A new and extreme summation method for breathtaking divergent series

1.1. The news

For a number-theorist, who is used to methods of divergent summation of series like

$$S = 1 + 2 + 3 + 4 + \ldots\ldots \quad = -1/12 \qquad \text{// Non-alternating; // Ramanujan-Summation}$$

$$AS = 1 - 2 + 4 - 8 + 16 \ldots \quad = 1/3 \qquad \text{// Eulersummation}$$

an algorithm to assign values to similar sums for power-towers may be an breathtaking idea - and that's just, what is presented here. It allows to assign values to alternating sums of powertowers of <u>infinitely increasing height</u> for conventionally summable series as well as for series, which are not summable by any currently available summation-technique (as to my knowledge).

The formal description for the hereby accessible series is

$$AS(s,x) = x - s^x + s^{s^x} - s^{s^{s^x}} + s^{s^{s^{s^x}}} - \ldots + \ldots.$$

or shorter, setting $x=1$,

$$AS(s) = 1 - s + s^s - s^{s^s} + s^{s^{s^s}} - \ldots + \ldots.$$

where the parameter s may be in the conventionally permitted range $e^{-e} < s < e^{1/e}$ for the individual infinite powertower (as given, for instance, by G. Eisenstein in 1844 see [Eis 1844]), and may also <u>exceed this upper bound</u>, thus solving the same problem for the divergent case as well.

The latter should explicitely be noted, because of the extreme and currently in no thinkable way accessible divergence of the occuring series.

While the long commonly known methods of L. Euler, E. Borel, and S. Ramanujan can handle series of the orders

$$AS = 1 - 2 + 4 - 8 + 16 - 32 + - \ldots$$
$$AS = 1 - 1 + 2 - 6 + 24 - 120 + -$$

or more generally of

$$AS = 1 - s + s^2 - s^3 + - \ldots \qquad \text{// } s>-1$$
$$AS = 0! - 1! + 2! - 3! + - \ldots \qquad \text{//}$$

maximally, where the quotient of two subsequent terms is either constant or increasing by a constant difference, the method proposed here can assign regular values to alternating series with the <u>exponentially increasing quotient</u> $s^{A-\log_s(A)}$. Thus sums including numbers like *googol* or *googolplex* just in their first four or five terms can be dealt with this method. For instance:

$$AS(10,1) = 1 - 10 + 10^{10} - 10^{10^{10}} + 10^{10^{10^{10}}} - \ \sim \quad 0.164280631274$$

or

$$AS(10,2) = 2 - 10^2 + 10^{10^2} - 10^{10^{10^2}} + 10^{10^{10^{10^2}}} - \ \sim \quad 0.336337317504$$

which means, using the common "***googol***"-terms, :

$$AS(10,2) = \ - 98 + googol - googolplex + 10^{googolplex} - \ \sim \quad 0.336337317504$$

where the *5'th* or *6'th* term already surpasses the giantic ***Skewes-number*** - and already at the very beginning of the series any other number which may ever have been considered to have a special meaning in numbertheory.

1.2. Deriving the method

The method was developed by analysis of iterative application of a matrix-operator to a powerseries, which transforms a powerseries in x:

$$V(x)\sim = [1, x , x^2 , x^3 , \dots]$$

into one of s^x :

$$V(s^x)\sim = [1, s^x , (s^x)^2 , (s^x)^3]$$

performed by a matrix-multiplication with an s-parametrized matrix $\mathbf{B_s}$:

$$V(x)\sim * \mathbf{B_s} = V(s^x)\sim$$

An iterative approach:

$$\begin{aligned} V(x)\sim * \mathbf{B_s} &= V(s^x)\sim \\ V(s^x)\sim * \mathbf{B_s} &= V(s^{s^x})\sim \\ V(s^{s^x})\sim * \mathbf{B_s} &= V(s^{s^{s^x}})\sim \\ &\dots \end{aligned}$$

or

$$\begin{aligned} V(x)\sim * \mathbf{B_s} &= V(s^x)\sim \\ V(x)\sim * \mathbf{B_s}^2 &= V(s^{s^x})\sim \\ V(x)\sim * \mathbf{B_s}^3 &= V(s^{s^{s^x}})\sim \\ &\dots \end{aligned}$$

introduces a geometric series of $\mathbf{B_s}$. Its alternating sum, where the powerseriesvector $\mathbf{V(x)\sim}$ is factored out, gives formally via

$$\begin{aligned} V(x)\sim * (\mathbf{I} - \mathbf{B_s} + \mathbf{B_s}^2 - \mathbf{B_s}^3 \dots) \\ = [(1 - 1 + 1 - 1 + \dots - \dots), \\ (x - s^x + s^{s^x} - s^{s^{s^x}} + \dots - \dots), \\ (x^2 - (s^x)^2 + (s^{s^x})^2 - (s^{s^{s^x}})^2 + \dots - \dots), \\ (\dots) , \dots] \end{aligned}$$

several infinite alternating sums as entries of the result vector, where the first entry may be Euler-summed (to $1/2$) , and where the second entry would represent our function-value. We cannot evaluate the sums of the entries of columns with index c>0 conventionally, but it is possible, to employ the shortcut-formula for the geometric series. If we assume, our result will be regular (in the sense as the term is used in divergent summation context), then the process should be invertible and a reciprocal should exist

$$\begin{aligned} (\mathbf{I} - \mathbf{B_s} + \mathbf{B_s}^2 - \mathbf{B_s}^3 \dots)(\mathbf{I} + \mathbf{B_s}) &= \mathbf{I} \\ (\mathbf{I} - \mathbf{B_s} + \mathbf{B_s}^2 - \mathbf{B_s}^3 \dots) &= (\mathbf{I} + \mathbf{B_s})^{-1} = \mathbf{M_s} \end{aligned}$$

The final step is then, to evaluate the geometric series in $\mathbf{B_s}$ by the geometric-series-sum-formula, applied to the matrix-argument:

$$\mathbf{M_s} = (\mathbf{I} + \mathbf{B_s})^{-1}$$

and use this matrix for multiplication with the powerseries vector:

$$V(x)\sim * \mathbf{M_s} = Y\sim$$

where the entries of $\mathbf{Y}$ are now:

$$Y\sim = [\Sigma_{k=0}^{oo} (-1)^k, \ AS(s,x) , \ y_2 , \ y_3 , \dots , \dots]$$

This can be done for parameters s in (for crossvalidation) and above the range of conventional summability of the powertower-series; and the relevant occuring coefficients in $\boldsymbol{M}$ allow even convergent series when multiplied with $V(x)$ with reasonable x (considering manageable dimension *(d~64)* of the ideally infinite matrices $\boldsymbol{M}$, which allow only -however good- numerical approximations for the final values).

1.3. Compatibility with convergent or regularly summable parameter settings

The method was checked for parameters s in $A(s,x)$ for cases, where conventional Euler-summation can be applied to a direct summation; this means for the above mentioned permitted range $e^{-e} < s < e^{1/e}$ for the individual infinite powertower. The method gave the same results in terms of reasonable numerical approximation.

If the parameter s was shifted above the upper bound, the summing behaves smoothly - the results are continuous continuations of the graph of the function $AS(s,x)$ from the conventionally cases. Also an analysis of the internal occurences of multiplications and summation when the matrix-product was performed, and the results are reasonable: either they come from convergent or at least Euler-summable series.

So the method has good numerical evidence even for the divergent cases, similar to the results, which the series of zeta-, geometric- or factorial terms provide when Euler-/Borel- or Ramanujan-summed.

1.4. further research/ formal verification needed

However some bounds for convergence or summability are not yet proven formally. The proposal here needs formal verification for the following aspects:

Since the matrices $\boldsymbol{B}_s$ are asssumed as square-matrices of infinite dimension:

1) *can the exponentiation of $\boldsymbol{B}_s$ to any finite power be justified?*

   here one needs the proof of convergence or summability of products of rows and columns of $\boldsymbol{B}_s$ resp its powers.

2) *can the inversion of $(\boldsymbol{I} + \boldsymbol{B}_s)$ be justified?*
   And for which range of the parameter s?

   The general idea of applying the powerseries-formula to matrix-arguments seems not questionable. Again it is only of interest, whether the terms and series, which occur in the intermediate matrixoperations, can formally be shown to be convergent or at least conventionally summable by regular summation-methods.

For *1)* I seem to have already a tool using the *k-fold* matrix-exponential of an appropriately scaled binomial-matrix, whose coefficients in its first columns must match the coefficients in the second columns of the *k'th* power $\boldsymbol{B}_s^k$. Since it is possible to replace the binomial-matrix by a diagonal-reduced version we have a nilpotent variant, whose entries are all composed of finitely many terms, and thus are all bounded dependend on the exponent k and the parameter s. However, also this needs a formal proof for its sufficiency.

A documentation for the coefficients for various parameters s in tables and graphs are provided, see links in "References"-appendix.

Gottfried Helms, 14.7.2007

2. References

[Helms2007] Gottfried Helms
A Powertower-application
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http://go.helms-net.de/math/binomial/PowertowerproblemDocSummation.htm

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Entwicklung von a^ (a^ (a ^ (a ^ ....))).
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Table of values for natural tetration: (excerpt of A.Robbins' page)

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y               E^^y
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-2              -00
-1.993817117    -5
-1.63635835     -1
-1              0
0               1
1               E
1.08824913      3
1.12805960      Pi
1.476292652     5
GoldenRatio     6.3214336
2               E^E
2.1             20.9213886
2.5             179.115515
E               2075.9682
3               E^E^E
3.529577845     Googol
4               E^E^E^E
4.529987400     GoogolPlex
5               Overflow
```


Recommendations by Eric Weissstein, Mathworld:

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R. A. Knoebel,
Exponentials Reiterated,
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