,1,1,...] is the first row of VZ*

*(1.3.2.)* *Let*

| | | |
|---|---|---|
| $B$ | $= {}^dF^{-1} * VZ$ | // a matrix, which will be constant all over this text |
| $B_s$ | $= {}^dV(\log(s))*B$ | // its parametrization with s |

| | |
|---|---|
| $b_{r,c}$ | the element in row r of column c in a current version $B_s$ |
| $b_r$ | the short form for the previous, where column 1 (second column) is assumed |
| $bb_r$ | like the previous, but of $B_s^2$ |
| $b^{(m)}_r$ | like the previous, but generalized to the y'th power $B_s^{\,y}$ |

Then we have the basic transformation and its obvious iterates:

*(1.3.3.)* $V(x)\sim\ * B_s = V(s^x)$
$V(x)\sim\ * B_s^2 = V(s^{s^x})$

*generally*

*(1.3.4.)* $V(x)\sim\ * B_s^y = V(\ \{s.x\}\text{^^}y\ )$
*(1.3.5.)* $V(1)\sim\ * B_s^y = V(\ s\text{^^}y\ )$
*(1.3.6.)* $V(0)\sim\ * B_s^y = V(\ s\text{^^}(y-1)\ )$

*(1.3.6)* is an obvious consequence and will not be explicitely proven here.

***Negative parameters x***

If we use negative parameters for $x$ for the simple case of using the first power of $\mathbf{B_s}$ we get according to the properties of the exponential series:

$$(1.3.6.1.)\qquad V(-x) \sim * B_s = V(s^{-x})\sim = V(1/s^x)\sim = [1, s^{-x}, s^{-2x}, s^{-3x},\ldots] = [1, 1/s^x, 1/s^{2x}, 1/s^{3x},\ldots]$$

Note also, that a vector $V(x)$ is translated into one of $V(-x)$ by the diagonal-unit-matrix $\mathbf{J}$ with alternating signs:

$$(1.3.6.2.)\qquad J = diag(1,-1,1,-1,\ldots) \qquad V(-x)\sim = V(x)\sim * J$$

and the sum

$$(V(x) + V(-x)) * B_s = V(s^x) + V(1/s^x) = [1+1, s^x+s^{-x}, s^{2x} + s^{-2x}, \ldots]$$

Setting $x=1$ we have

$$[2, 0, 2, 0, 2, \ldots] * B_s = [\,1 + 1\,, s + 1/s,\ s^2+1/s^2, s^3+1/s^3, \ldots]$$

and analoguously for higher powers of $\mathbf{B_s}$:

$$[2, 0, 2, 0, 2, \ldots] * B_s^2 = [\,1 + 1\,, s^s + s^{1/s},\ s^{2s}+s^{2/s}, s^{3s}+s^{3/s}, \ldots]$$
$$[2, 0, 2, 0, 2, \ldots] * B_s^y = [\,1 + 1\,, s\text{^^}y + \{s,-1\}\text{^^}y, (s\text{^^}y)^2 + (\{s,-1\}\text{^^}y)^2,\ \ldots.]$$

which shall be of relevance later.

**Extracting the scalar result of the computations**

The interesting result in our computation is then usually in the first column of $V(y)\sim$ , so we may write

$$(1.3.7.)\qquad s^x = V(y)\sim [1] = (V(x)\sim *B_s)\,[1] = V(x)\sim * (B_s[,1])$$

$$(1.3.8.)\qquad \{s.x\}\text{^^}y = V(x)\sim * (B_s^{\,y}[,1])$$

in conventional notation for powerseries using the matrix-entries as coefficients, with $b_r$ indicating the $r'th$ entries of $\mathbf{B_s}$ in the second column, $bb_r$ of $\mathbf{B_s}^2$ and generally $b^{(y)}{}_r$ of $\mathbf{B_s}^y$ $2^{'nd}$ column-entries:

$$(1.3.9.)\qquad s^x = \Sigma_{r=0..inf}\ b_r * x^r$$
$$(1.3.10.)\qquad s^{s^x} = \Sigma_{r=0..inf}\ bb_r * x^r$$
$$(1.3.11.)\qquad \{s.x\}\text{^^}y = \Sigma_{r=0..inf}\ b^{(y)}{}_r * x^r$$

For $y=1$, we consider the coefficients $b_r$ only, and since this is an important example, I give its decomposition here, as it is clear from the matrixdefinition of $\mathbf{B}$ and $\mathbf{B_s}$:

$$(1.3.12.)\qquad B_s = {}^dV(log(s)) * {}^dF^{-1} * VZ$$
$$b_r = \frac{log(s)^r}{r!}$$

since in $\mathbf{VZ}$ the second column is $columnvector(1,1,1,1,\ldots)$.

The numerical evaluation of some of the final or even intermediate series is not always possible, especially with high $y$:

either because the convergence is slow,

often we have series with unbounded size of the entries.
However, having alternating signs makes this (sometimes) summable by
for instance Euler-summation.

For numerical approximations of examples in all such cases summation with the Euler-summation was applied, either simply to accelerate convergence or to transform the alternating divergent series into a convergent one, where this is possible. I won't go deeper into this detail, since the main result is finally analytical.

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### 1.4. Alternating series of powertowers

Since the tetration-operation is expressed as a matrix-operation, it is possible to apply some linear combinations to the operands.

Here we want to approach sums of powertowers of like height in expressions like

*(1.4.1.)* *Definition:*

$$AS_1(s) = s^0 - s^1 + s^2 - s^3 + \ldots -$$

$$AS_2(s) = s^{s^0} - s^{s^1} + s^{s^2} - s^{s^3} + \ldots - \ldots$$

The first sum $AS_1(s)$ is also the ordinary alternating geometric sum of $s$.

Interestingly, and unexpected, there are curious identities with the related type of series:

*(1.4.2.)* *Definition:*

$$BS_1(s) = s^{-0} - s^{-1} + s^{-2} - s^{-3} + \ldots - = \frac{1}{s^0} - \frac{1}{s^1} + \frac{1}{s^2} - \frac{1}{s^3} + \ldots -$$

$$BS_2(s) = s^{s^{-0}} - s^{s^{-1}} + s^{s^{-2}} - s^{s^{-3}} + \ldots - = \sqrt[s^0]{s} - \sqrt[s^1]{s} + \sqrt[s^2]{s} - \sqrt[s^3]{s} + \ldots -$$

These sums can easily be rewritten by the matrix-notation of powertowers.

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### 1.5. A sum of operands on the lhs

Recall the basic identity for any vandermonde-vector of $x$, given a parameter $s$:

(1.5.1.) $V(x)\sim * B_s = V(s^x)\sim$

Since this is a ordinary matrix-operation, we may apply linear combinations of different $V(x)$ at the lhs,

(1.5.2.) $(V(x)+ V(y))\sim * B_s =( V(s^x) +V(s^y))\sim$

where also the result will be just a linear combination of the two single results; for instance

(1.5.3.) $(V(1) -V(2) + V(3))\sim * B_s = Z_1\sim = ( V(s^1) -V(s^2) + V(s^3) )\sim$

and iterated:

$( V(s^1) -V(s^2) + V(s^3))\sim * B_s = Z_2\sim = ( V(s^{s^1}) - V(s^{s^2}) + V(s^{s^3}))\sim$

...

and the first scalar result were:

$Z_1[1] = s^1 - s^2 + s^3$

The conditions for linear composition are the same all iterations (or powers) of $\boldsymbol{B_s}$.

In conventional notation we have for all heights of powertowers the result by evaluating the second column of the appropriate power of $\boldsymbol{B_s}$:

(1.5.4.)
$$\sum_{r=0}^{oo} b_r( x^r + y^r ) = s^x + s^y$$
$$\sum_{r=0}^{oo} bb_r( x^r + y^r ) = s^{s^x} + s^{s^y}$$
$$\sum_{r=0}^{oo} b^{(m)}{}_r( x^r + y^r ) = \{s.x\}^{\wedge\wedge} m + \{s.y\}^{\wedge\wedge} m$$

and the analoguous is similarly true for arbitrarily many summands $V(x_k)\sim$ on the lhs in *(2.1.2.2)*

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### 1.6. an alternating series operands $V(x_k)\sim$ on the lhs

If we introduce infinite sums on the left hand, this should not be different as long as convergence for the emerging sums is given.

We may loosen this condition a bit and may allow also infinite sums of alternating signs[1], as far as we know that they can regularly be summed, for instance by Euler-summation.

Assume we apply the alternating sum of $V(x)$-vectors with ascendent arguments $x=0,1,2,3,...$ on the lhs, then on the rhs we should get

(1.6.1.)
$$(V(0)-V(1)+V(2)-V(3)+....)\sim * B_s = (V(s^0)- V(s^1) + V(s^2) - V(s^3)-...)\sim$$
$$= Z_1\sim \qquad \text{//(write } Z_1 \text{ for the rhs)}$$

[1] I was not able to find a relation to non-alternating sums like zeta(-n), so in general I only consider alternating sums.

**First let's look at the rhs of the equation.**

In the columns of $Z_1\sim$ we have the alternating sums (per entry vertically noted here):

$$Z_1\sim \quad = [ \begin{matrix} 1, & 1, & 1, & 1 & \dots \\ -1, & -s, & -s^2, & -s^3 & \dots \\ +1, & +s^2, & +s^4, & +s^6 & \dots \\ -1, & -s^3, & -s^8, & -s^9 & \dots \\ \dots & \dots \end{matrix} ]$$

and these column-sums are just the geometric series of negatives of powers of s , where we are only interested in the value of the second column [2]:

(1.6.2.) $Z_1[1] \quad = s^0 - s^1 + s^2 - s^3 + - \dots \qquad =(1 + s)^{-1}$ // by geometric series

Decomposing the matrix-operation into the terms of the sum in the conventional notation this is, for the second column of $B_s$ only:

(1.6.3.)

$$\sum_{r=0}^{oo} b_r(0^r - 1^r + 2^r - 3^r + \dots - \dots) = s^0 - s^1 + s^2 - s^3 + \dots - \dots = \frac{1}{1+s} \qquad \text{// where } 0^0 = 1$$

**Second, let's look at the lhs of the equation.**

The sums in the lhs-parenthese obviously refer $\eta()$[3]-values at nonpositive exponents. But since the first element is an additional power of zero we need a hurwitz-analogon for the $\eta$-function and define

(1.6.4.)

$$\eta(n, a) = (-1)^a * ( (0+a)^{-n} - (1+a)^{-n} + (2+a)^{-n} - (3+a)^{-n} + \dots - \dots) \qquad \text{// where } 0^0 = 1$$

where also if $n>0$ or noninteger then $a =/= 0,-1,-2,-3,\dots$

With this we get:

(1.6.5.)

$$\sum_{r=0}^{oo} b_r * \eta(-r, 0) = \frac{1}{1+s} \qquad \text{// where } 0^0 = 1$$

and further decomposing $b_r$ in its components:

(1.6.6.)

$$\sum_{r=0}^{oo} \frac{\log(s)^r}{r!} * \eta(-r, 0) = \frac{1}{1+s} \qquad \text{// where } 0^0 = 1$$

For the matrix-notation we introduce the vector $H_0$ containing the $\eta(-r,0)$-values as replacement for the sum $V(0)-V(1)+V(2) - + \dots$

(1.6.7.) $H_0 = \text{columnvector}( [ \eta(0,0), \; \eta(-1,0), \; \eta(-2,0),\dots])$

which is also equivalent to:

(1.6.8.) $H_0 = (V(0) - V(1) + V(2) - V(3)+\dots.)$

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[2] in the first column we had:

$Z_1[0] = s^0 - s^0 + \dots = 1 - 1 + 1 \dots = ½$ // Eulersummation

[3] $\eta(s)$ means the sign-alternating version of the zeta-function $\zeta(s)$

The matrix-formula looks then like

(1.6.8.1.)

$$H_0\sim * B_s = Z_1\sim$$

$$Z_1\sim = \text{rowvector}(\ (1+1)^{-1},(1+s)^{-1},(1+s^2)^{-1},\dots,(1+s^c)^{-1},\dots)$$

// where c is the column-number, starting c=0

We have then an expression, which converts infinite sums of $\eta(-n,0)$-values into geometric series of $(-s^c)$. Note that the resulting- $\mathbf{Z_1}$-vector on the rhs is now not a vandermonde-vector!

Since from the construction of $\mathbf{B_s}$ the entries $b_r$ for each column $c$ have a known structure (and $b_0=1$)we may decode this into fully explicite conventional notation (using $0^0=1$ )

(1.6.8.2.)

$$\sum_{k=0}^{inf} \eta(-k,0) * \frac{(\log(s)c)^k}{k!} = \frac{1}{1+s^c} \qquad \text{// for column } c;\ 0^0=1$$

Back to matrix-notation:

(1.6.8.3.)

$$H_0\sim * B_s = Z_1\sim = \text{rowvector}(\ (1+s^c)^{-1})$$

// where c is the column-number, starting c=0

and in column $1$ of the result we get the alternating sum of the powers of $s$

$$AS_1(s) = s^0 - s^1 + s^2 - s^3 + \dots - \dots$$

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*Example: Let t=2 and* $s = t^{1/t} = 2^{1/2} = \text{sqrt}(2)$.

$1+s^2=3$ $\qquad 1/(1+s^2)=1/3=0.333\dots$

$\qquad\qquad 1/(1+s^4)=1/5=0.2$

(1.6.8.4.)

$$H_0\sim * B_s = Z_1\sim$$

$$* \begin{bmatrix} 1.00000000000 & 1.00000000000 & 1.00000000000 & 1.00000000000 & 1.00000000000 \\ 0 & 0.346573590280 & 0.693147180560 & 1.03972077084 & 1.38629436112 \\ 0 & 0.0600566267398 & 0.240226506959 & 0.540509640658 & 0.960906027836 \\ 0 & 0.00693801358310 & 0.0555041086648 & 0.187326366744 & 0.444032869319 \\ 0 & 0.000601133069227 & 0.00961812910763 & 0.0486917786074 & 0.153890065722 \end{bmatrix}$$

....

$$.\begin{bmatrix} 1/2 & -1/4 & 0 & 1/8 & 0 & -1/4 & 0 & 17/16 \end{bmatrix} = \begin{bmatrix} 0.500000000000 & 0.414213562373 & 0.333333333333 & 0.261203874964 & 0.200000000000 \end{bmatrix}$$

$$\begin{bmatrix} 1/(1+s\text{^}0) & 1/(1+s\text{^}1) & 1/(1+s\text{^}2) & 1/(1+s\text{^}3) & 1/(1+s\text{^}4) \end{bmatrix}$$

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### *Negative parameters x*

Again we discuss negative parameters, now of its analoguos sums. Since

(1.6.8.5.) $H_0 = (V(0) - V(1)+ V(2)- V(3) +...-...)$

we may contruct the sum of the different $V(x)$ with negative arguments by simply postmultiply $H_0$ by $J$:

(1.6.8.6.) Let $JH_0 := J * H_0$

and

$JH_0 \sim = H_0 \sim * J = (V(0) - V(1) + V(2) - ...)\sim * J = (V(-0) - V(-1) + V(-2) - ...)\sim$

Then we have, each of the entries of $JH_0$ displayed in rows here

(1.6.8.7.)

$$JH_0 = [ (1 -1 + 1 - 1...), -(0 -1 + 2 - 3...), (0^2 -1^2 + 2^2 - 3^2...), -(0^3 -1^3 + 2^3 - 3^3...), +..., -...]$$
$$= [\eta(0,0), -\eta(1,0), \eta(2,0), -\eta(3,0), ... ,.. ]$$

Because of the construction of the columns of $B_s$ as coefficients of an exponential series the effect of using the negative variant of $x$ is, that the result is the reciprocal, so we have:

(1.6.8.8.)

$$JH_0 \sim * B_s = Z_1 \sim = rowvector( (1 + 1/s^c)^{-1})$$
// where c is the column-number, starting c=0

and in column $1$ of the result we get the alternating sum of the powers of reciprocals of $s$

(1.6.8.9.)

$$BS_1(s) = 1/s^0 -1/s^1 +1/s^2 -1/s^3 +... -....$$

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### *The sum of $AS_1 + BS_1$*

A surprising, although simple, identity occurs here, since each $\eta(-2*k)$-value is zero $(k=1,2,3,...)$. If we recall, that

(1.6.8.10.)

$$H_0 \sim = [\eta(0,0), \eta(-1,0), \eta(-2,0), \eta(-3,0), ... ]$$
$$JH_0 \sim = [\eta(0,0), -\eta(-1,0), \eta(-2,0), -\eta(-3,0), ... ]$$

and the fact, that each second occuring $\eta()$-value is zero, we may first write:

(1.6.8.11.)

$$H_0 \sim = [\eta(0,0), \eta(1,0), 0, \eta(3,0), 0... ,+... ]$$
$$JH_0 \sim = [\eta(0,0), -\eta(1,0), 0, -\eta(3,0), 0... ,+... ]$$

then adding them we have

(1.6.8.12.)

$$(H_0+JH_0)\sim = [ 1 , 0 , 0, 0, 0, ... ] = V(0)\sim$$

Let's denote the value of the second column of the result in these equations as $AS_1(s)$:

**(1.6.8.13.)**

$$H_0 \sim * B_s = Z_1\sim \quad = Z_{1,1} = [(1+s^0)^{-1}, (1+s^1)^{-1}, (1+s^2)^{-1},...]$$

$$AS_1(s) \quad = Z_{1,1}[1]$$
$$= 1 - s + s^2 - s^3 + s^4 ... = (1+s)^{-1}$$

**(1.6.8.14.)**

$$JH_0 \sim * B_s = Z_2\sim \quad = Z_{1,2} = [(1+1/s^0)^{-1}, (1+1/s^1)^{-1}, (1+1/s^2)^{-1},...]$$

$$BS_1(s) \quad = Z_{1,2}[1]$$
$$= 1 - 1/s + 1/s^2 - 1/s^3 + 1/s^4 ... = (1+1/s)^{-1}$$

From the previous we have then the expression for the sum:

$$(H_0+JH_0)\sim * B_s = (Z_{1,1} + Z_{1,2})\sim$$
$$V(0)\sim * B_s = [(1+s^0)^{-1}, (1+s^1)^{-1}, (1+s^2)^{-1},...] + [(1+1/s^0)^{-1}, (1+1/s^1)^{-1}, (1+1/s^2)^{-1},...]$$

$$[1, 1, 1, 1,...] = [(1+s^0)^{-1}+(1+1/s^0)^{-1},$$
$$(1+s^1)^{-1} +(1+1/s^1)^{-1},$$
$$(1+s^2)^{-1} + (1+1/s^2)^{-1},$$
$$...]$$

and each entry of the rhs in the last equation simplifies to *1*, so we have

**(1.6.8.15.)**

$$(H_0+JH_0)\sim * B_s = V(1)\sim = [1,1,1,1,...]$$

and from the second column of the result we have

**(1.6.8.16.)**

$$AS_1(s) + BS_1(s) = 1$$

also by means of this matrix-method.

### 1.6.9. Powertowers of height 2, increasing exponent x, gives geometric series analogons

Now we'll completely analoguously consider the alternating sums of powertowers of height 2.

We wish to compute

(1.6.9.1.) $$AS_2(s) = \sum_{k=0}^{inf} (-1)^k * s^{s^k} = s - s^s + s^{s^2} - s^{s^3} + s^{s^4} + ... - ...$$

This is in matrix-notation, using the eta-vector ***H*** resp. $\boldsymbol{H_0}$ again, where

(1.6.9.2.) $H_0 = (V(0)-V(1)+V(2)-V(3)+...-...) = columnvector(r=0,inf, eta(-r,0))$

for the formula of powertowers of height 2:

(1.6.9.3.) $H_0\sim * B_s^2 = Z_2\sim$

and only considering the second column $c=1$ of the result vector

(1.6.9.4.) $AS_2(s) = Z_2[1]$

we have, written as series involving the column-coefficients $bb$,

(1.6.9.5.) $$s^{s^0} - s^{s^1} + s^{s^2} - s^{s^3} + ... - ... = \sum_{r=0}^{oo} ( bb_r (0^r - 1^r + 2^r - 3^r + ... - ...))$$

This is then finally, using the eta-vector $\boldsymbol{H_0}$,

(1.6.9.6.) $$s^{s^0} - s^{s^1} + s^{s^2} - s^{s^3} + ... - ... = \sum_{r=0}^{oo} bb_r * \eta(-r,0) \quad // using\ 0^0 = 1$$

then we get for some $s$:

| $s$ | $AS_2(s)$ | |
|---|---|---|
| 0.2 | -0.0804895642300 | questionable, dubious approximation |
| 0.3 | 0.0250077352382 | |
| 0.5 | 0.188643256530 | |
| 0.7 | 0.327755444502 | |
| 1.0 | 0.500000000000 | |
| 1.5 | 0.690214956597 | |
| 2.0 | 0.785870155025 | |
| 2.5 | 0.828203629549 | |
| 3.0 | 0.843690663363 | |
| 3.1 | 0.844901708538 | dubious approximation using dim=32 |

Note, that with the classical methods of Euler-summation the series with parameter $s>2.0$ should hardly be achievable, since the quotient of the absolute values of two consecutive terms is $s^{(s-1)s^{\wedge}k}$ at index $k$, and with $s=2$ it is then $2^{2^{\wedge}k}$

## Negative exponent x

This represents then

(1.6.9.7.)

$$BS_2(s) = \sum_{k=0}^{inf} (-1)^k * s^{s^{-k}} = s^{s^{-0}} - s^{s^{-1}} + s^{s^{-2}} - s^{s^{-3}} + s^{s^{-4}} + \dots - \dots$$
$$= \sum_{k=0}^{inf} (-1)^k * \sqrt[s^k]{s} = \sqrt[s^0]{s} - \sqrt[s^1]{s} + \sqrt[s^2]{s} - \sqrt[s^3]{s} + \sqrt[s^4]{s} + \dots - \dots$$
$$= ?$$

which we want to determine.

Some results are:

| s | $AS_2(s)$ | $BS_2(s)$ |
|---|---|---|
| 0.5 | 0.188643256530 | 0.311356741384 |
| 1.5 | 0.690214956597 | 0.809785041588 |
| 2.0 | 0.785870155025 | 1.21412984317 |

The entries in this table look suspiciously to sum up to its parameter *s*.

---

## The sum $AS_2$+$BS_2$ from powertowers of height = 2

The sum of $AS_2(s) + BS_2(s)$ is again accomplished by applying the vanishing sum of $H_0$ and $JH_0$ to the matrix-formula:

(1.6.9.8.) $AS_2(s) + BS_2(s) = (H_0 + JH_0)\sim * B_s^2 [,1] = V(0)\sim \ B_s^2 [,1]$

and because

(1.6.9.9.) $V(0)\sim \ B_s = V(1)\sim$
$V(1)\sim \ B_s = V(s)\sim$

and

(1.6.9.10.) $V(0)\sim \ B_s^2 = V(s)\sim$

we have the result in the second entry of the result vector:

(1.6.9.11.) $AS_2(s) + BS_2(s) = V(s)[1] = s$

So the surprising result is, that the sum $CS_2(s) = AS_2(s) + BS_2(s)$

(1.6.9.12.)

$$CS_2(s) = \sum_{k=0}^{inf} (-1)^k * (s^{s^k} + s^{s^{-k}}) = \sum_{k=0}^{inf} (-1)^k * (s^{s^k} + \sqrt[s^k]{s})$$
$$= s^{s^0} + \sqrt[s^0]{s} - s^{s^1} - \sqrt[s^1]{s} + s^{s^2} + \sqrt[s^2]{s} - s^{s^3} - \sqrt[s^3]{s} + + \dots - - \dots$$
$$= s$$

This can again be resolved in conventional notation for the explicite matrix-multiplication :

(1.6.9.13.)

$$CS_2(s) = \sum_{r=0}^{oo} bb_r * \eta(-r,0) + \sum_{r=0}^{oo} bb_r * (-1)^r \eta(-r,0)$$

$$= \sum_{r=0}^{oo} bb_r * (1+(-1)^r) * \eta(-r,0)$$

$$= bb_0 * 2 * (1-1/2) + \sum_{r=1}^{oo} bb_{2r} * 2 * \eta(-2r,0) + \sum_{r=1}^{oo} bb_{2r-1} * 0 * \eta(1-2r,0)$$

$$= bb_0 + \sum_{r=1}^{oo} 0 + \sum_{r=1}^{oo} 0$$

$$= bb_0$$

or

(1.6.9.14.) $$BS_2(s) = s - AS_2(s)$$

The results for the previous table are:

| *s* | *AS₂(s)* | *BS₂(s)* | *AS₂(s)+BS₂(s)* |
|---|---|---|---|
| 0.5 | 0.188643256530 | 0.311356741384 | 0.5 |
| 1.5 | 0.690214956597 | 0.809785041588 | 1.5 |
| 2.0 | 0.785870155025 | 1.21412984317 | 2.0 |

and is easily checked for other parameters *s*.

---

### 1.6.10. Generalization to greater heights of powertowers

The generalization is obvious. Since the basic structure of the powertower-formula remains the same, and we simply increase the power of the matrix $\mathbf{B_s}$ we have for any power *k* of $\mathbf{B_s}$:

(1.6.10.1.) $CS_m(s) = (H_0 + JH_0)\sim * (B_s^m [,1] )$ // using second column of $B_s^m$ only

and this is then by

(1.6.10.2.) $(H_0 + JH_0)\sim * B_s^m = V(0) \sim * B_s^m = Z_m \sim$

where $\mathbf{Z_m}\sim$ contains exactly the first row of $\mathbf{B_s}^m$

It is also

(1.6.10.3.) $V(0)\sim * B_s^m$ = [1, {s.0}^^m, ({s.0}^^m)$^2$ , ({s.0}^^m)$^3$ , ...]
= [1, s^^(m-1), {s^^(m-1))$^2$ , {s^^(m-1))$^3$ , ...]

and thus the most interesting general formula pops up:

***Conjecture:***

(1.6.10.4.)

$$\sum_{k=0}^{oo} (-1)^k (\{s.k\}^{\wedge\wedge} m + \{s.-k\}^{\wedge\wedge} m) = s^{\wedge\wedge}(m-1)$$

*Example: the first three instances:*

| | | |
|---|---|---|
| (1.6.10.5.) | 1) | $CS_1(s)=\sum_{k=0}^{inf}(-1)^k*(s^k+s^{-k})=\sum_{k=0}^{inf}(-1)^k*(s^k+\frac{1}{s^k})=s\hat{}\hat{}0$ <br> $(s^0+\frac{1}{s^0})-(s^1+\frac{1}{s^1})+(s^2+\frac{1}{s^2})-(s^3+\frac{1}{s^3})+\ldots- \qquad =1$ |
| (1.6.10.6.) | 2) | $CS_2(s)=\sum_{k=0}^{inf}(-1)^k*(s^{s^k}+s^{s^{-k}})=\sum_{k=0}^{inf}(-1)^k*(s^{s^k}+\sqrt[s^k]{s})=s\hat{}\hat{}1$ <br> $(s^{s^0}+\sqrt[s^0]{s})-(s^{s^1}+\sqrt[s^1]{s})+(s^{s^2}+\sqrt[s^2]{s})-(s^{s^3}+\sqrt[s^3]{s})+\ldots-\ldots \qquad =s$ |
| (1.6.10.7.) | 3) | $CS_3(s)=\sum_{k=0}^{inf}(-1)^k*(s^{s^{s^k}}+s^{s^{s^{-k}}})=\sum_{k=0}^{inf}(-1)^k*(s^{s^{s^k}}+s^{\sqrt[s^k]{s}})=s\hat{}\hat{}2$ <br> $(s^{s^{s^0}}+s^{\sqrt[s^0]{s}})-(s^{s^{s^1}}+s^{\sqrt[s^1]{s}})+(s^{s^{s^2}}+s^{\sqrt[s^2]{s}})-(s^{s^{s^3}}+s^{\sqrt[s^3]{s}})+\ldots-.. \qquad =s^s$ |

**Note:** *the above example-design of the formulae should give only an impression here; caution concerning the ordering of the summation-terms is needed since we deal with divergent series with alternating signs. I'll consider this requirement later.*

Gottfried Helms , Kassel

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**Further links to online-resources**

[Galidakis] Ioannis Galidakis, **Tetration, LambertW-function** and more; online at
http://ioannis.virtualcomposer2000.com/math/

[Geiser] Daniel Geiser, **Tetration-pages**; online at
http://www.tetration.org

[McDonnell]
http://www.faculty.fairfield.edu/jmac/ther/tower.htm

[Robbins] Andrew Robbins, **Tetration-pages**, online at:
http://tetration.itgo.com/txt/table-tetnat.txt

[Weissstein] http://mathworld.wolfram.com/LambertW-Function.html

**From my own project (06-07 2007):**

[PowTowSum] An infinite alternating sum of powertowers of increasing height (the initial heuristic)
*http://go.helms-net.de/math/binomial_new/10_4_Powertower.pdf*

[PowTowCrit] Critical point for this method of summation at *s=exp(-1)*
*http://go.helms-net.de/math/binomial_new/PowertowerproblemDocSummation.htm*

[SumLikePow] Summing of like powers (zeta and hurwitz-zeta using matrices)
*http://go.helms-net.de/math/binomial_new/04_3_SummingOfLikePowers.pdf*