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Continuous iteration of powerseries-defined functions

A guide to a general powerseries for continuous tetration

Abstract: In this article I discuss the formal process to determine integer, fractional and general iterates of a function which is given in a powerseries representation.

The article is a basic introduction; I give some examples of powerseries, which are nicely configured to give the idea of such iterations and arrive at the powerseries representation of tetration (b^x iterated) understood as exponential iteration beginning at a start-parameter x, and of the decremented iterated exponentiation ($dxp_b(x)$ = b^x–1) which I call U-tetration here.

Fractional iteration is then introduced as interpolation of the coefficients of the powerseries of the integer iterations. Here my examples employ polynomial interpolation only; so the coefficients of an integer iterate of a powerseries-function are the interpolation at integer heights of the height-parameter h (which naturally are the original coefficients) and fractional/general iteration are then just using the fractional parameter h for the polynomials to construct the required coefficients.

The mathematical "engine" for this employs formal powers of the powerseries of the function f under discussion, and, although this can all be expressed by the functional notation I find it much more convenient, to represent all coefficients of the formal powerseries in one matrix-form and operate with these matrices. This will prove extremely useful for the discussion of fractional iteration, since this can then be expressed by fractional powers of these matrices, for which a reliable and well-defined instrumentarium is already existent (and was in fact used by many authors, for instance L.Comtet and more recently P.Walker, Aldrovandi/Freitas and S.C.Woon and others).

However – the polynomial interpolation of coefficients is not the only solution for a concept of fractional iteration; there are various approaches to the (best) interpolation-method; one may refer to the discussion of various appoaches to define an interpolation for the factorial function, where one method was singled out to have the "best"/most consistent properties for the use in numbertheory, namely the gamma-function for fractional arguments of the factorial as defined by L. Euler.

So even if the polynomial approach may not be "the best" – the discussion here may be fruitful for the general introduction into the concept (and its possible shortcomings).

Contents

1. Iteration of functions, serial and matrix approach

1.1. Formal powerseries and their iterations

Consider a general function $f(x)$, which is implemented as a powerseries

(1.1.1.) $f(x) = K + a x + b x^2 + c x^3 + d x^4 + \dots$

Iterations of f, where the function-value is reintroduced as its argument, like $f(x), f(f(x)), f(f(f(x)))$ may be indicated by an iterator-index h and the iteration may be denoted by a superscript-circle:

(1.1.2.) $f^{\circ h}(x) = f(f(f(\dots f(x))$ // with h-fold occurence of f

The following discussion of iterations is done by consideration of f as formal powerseries (see...). In that context of formal powerseries the coefficients $(K,a,b,c,\dots)$ are discussed without respect to actual values of x, and thus considerations of convergence-radius wrt x are omitted here.

To compute iterates of f, we insert $f(x)$ instead of x at each term in (1.1.1).

(1.1.3.) $f^{\circ 2}(x) = K + a f(x) + b f(x)^2 + c f(x)^3 + \dots$

Thus we need the formal expansions of each power of $f(x)$ in terms of its coefficients $(K,a,b,c,\dots)$ first, since now:

(1.1.4.)
$$\begin{aligned} f^{\circ 2}(x) &= K \\ &+ a (K + a x + b x^2 + c x^3 + d x^4 + \dots) \\ &+ b (K + a x + b x^2 + c x^3 + d x^4 + \dots)^2 \\ &+ c (K + a x + b x^2 + c x^3 + d x^4 + \dots)^3 \end{aligned}$$

and the resulting powerseries for $f^{\circ 2}(x)$ can be given, if the powers of the parentheses are expanded and equal powers of x are collected.

For instance the formal powerseries $f(x)^2$, which is the third term only (in the above formula), begins with

(1.1.5.)
$$\begin{aligned} f(x)^2 = \; & K^2 \\ & +(\quad 2(Ka))\, x \\ & +(a^2+2(Kb))\, x^2 \\ & +(\quad 2(Kc+ab))\, x^3 \\ & +(b^2+2(Kd+ac))\, x^4 \\ & +(\quad 2(Ke+ad+bc))\, x^5 \\ & +(c^2+2(Kf+ae+bd))\, x^6 \\ & + \dots \end{aligned}$$

and this may be thought continued analoguously to higher powers of $f(x)$.

For the second iterate we get by such expansions of powers of $f(x)$:

(1.1.6.)
$$\begin{aligned} f^{\circ 2}(x) &= K \\ &+ a\,(K + \quad a x \; + \quad b x^2 \quad + \quad c x^3 \quad + \quad d x^4 \quad + \dots) \\ &+ b\,(K^2+2Ka\, x \; + \; (a^2+2Kb)\, x^2 \; + \; (2Kc+2ba)\, x^3 \; + (2ac+b^2+2Kd)\, x^4 + \;) \\ &+ c\,(K^3+3K^2a\, x + (3Ka^2+3K^2b)\, x^2+(3K^2c+(a^3+6Kba))\, x^3+\dots) \\ &+ \dots \end{aligned}$$

and if the coefficients are collected according to their like powers of x we get the following powerseries

(1.1.7.)
$$\begin{aligned} f^{\circ 2}(x) &= K(1 \; + \; a \; + Kb \; + K^2c + \dots) \\ &+(a(\quad a \; + 2bK + 3cK^2 + \quad \dots)) \quad x \\ &+(\quad b(\quad a \; + 2bK + 3cK^2 + \; 4dK^3 + \dots) \\ & \quad + \; a^2(\quad 1b \quad + 3cK + \; 6dK^2 + \dots \;)) \;\; x^2 \\ &+(\quad c(\quad a \; + 2bK + 3cK^2 + \; 4dK^3 + \dots) \\ & \quad +2ab(\quad 1b \quad + 3cK + \; 6dK^2 + \; \dots \;) \end{aligned}$$

$+ a^3(\quad 1c \quad + 4dK \quad + \ldots) \quad x^3$

$\ldots$

whose final coefficients must be determined by evaluation of the parentheses. The parentheses – btw – show the formal expansions of $f(x)$ and its derivatives at $x=K$, so it is also convenient, to express these parentheses as $f(K), f'(K), f''(K)/2!, \ldots$ and have a much shorter form at hand.

(1.1.8.) $f^{\circ 2}(x) \quad = f(K) + a\,f'(K)\,x \; + (a^2 f''(K)/2! + b\,f'(K)\,)\,x^2 +$

but I'm not going to discuss this here in more detail. (see 3.2)

1.2. Simpler powerseries, having K=0 and/or a=1

The sheer massiveness of such a formula as (1.1.7) explains, why we discuss simpler powerseries first. Most discussions about fractional iteration focus powerseries, whose K-term is zero. We get then the remarkable reduction:

(1.2.1.) $f_{K=0}(x) \quad = a\,x + b\,x^2 + c\,x^3 + \ldots$

$$
\begin{aligned}
f_{K=0}{}^{\circ 2}(x) = &+(a(\quad a\,)) \quad x \\
&+(\quad b(\quad a\;) \\
&\quad + a^2(\quad 1b\;)) \quad x^2 \\
&+(\quad c(\quad a\;.) \\
&\quad +2ab(\quad 1b\;\;) \\
&\quad + a^3(\quad 1c\;\;) \quad x^3 \\
&\ldots \\
&= a^2\,x \; + (ba + ba^2\,)x^2 + (ca + 2ab^2 + ca^3)\,x^3 + \ldots
\end{aligned}
$$

and a further reduction occurs for functions, where $a=1$:

(1.2.2.) $f_{K=0,a=1}(x) \quad = x + b\,x^2 + c\,x^3 + \ldots$

$f_{K=0,a=1}{}^{\circ 2}(x) \quad = x \; + 2b\,x^2 + (2c + 2b^2)\,x^3 + (b^3 + 5cb + 2d)x^4 + \;\ldots$

(see further development at 3.1)

Because of these reductions we discuss only such functions in the following examples, where we want to explain the general idea of the interpolation to fractional iterates. Only in the description of the powerseries and iteration of (U-) tetration we'll reintroduce functions with a general coefficient a.

1.3. Matrices composed by consecutive powers of formal powerseries

The concept of formal powerseries ***and their powers*** is needed for the study of functional iteration.

We may write and analyze their compositions step by step – but I find this a tedious method. If instead we collect the coefficients of a powerseries ***and its powers*** into a ***matrix***, such that we have

1	.	.	.	.	...
0	a	.	.	.	...
0	b	a^2	.	.	...
0	c	2ab	a^3	.	...
0	d	$2ac+b^2$	$3a^2b$	a^4	...
...	...	...	...	...	...

where in column *1* (counting begins at zero) occur the original coefficients of *f(x)*, in column *2* the coefficients of $(f(x))^2$ and so on, then this – together with the notation of matrix-algebra – allows us to make analysis of iteration much more readable: an explicite handling of the coefficents of higher iterates is very soon impossible otherwise.

So here is a short introduction into the matrix-notation of the coefficients of the formal powerseries.

First write the coefficients of the function *f* in a columnvector and the powers of *x* as a rowvector to denote the series as vector-multiplication.

Here I introduce a vector of consecutive powers of a variable *x* as a symbolic "vandermondevector" *V(x)*:

(1.3.1.) $V(x) = column(1,x,x^2,x^3,...)$

and with a column-vector A_1 of the original coefficients

(1.3.2.) $A_1 = column(K,a,b,c,d,...)$

we have then (where ~ denotes the transpose):

(1.3.3.) $f(x) = V(x)\sim * A_1$

Since for iteration we need the vectors $A_0,A_1,A_2,...$ of coefficients for the consecutive powers of *f(x)* as well, we may arrange them in a matrix ***A*** to write

(1.3.4.) $A = concatenate(A_0,A_1,A_2,...)$

and then

(1.3.5.) $[1, x, x^2, x^3,...] * A = [1, f(x), f(x)^2, f(x)^3, ...]$

The most interesting aspect of this is that we see that in the result vector we have the same structure of consecutive powers of a certain value (here of *f(x)*), so the output-vector is again of vandermonde-type and can be reused as new input vector again:

$V(x)\sim \quad * A = V(f(x))\sim$
$V(f(x))\sim * A = V(f(f(x))) \sim$
and so on.

The top-left of ***A*** for a general powerseries-defined function *f* looks like

(1.3.6.)

A =

1	K	K^2	K^3	K^4
0	a	$2Ka$	$3K^2a$	$4K^3\ a$
0	b	a^2+2Kb	$3Ka^2+3K^2b$	$6K^2a^2+4K^3\ b$
0	c	$2ab+2Kc$	$a^3+6Kab+3K^2c$	$4Ka^3+6K^2(2ab)+4K^3\ c$
0	d	$b^2+2ac+2Kd$	$3a^2b+3K(b^2+2ac)+3K^2d$	$a^4+12Ka^2b+6K^2(b^2+2ac)+4K^3\ d$

and in the columns we find the coefficients for the formal powerseries for $f(x)^0, f(x)^1, f(x)^2, ...$.

I call matrices, which can be used to transform a Vandermonde-vector (of an argument x) into another Vandermondevector (of another argument y of simply of the argument $f(x)$) a "***matrix-operator***" or simply "***operator***" here. So A is an operator, which transforms a Vandermondevector in x into one in $f(x)$, which is expressed in the matrix-formula (implying infinite size)

(1.3.7.) $V(x)\sim * A = V(f(x)) \sim$ // then A is called an "operator"

For a better picture I like to write examples for matrix-multiplications in a certain scheme, so we get

Example:

(1.3.8.)
$V(x)\sim * A = V(f(x))\sim$

A_0	A_1	A_2	A_3	A_4	...
1	K	K^2	K^3	K^4	...
0	a	2Ka	$3K^2a$	$4K^3a$	...
0	b	a^2+2Kb	$3Ka^2+3K^2b$	$6K^2a^2+4K^3b$	...
0	c	$2Kc+2ba$	$3K^2c+a^3+6Kba$	$4K^3c+4Ka^3+12K^2ba$	...
...	...	...	...	...	

*

$[1, x, x^2, x^3, ...] = [1, f(x), f(x)^2, f(x)^3, f(x)^4, ...]$

1.4. Iteration via matrix-operator

Iterations of $f(x)$ can then be computed, if in the lhs-vector $V(x)$, $f(x)$ is inserted for x and the matrix-multiplication is repeated.

(1.4.1.)
$V(x)\sim * A = V(f(x)) \sim$
$V(f(x))\sim * A = V(f(f(x))) \sim = (V(x)\sim * A)*A$
$V(f^{\circ h}(x))\sim * A = V(f^{\circ h+1}(x)) \sim = (((V(x)\sim * A)*A*...*A)*A$

By associativity we may rewrite this in terms of powers of A, again formally,

(1.4.2.)
$V(x)\sim * A = V(f(x)) \sim$
$V(x)\sim * A^2 = V(f(f(x))) \sim$
$V(x)\sim * A^h = V(f^{\circ h}(x)) \sim$

But now, since the computation of powers of A, (where A has infinite size in each dimension), means evaluation of infinite series itself, such operations require considerations of convergence on a new level.

1.5. K=0, triangular matrices

If in the powerseries for $f(x)$ the coefficient $K=0$ (also[1] written as $f(0)=0$), this simplifies considerably, and we get triangular matrices $A_{k=0}$ for such powerseries:

Example:

(1.5.1.)
$V(x)\sim * A_{k=0} = V(f_{k=0}(x))\sim$

	A_0	A_1	A_2	A_3	A_4
	1	.	.	.	.
	0	a	.	.	.
*	0	b	a^2	.	.
	0	c	2ab	a^3	.
	...	...	...	...	...

$[1, x, x^2, x^3, ...] = [1, f(x), f(x)^2, f(x)^3, f(x)^4, ...]$

Since we have finitely many terms in each row now[2], powers of A are not affected by new problems of non-convergence and the behave of the iteration of such functions is much easier to study. Infinite matrices of this type are called "rowfinite", and the "rowfiniteness" allows to compute the terms of powers of this matrix exactly up to the (finitely truncated) size of the matrix.

A typical property of these matrices is, that in the diagonal we have the consecutive powers of the second coefficient of the powerseries, a. Since the eigenvalues of a triangular matrix are its diagonal entries, we already know, that the eigenvalues of our triangular matrix-operator shall be the set of consecutive powers of a. (This shall be discussed in context with fractional and general continuous iteration).

Further reduced subcase a=1 (equivalent to "$f(0)=0$ and $f'(0)=1$ ")

A subcase of this $K=0$-type functions $f_{k=0}(x)$ are then functions, where $a=1$: then also the whole diagonal has the value 1, and powers of the matrix (and thus iterations of the function) are again more simple to discuss.

Example:

(1.5.2.)
$V(x)\sim * A_{k=0,a=1} = V(f_{k=0,a=1}(x))\sim$

	A_0	A_1	A_2	A_3	A_4
	1	.	.	.	.
	0	1	.	.	.
*	0	b	1	.	.
	0	c	2b	1	.
	...	...	...	...	...

$[1, x, x^2, x^3, ...] = [1, f(x), f(x)^2, f(x)^3, f(x)^4, ...]$

[1] see for instance [AF97]

[2] see "rowfiniteness" of infinite matrices in [???]

2. Examples for K=0 functions

2.1. Geometric series, a=1

2.1.1. The functional approach

The most simple example is the geometric series:

(1.1.1.1.) $f(x) = 1 x + 1 x^2 + 1 x^3 + ... = x/(1-x)$

The radius of convergence is $|x|<1$, and with the tools of divergent summation (Euler-summation) we may extend its domain to $x\le -1$. Also we know, that we can extend its domain to all $x\ne 1$ due to analytic continuation. But this shall be of concern only as a sidenote since primarily we want to study the expansion of iterations into formal powerseries and the conversion into a matrix-problem.

Iteration means to substitute x by $f(x)$ and this means application of the binomial-theorem.

First we get

$$\begin{aligned}(1.1.1.2.)\ f^{\circ 2}(x) &= 1 f(x) + 1 f(x)^2 + 1 f(x)^3 + ... \\ &= 1 (1 x + 1 x^2 + 1 x^3 + ...) \\ &+ 1 (1 x + 1 x^2 + 1 x^3 + ...)^2 \\ &+ 1 (1 x + 1 x^2 + 1 x^3 + ...)^3 \\ &+ ..\end{aligned}$$

Powers of $f(x)$ expanded

$$\begin{aligned}(1.1.1.3.)\ f^{\circ 2}(x) &= 1 (1 x + 1 x^2 + 1 x^3 + 1 x^4 + 1 x^5 + ...) \\ &+ 1 (1 x^2 + 2 x^3 + 3 x^4 + 4 x^5 + ...) \\ &+ 1 (1 x^3 + 3 x^4 + 6 x^5 + ...) \\ &+ 1 (1 x^4 + 4 x^5 + ...) \\ &+ ....\end{aligned}$$

Equal powers of x collected

(1.1.1.4.) $f^{\circ 2}(x) = 1 x + 2 x^2 + 4 x^3 + ... + 2^k x^{k+1} ...$

For $x=1/4$ we get

$$\begin{aligned}(1.1.1.5.)\ f(1/4) &= 1/4(1+1/4+1/4^2+...) = 1/4 * 1/(1-1/4) = 1/3 \\ f^{\circ 2}(1/4) = f(1/3) &= 1/3 (1+1/3+1/3^2+...) = 1/3*1/(1-1/3) = 1/2\end{aligned}$$

other way:

$$\begin{aligned}(1.1.1.6.)\ f^{\circ 2}(1/4) &= 1/4(1 + 2/4 + 2^2/4^2 + 2^3/4^3 + ...) \\ &= 1/4(1 + 1/2 + 1/2^2 + 1/2^3 + ...) = 1/4 * 2 = 1/2\end{aligned}$$

For the next iteration we need also the powers of $f^{\circ 2}(x)$ which becomes tedious to write down explicitely.

In a view of the closed form of the function $f(x) = x/(1-x)$

$$(1.1.1.7.)\ f^{\circ 2}(x) = \frac{f(x)}{1-f(x)} = \frac{\frac{x}{1-x}}{1-\frac{x}{1-x}} = \frac{\frac{x}{1-x}}{\frac{1-2x}{1-x}} = \frac{x}{1-x} * \frac{1-x}{1-2x} = \frac{x}{1-2x}$$

this means, that the powerseries is

(1.1.1.8.) $f^{\circ 2}(x) = 1*x + 2*x^2 + 4*x^3 + 8*x^4 + ... = x * ((2x)^0 + 2x + (2x)^2 + (2x)^3 + ...)$

and by induction this can then be generalized to any positive integer power:

$$(1.1.1.9.)\ f^{\circ h}(x) = \frac{x}{1-hx} = x(1+hx+(hx)^2+...) = f(hx)/h$$

We may rewrite this as a recursion formula

$$\begin{aligned}(1.1.1.10.)\ 1/f^{\circ 0}(1/x) &= x \\ 1/f^{\circ h+1}(1/x) &= 1 / f^{\circ h}(1/x) - 1\end{aligned}$$

Since h is here a simple parameter, from here we may also conclude a version of fractional iteration.

2.1.2. Matrix-approach

If we use the matrix-notation for the coefficients of all formal powerseries $f(x)^0, f(x), f(x)^2, \ldots$ as columns, we get the (shifted) pascal matrix ***P***[3] and then by matrix-multiplication the powers of $f(x)$:

(1.2.1.1.)
$$V(x)\sim * P = V(f(x))\sim$$

$$P = \begin{bmatrix} 1 & . & . & . & . \\ 0 & 1 & . & . & . \\ 0 & 1 & 1 & . & . \\ 0 & 1 & 2 & 1 & . \\ 0 & 1 & 3 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & x & x\text{^}2 & x\text{^}3 & x\text{^}4 \end{bmatrix} \qquad \begin{bmatrix} 1 & f(x) & f(x)\text{^}2 & f(x)\text{^}3 & f(x)\text{^}4 \end{bmatrix}$$

and for $x=1/4$ we get the geometric series with $q=1/4$

(1.2.1.2.) $f(1/4) = 1/4(1 + 1*1/4 + 1*1/4^2 + \ldots) = 1/4*(1/(1 - 1/4)) = 1/3$

The next iteration is perfomed as repeated matrix-multiplication using associativity of matrix operations

(1.2.1.3.)
$$\begin{aligned} &V(x) * P = V(f(x))\sim \\ &(V(x) * P)*P = V(x) * (P*P) = V(x)\sim*P^2 = V(f(f(x)))\sim \\ &\ldots \\ &V(x) * P^h = V(f^{\circ h}(x))\sim \end{aligned}$$

The coefficients of the formal powerseries $f^{\circ 2}(x)$ are then in the second column of ***P***^{2}

(1.2.1.4.)
$$V(x)\sim * P^2 = V(f^{\circ 2}(x))\sim$$

$$P^2 = \begin{bmatrix} 1 & . & . & . & . \\ 0 & 1 & . & . & . \\ 0 & 2 & 1 & . & . \\ 0 & 4 & 4 & 1 & . \\ 0 & 8 & 12 & 6 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & x & x\text{^}2 & x\text{^}3 & x\text{^}4 \end{bmatrix} \qquad \begin{bmatrix} 1 & f^2(x) & f^2(x)\text{^}2 & f^2(x)\text{^}3 & f^2(x)\text{^}4 \end{bmatrix}$$

and this agrees with the formula in the previous paragraphs.

Inverse iteration

The simpliness of the function $f(x)$ and of the associated matrix allows to introduce inverse iteration in a few lines of text.

By the functional approach we ask:

(1.2.1.5.) $f(f^{\circ -1}(x)) = x$

Setting y for $f^{\circ -1}(x)$

(1.2.1.6.)
$$\begin{aligned} y/(1-y) &= x \\ 1/(1/y-1) &= x \\ 1/y - 1 &= 1/x \\ 1/y &= 1+1/x = (x+1)/x \\ y &= x/(1+x) \end{aligned}$$

and this is then

(1.2.1.7.) $f^{\circ -1}(x) = x(1 - 1*x + 1*x^2 - 1*x^3 + \ldots\text{-}\ldots)$

By the matrix-approach, the inverse, or better (since we have the case of infinite size) a matrix-reciprocal, is defined, if

(1.2.1.8.) $P * P^{-1} = I$

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[3] In my other texts I refer to the unshifted Pascalmatrix as ***P***. I do it here only in this example for simplicitiness. I hope, this does not introduce too much confusion

can be found.

Since ***P*** is triangular, the reciprocal must also be triangular

(1.2.1.9.)

$$\begin{array}{cc} & \underbrace{\begin{bmatrix} ? & . & . & . & . \\ ? & ? & . & . & . \\ ? & ? & ? & . & . \\ ? & ? & ? & ? & . \\ ? & ? & ? & ? & ? \end{bmatrix}}_{P^{-1}} \\ \underbrace{\begin{bmatrix} 1 & . & . & . & . \\ 0 & 1 & . & . & . \\ 0 & 1 & 1 & . & . \\ 0 & 1 & 2 & 1 & . \\ 0 & 1 & 3 & 3 & 1 \end{bmatrix}}_{P} * & = \underbrace{\begin{bmatrix} 1 & . & . & . & . \\ . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 1 \end{bmatrix}}_{I} \end{array}$$

This system of equations can iteratively be solved by evaluating rowwise to get

(1.2.1.10.)

$P*P^{-1} = I$

$$\begin{array}{cc} & \underbrace{\begin{bmatrix} 1 & . & . & . & . \\ 0 & 1 & . & . & . \\ 0 & -1 & 1 & . & . \\ 0 & 1 & -2 & 1 & . \\ 0 & -1 & 3 & -3 & 1 \end{bmatrix}}_{P^{-1}} \\ \underbrace{\begin{bmatrix} 1 & . & . & . & . \\ 0 & 1 & . & . & . \\ 0 & 1 & 1 & . & . \\ 0 & 1 & 2 & 1 & . \\ 0 & 1 & 3 & 3 & 1 \end{bmatrix}}_{P} * & = \underbrace{\begin{bmatrix} 1 & . & . & . & . \\ . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 1 \end{bmatrix}}_{I} \end{array}$$

and in the second column of $\boldsymbol{P}^{-1}$ we find the same coefficients as in the functional approach above.

The inverse function is then $f^{\circ-1}(x)$

(1.2.1.11.) $V(x)\sim * P^{-1} = V(f^{\circ-1}(x))\sim$

which occurs obviously, if the multiplication-scheme with $\boldsymbol{P}^{-1}$ is displayed (I abbreviated $f^{\circ-1}(x)$ by $F(x)$ here because of limitations of the bitmap):

(1.2.1.12.)

$V(x)\sim * P^{-1} = V(f^{\circ-1}(x))\sim \; = V(F(x))\sim$

$$\begin{array}{cc} & \underbrace{\begin{bmatrix} 1 & . & . & . & . \\ 0 & 1 & . & . & . \\ 0 & -1 & 1 & . & . \\ 0 & 1 & -2 & 1 & . \\ 0 & -1 & 3 & -3 & 1 \end{bmatrix}}_{P^{-1}} \\ \begin{bmatrix} 1 & x & x^2 & x^3 & x^4 \end{bmatrix} * & = \begin{bmatrix} 1 & F(x) & F(x)^2 & F(x)^3 & F(x)^4 \end{bmatrix} \end{array}$$

For $x=1/4$ we get

(1.2.1.13.) $f^{\circ-1}(1/4) = 1/4 \,/\, (1 + 1/4) = 1/(4+1) = 1/5$

Since this is assumed as the result of the inverse operation, we should get $1/4$ by applying the iteration on the result:

(1.2.1.14.) $f(1/5) = 1/5*(1 + 1/5 + (1/5^2 + \dots) = 1/5* 1/(1-1/5) = 1/5 * 5/4 = 1/4$

which is the expected result.

2.1.3. Fractional and general continuous iteration

The computation of the reciprocal in the previous is just a special case, of first negative power. This shall now be generalized to arbitrary powers / iteration heights.

Fractional powers of ***P*** are not obviously constructable. For finite matrices we have three options:

a) (meaningful) interpolation of the consecutive integer powers of ***P***
b) use of Matrix-logarithm
c) Eigensystem-decomposition

For the matrix ***P*** we even have a fourth option

d) similarity scaling by diagonal-matrices (comes out the be equivalent to matrix-logarithm-method)

Since ***P*** has a degenerated eigensystem, option c) is not applicable here.

a) Interpolation of matrix-powers

Since we know, that the coefficients of the powerseries of $f^{\circ h}(x)$ occur in the 2'nd column of the h'th power of ***P***, we may collect all these columns and try, whether we can find a meaningful interpolation based on the progression of coefficients in equal rows.

A list of these second columns for iteration heights $h=0,1,2,3,4,5,...$ is

(1.3.1.1.)

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 4 & 9 & 16 & 25 \\ 0 & 1 & 8 & 27 & 64 & 125 \\ 0 & 1 & 16 & 81 & 256 & 625 \end{bmatrix}$$

Now we might apply a technique to find continuous polynomials in h, which interpolate each row, beginning with index $h=0$. What we get by any polynomial interpolation-procedure is the following matrix of coefficients for polynomials in h, where the first column is associated with h^0, the second column with h^1 and so on

(1.3.1.2.)

$$Poly = \begin{bmatrix} 0 & . & . & . & . & . \\ 1 & 0 & . & . & . & . \\ 0 & 1 & 0 & . & . & . \\ 0 & 0 & 1 & 0 & . & . \\ 0 & 0 & 0 & 1 & 0 & . \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

This is a very simple solution; it means, that for the interpolation of the rows we have for the h'th entry:

$row_1[h] = 1$
$row_2[h] = 0 + 1*h$
$row_3[h] = 0 + 0 \quad + 1*h^2$
...

and since the entries in the r'th row in ***L*** are the coefficients of x^r in the powerseries of $f^{\circ h}(x)$, we can construct this powerseries by

(1.3.1.3.) $$f^{\circ h}(x) = 1x + hx^2 + h^2 x^3 + ... = x(1 + hx + (hx)^2 + (hx)^3 + ...)$$

as we had by expansion and collection with the method of recursive series-substitution for the integer case.

Since the coefficients in ***Poly*** represent interpolation-polynomials, we may feel enabled to declare this as one meaningful interpolation-technique for fractional or even continuous and complex h, which means then the same type of iteration as well as an interpolation-technique for the coefficients in 2'nd column of an arbitrary power of ***P***.

So, $f^{\circ 1/2}(x)$ and its associated powerseries may now simply be determined by inserting $h=1/2$ in the above formula.

b) Use of matrix-logarithm

The logarithm of a scalar $log(1+x)$ is defined by a powerseries

(1.3.1.4.) $log(1+x) = x/1 - x^2/2 + x^3/3 - x^4/4 ...$
$log(x) = (x-1)/1 - (x-1)^2/2 + (x-1)^3/3 - ...$

and a fractional h'th power is then defined as

(1.3.1.5.) $x^h = exp(h * log(x))$

The formula of the powerseries for the logarithm as well for the exponential can formally be extended to have matrices as their argument.

(1.3.1.6.) $log(P) = (P-I)/1 - (P-I)^2/2 + (P-I)^3/3 - ...$

Since the diagonal of $(\mathbf{P}-\mathbf{I})$ is zero, $(\mathbf{P}-\mathbf{I})$ is nilpotent to the order of its size and we may approximate/extrapolate the case of infinite size by finite matrices of increasing size. We will always have only finitely many terms in the logarithm-series for consecutive sizes, and increasing the size does not affect the earlier computed results:

(1.3.1.7.)

$log(P_{2x2}) = (P_{2x2} - I)/1 - (P_{2x2} - I)^2/2 + \mathbf{0} - \mathbf{0} + ... - ....$ $\begin{bmatrix} 0 & . \\ . & 0 \end{bmatrix}$

$log(P_{3x3}) = (P_{3x3} - I)/1 - (P_{3x3} - I)^2/2 + (P_{3x3} - I)^3/3$ $\begin{bmatrix} 0 & . & . \\ 0 & 0 & . \\ 0 & 1 & 0 \end{bmatrix}$

$log(P_{4x4}) = \Sigma_{k=1..4} (-1)^{k-1} * (P_{4x4} - I)^k/k$ $\begin{bmatrix} 0 & . & . & . \\ 0 & 0 & . & . \\ 0 & 1 & 0 & . \\ 0 & 0 & 2 & 0 \end{bmatrix}$

$log(P_{5x5}) = \Sigma_{k=1..5} (-1)^{k-1} * (P_{5x5} - I)^k/k$ $\begin{bmatrix} 0 & . & . & . & . \\ 0 & 0 & . & . & . \\ 0 & 1 & 0 & . & . \\ 0 & 0 & 2 & 0 & . \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix}$

We may multiply $log(P)$ by an arbitray constant h and compute the exponential again. This can also be done symbolically and we get

(1.3.1.8.)
$P^h = exp(h*log(P)) =$ $\begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & h & 1 & . & . & . \\ 0 & h^2 & 2*h & 1 & . & . \\ 0 & h^3 & 3*h^2 & 3*h & 1 & . \\ 0 & h^4 & 4*h^3 & 6*h^2 & 4*h & 1 \end{bmatrix}$ P^h

from which we find, using

(1.3.1.9.) $V(x)\sim * P^h = V(f^{\circ h}(x))\sim$

and the vectorial product of the rowvector $V(x)\sim$ with the columns of $\mathbf{P}^h$ that

(1.3.1.10.) $f^{\circ h}(x)^0 = 1$
$f^{\circ h}(x)^1 = 0 + 1x + hx^2 + h^2x^3 + ... = x(1 + 1hx + 1(hx)^2 + 1(hx)^3 + ...)$
$f^{\circ h}(x)^2 = 0 + 0x + 1x^2 + 2hx^3 + ... = x^2(1 + 2hx + 3(hx)^2 + 4(hx)^3 + ...)$
$f^{\circ h}(x)^3 = 0 + 0x + 0x^2 + 1x^3 + ... = x^3(1 + 3hx + 6(hx)^2 + 10(hx)^3 + ...)$
...

where the expression for $f^{\circ h}(x)^1 = f^{\circ h}(x)$ is the same result as we got with the computation via iterated substitution of the powerseries and the polynomial matrix-interpolation.

d) similarity scaling

Using the binomial-theorem one can also show, that the similarity-scaling by a diagonal-vector holds:

(1.3.1.11.) $P^h = {}^dV(h) * P * {}^dV(h)^{-1}$

Again fractional and even complex iterates can be defined, since the similarity scaling of ***P*** to get powers of itself needs only diagonal matrices. For diagonal-matrices any complex power is defined by the same power of the scalar entries of its diagonal. So we may describe any fractional or even complex "height" *h* of iteration by the matrix-formula

(1.3.1.12.)
$$\begin{aligned} P^h &= {}^dV(h) * P * {}^dV(h)^{-1} \\ V(x)\sim * P^h &= V(x)\sim * {}^dV(h) * P * {}^dV(1/h) \\ &= V(hx)\sim * P * {}^dV(1/h) \end{aligned}$$

and since $f^{\circ h}(x)$ occurs in the second column of the result (column-index *1*)

(1.3.1.13.) $f^{\circ h}(x) \quad = V(f^{\circ h}(x))\sim [1]$

it is also

(1.3.1.14.)
$$\begin{aligned} f^{\circ h}(x) &= V(hx)\sim *P\ [1]\ *1/h \\ &= V(hx)\sim * [0,1,1,....]\sim /h \end{aligned}$$

(1.3.1.15.)
$$\begin{aligned} &= 1*0 + hx/h + (hx)^2/h + (hx)^3/h + ... \\ &= \quad x(1 + (hx) + (hx)^2 + ...) \end{aligned}$$

for any fractional or continuous value of *h*. Note, that the similarity-scaling is essentially equivalent to the matrix-logarithm-method, since

(1.3.1.16.) ${}^dV(h)*P*{}^dV(1/h) = {}^dV(h)*exp(log(P))*{}^dV(1/h) = exp({}^dV(h)*log(P)*{}^dV(1/h))$

and the inner part comes out the be equivalent to the scalar-multiplication of *log(**P**)* by *h*, since *log(**P**)* is the first principal subdiagonal matrix.

Conclusion

All shown methods ***a)***, ***b)***, ***d)*** give the same result for the fractional iteration for the geometric series

(1.3.1.17.) $f(x) = x + x^2 + x^3 + \dots$
$f^{\circ h}(x) = x(1 + (hx) + (hx)^2 + (hx)^3 + \dots$

such that, for instance for $x=1/2$ the first half-iterate is

(1.3.1.18.) $f^{\circ 1/2}(1/2) = 1/2\ (1 + 1/4 + 1/4^2 + \dots) = 1/2/(1-1/4) = 1/2\ /(3/4) = 2/3$

and the next half-iterate is

(1.3.1.19.) $f^{\circ 1/2}(2/3) = 2/3\ (1 + 1/3 + 1/3^2 + \dots) = 2/3/(1-1/3) = 2/3\ /(2/3) = 1$

From the view of iteration we have then the following table

(1.3.1.20.) $f^{\circ 0}\ (1/2) = 1/2 \qquad f^{\circ 1/2}(1/2) = 2/3 \qquad f^{\circ 2/2}(1/2) = 1$

where the last equation shows coincidence with

(1.3.1.21.) $f^{\circ 2/2}(1/2) = 1 = f^{\circ 1}(1/2)$

which can be generalized to all x and h, as far convergence of the final powerseries in x is given.

Also a first general law for the iterator-index can be formulated (additivity of iterations):

(1.3.1.22.) $f^{\circ a}(f^{\circ b}(x)) = f^{\circ a+b}(x)$

which is compatible with the matrix-power-approach:

(1.3.1.23.) $V(x)\sim * P^a = V(f^{\circ a}(x))\sim$
$(V(x)\sim * P^a\ ) *P^b = V(x)\sim * (P^a *P^b)$
$= V(x)\sim * P^{a+b}$
$= V(f^{\circ a+b}(x))\sim$

according to the general rules of matrix-algebra and is fundamental for this approach to fractional iteration.

Note, that the various methods, interpolation of powers, matrix-logarithm and similarity-scaling allow (and provide) only interpolation based on fractional **powers** of the matrix, giving polynomials and powerseries of a parameter h, and is only "exact" as far as the coefficient a in the basic powerseries is $a=1$. For other conditions (and thus matrices) they are possibly uncomfortable and require considerations of convergence in the sequence of matrix-powers itself. The eigensystem-approach, if it is available, gives more flexibility and the convergence-criteria are much more obvious.

2.2. *The sine-function,* a=1

The formal powerseries for the sine-function is

(2.2.1.1.) $\sin(x) = x/1! - x^3/3! + x^5/5! - x^7/7! + \dots - \dots$

its coefficients can be written as infinite vector

(2.2.1.2.) $[0, 1, 0, -1/3!, 0, 1/5!, 0, -1/7!, \dots]$

So we use another example of the type $K=0$, $a=1$ (or $f(0)=0$,$f'(0)=1$). The matrix-operator, computed by the method described above is

(2.2.1.3.) $$SIN = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 0 & 1 & . & . & . \\ 0 & -1/6 & 0 & 1 & . & . \\ 0 & 0 & -1/3 & 0 & 1 & . \\ 0 & 1/120 & 0 & -1/2 & 0 & 1 \end{bmatrix}$$

Its application as coefficients for powerseries in x gives the powers of $\sin(x)$:

(2.2.1.4.) $V(x)\sim * SIN = V(\sin(x))\sim = [1, \sin(x), \sin(x)^2 , \sin(x)^3, \dots]$

which is iterable, since the form of the result is of the same form as the left-multiplicator: a vandermonde vector consisting of consecutive powers of a parameter.

So we extend the notation for the iterable sin-function to

(2.2.1.5.) $\sin^{\circ h}(x) = \sin(\sin(\dots\sin(x)\dots))$ //h iterations

For instance, the second power of ***SIN*** gives $\sin^{\circ 2}(x) = \sin(\sin(x))$, with an actual value for x, $x=1$ this should give

(2.2.1.6.) $\sin^{\circ 2}(1) = 0.745624141666\dots$

The second power of ***SIN*** looks like

(2.2.1.7.) $$SIN^2 = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 0 & 1 & . & . & . \\ 0 & -1/3 & 0 & 1 & . & . \\ 0 & 0 & -2/3 & 0 & 1 & . \\ 0 & 1/10 & 0 & -1 & 0 & 1 \end{bmatrix}$$

and used as matrix-operator this gives

(2.2.1.8.)

$$V(1)\sim *SIN^2 = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 0 & 1 & . & . & . \\ 0 & -1/3 & 0 & 1 & . & . \\ 0 & 0 & -2/3 & 0 & 1 & . \\ 0 & 1/10 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1.000 & 1.000 & 1.000 & 1.000 & 1.000 \end{bmatrix} \quad \begin{bmatrix} 1 & 0.7456 & 0.5560 & 0.4145 & 0.3091 & 0.2304 \end{bmatrix}$$

where the second column of the result is the above value $\sin^{\circ 2}(1) = 0.7456$ and the remaining entries its consecutive powers.

The inverse of ***SIN*** should give the coefficients for the powerseries of $\sin^{\circ -1}(x)$ (or $\arcsin(x)$) and the top left looks like:

(2.2.1.9.) $$SIN^{-1} = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 0 & 1 & . & . & . \\ 0 & 1/6 & 0 & 1 & . & . \\ 0 & 0 & 1/3 & 0 & 1 & . \\ 0 & 3/40 & 0 & 1/2 & 0 & 1 \end{bmatrix}$$

where the entries of the second column are that of the Taylor-series for $\sin^{-1}(x)$ according to the "Handbook of mathematical functions" *([A&S] pg 81)*

(2.2.1.10.) $\sin^{-1}(x) = x + 1/(2*3)\, x^3 + (1*3)/(2*4*5)\, x^5 + (1*3*5)/(2*4*6*7)\, x^7 + \dots$

The continuous version of height/iteration

The polynomial-interpolation-approach gives the matrix of coefficients ***POLY*** for the interpolation-polynomials (read by rows, first column associated to h^0)

(2.2.1.11.) $$POLY = \begin{bmatrix} 0 & . & . & . & . & . \\ 1 & 0 & . & . & . & . \\ 0 & 0 & 0 & . & . & . \\ 0 & -1/6 & 0 & 0 & . & . \\ 0 & 0 & 0 & 0 & 0 & . \\ 0 & -1/30 & 1/24 & 0 & 0 & 0 \end{bmatrix}$$

so, assuming coefficients $a_{h,k}$ as entries of a vector A_h building the powerseries

(2.2.1.12.) $\sin^{\circ h}(x) = a_{h,1}\, x + a_{h,3}\, x^3 + a_{h,5}\, x^5 + \dots = V(x)\sim * A_h$

then

(2.2.1.13.) $a_{h,1} = 1 \qquad a_{h,3} = -1/6\, h \qquad a_{h,5} = -1/30\, h + 1/24\, h^2$

and so on. The powerseries for general h is then

(2.2.1.14.)
$$\begin{aligned} \sin^{\circ h}(x) = {} & 1 * x \\ & + (0 - 1/6\ h) * x^3 \\ & + (0 - 1/30\ h + 1/24\ h^2) * x^5 \\ & + (0 - 41/378\ h + 1/45\ h^2 - 5/432\ h^3) * x^7 \\ & + (0 - 4/945\ h + 67/5670\ h^2 - 71/6480\ h^3 + 35/10368\ h^4) * x^9 \\ & \dots \end{aligned}$$

The half-iterate uses $h=1/2$ and we get the powerseries

(2.2.1.15.) $\sin^{\circ 1/2}(x) = x - 1/12\, x^3 - 1/160\, x^5 - 53/40320\, x^7 - 23/71680\, x^9 - 92713/1277337600\, x^{11} + O(x^{17})$

For $x=1$ we get approximately

(2.2.1.16.)
$$\begin{aligned} \sin^{\circ 1/2}(1) &= 0.908708\dots \\ \sin^{\circ 1/2}(0.908708\dots) &= 0.841471.. \\ &= \sin^{\circ 2/2}(1) = \sin^{\circ 1}(1) = \sin(1). \end{aligned}$$

This agrees with the direct computation $\sin(1) = 0.841471\dots$

Interpolation using the matrix-logarithm

The matrix-logarithm of ***SIN*** can easily be determined, since its diagonal contains the unit only and so the matrix-terms for the logarithm-series are nilpotent to the order of matrix-size. The top-left edge of the matrix-logarithm is then:

(2.2.1.17.) $$\log(SIN) = \begin{bmatrix} 0 & . & . & . & . & . \\ 0 & 0 & . & . & . & . \\ 0 & 0 & 0 & . & . & . \\ 0 & -1/6 & 0 & 0 & . & . \\ 0 & 0 & -1/3 & 0 & 0 & . \\ 0 & -1/30 & 0 & -1/2 & 0 & 0 \end{bmatrix}$$

and the general h'th power of SIN according to $\mathbf{SIN}^h = \exp(h*\log(\mathbf{SIN}))$ is

(2.2.1.18.) $$\exp(h*\log(SIN)) = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 0 & 1 & . & . & . \\ 0 & -1/6*h & 0 & 1 & . & . \\ 0 & 0 & -1/3*h & 0 & 1 & . \\ 0 & 1/24*h\verb|^|2-1/30*h & 0 & -1/2*h & 0 & 1 \end{bmatrix}$$

Here the entries of the second column provide the (polynomial) coefficients for the $\sin^{\circ h}()$-powerseries in x and we find, that this agrees with the solution by polynomial interpolation.

Then the half-power ***SIN***$^{1/2}$ is

(2.2.1.19.) $SIN^{1/2} = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 0 & 1 & . & . & . \\ 0 & -1/12 & 0 & 1 & . & . \\ 0 & 0 & -1/6 & 0 & 1 & . \\ 0 & -1/160 & 0 & -1/4 & 0 & 1 \end{bmatrix}$

The powerseries for $sin^{\circ 1/2}(x)$ according to this is as in the previous example

(2.2.1.20.) $sin^{\circ 1/2}(x) = x - 1/12\ x^3 - 1/160\ x^5 - 53/40320\ x^7 - 23/71680\ x^9 - 92713/1277337600\ x^{11} + O(x^{17})$

and we get from the matrix-formula in numbers

(2.2.1.21.)

$V(1)\sim * SIN^{1/2} = V(y)\sim$
$V(y)\sim * SIN^{1/2} = V(z)\sim$

$y = sin^{\circ 1/2}(1) \qquad z = sin^{\circ 2/2}(1)$

$\begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 0 & 1 & . & . & . \\ 0 & -1/12 & 0 & 1 & . & . \\ 0 & 0 & -1/6 & 0 & 1 & . \\ 0 & -1/160 & 0 & -1/4 & 0 & 1 \end{bmatrix}$

$[\ 1 \quad 1.000 \quad 1.000 \quad 1.000 \quad 1.000 \quad 1.000\ ] \qquad [\ 1.000 \quad 0.9087 \quad 0.8258 \quad 0.7504 \quad 0.6819 \quad 0.6196\ ]$

Next iteration:

$[\ 1.000 \quad 0.9087 \quad 0.8258 \quad 0.7504 \quad 0.6819 \quad 0.6196\ ] \qquad [\ 1 \quad 0.8415 \quad 0.7081 \quad 0.5958 \quad 0.5014 \quad 0.4219\ ]$

The inverse of *sin(x)* (= $sin^{-1}(x)$, *arcsin(x)* or "negative height" *–h*)

In the previous paragraphs we also introduced the matrix ***SIN***$^{-1}$ by use of matrix-logarithm and *h=-1*

(2.2.1.22.) $SIN^{-1} = exp(- log(SIN))$

(2.2.1.23.) $SIN^{-1} = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 0 & 1 & . & . & . \\ 0 & 1/6 & 0 & 1 & . & . \\ 0 & 0 & 1/3 & 0 & 1 & . \\ 0 & 3/40 & 0 & 1/2 & 0 & 1 \end{bmatrix}$

where the second column precisely provides the known coefficients for *arcsin*-function, for instance according to Abramowitsch/Stegun (*Pg 81*):

4.4.40

$$\arcsin z = z + \frac{z^3}{2\cdot 3} + \frac{1\cdot 3 z^5}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5 z^7}{2\cdot 4\cdot 6\cdot 7} + \dots \qquad (|z|<1)$$

3. Symbolic fractional iteration

3.1. Formula for interpolation for a general powerseries $f(x)= 1 x + b x^2 + c x^3 + \dots$

We restrict ourselves to the case $K=0$, $a=1$ here (or: $f(0)=0, f'(0)=1$)

(3.1.1.1.) $f(x) = 1 x + b x^2 + c x^3 + d x^4 + e x^5 + \dots$

First we build a table of differences of the second columns of powers of the associated matrix-operator, see table below.

In the first row we get the original coefficients of the powerseries of

(3.1.1.2.)
$$f^{\circ 0}(x) = 1 x$$
$$f^{\circ 1}(x) = 1 x + 1b x^2 + 1c x^3 + \dots$$
$$f^{\circ 2}(x) = 1 x + 2b x^2 + \dots$$
...

where the table-columns are associated with the according powers of the iteration-height h.

In the subsequent rows the forward-differences of the coefficients, where the forward-differences are of the order of rowindex r.

We find mixtures of differences of different order of progression for each symbolic coefficient (and their composites), where I highlighted different orders of progression by different colors.

(3.1.1.3.) *Table of differences of symbolic coefficients along iterates of $f^{\circ h}(x)$, using a=1*

diff. index	$f^{\circ 0}(x)$	$f^{\circ 1}(x)$	$f^{\circ 2}(x)$	$f^{\circ 3}(x)$
Δ^0	$1 x$ 0 0 0 0	$1 \ {}^*x$ $1b * x^2$ $1c * x^3$ $1d * x^4$ $1e * x^5$ ...	$1 \ {}^*x$ $(2b) * x^2$ $(2c+2b^2) * x^3$ $(2d+5bc+b^3) * x^4$ $(2e+3(2bd+c^2)+5b^2c) * x^5$ ...	$1 x$ $(3b) * x^2$ $(3c + 6b^2) * x^3$ $(3d + 15bc + 9b^3) * x^4$ $(3e + 9(2bd + c^2) + 41b^2c + 10b^4) * x^5$ . ...
Δ^1		$0 x$ $1b x^2$ $1c x^3$ $1d x^4$ $1e x^5$	$0 x$ $(1b) x^2$ $(1c + 2b^2) * x^3$ $(1d + 5bc + b^3) * x^4$ $(1e + 3(2bd + c^2) + 5b^2c) * x^5$	$0 x$ $(1b) x^2$ $(1c + 4b^2) * x^3$ $(1d + 10bc + 8b^3) * x^4$ $(1e + 6(2bd + c^2) + 36b^2c + 10b^4) * x^5$
Δ^2			$0 x$ $0 x^2$ $(2b^2) x^3$ $(5bc + b^3) * x^4$ $(3(2bd+c^2)+5b^2c) * x^5$	$0 x$ $0 x^2$ $(2bb) * x^3$ $(5bc + 7b^3) * x^4$ $(3(2bd + c^2) + 31b^2c + 10b^4) * x^5$
Δ^3				$0 x$ $0 x^2$ $0 \ {}^*x^3$ $(6b^3) * x^4$ $(26b^2c + 10b^4) * x^5$

Legend: linear progression quadratic progression cubic progression biquadratic progression of coefficients

If we write the numeric coefficients at each (combination) of the symbolic coefficients as function of the iteration-index h (which indexes also the columns in the table), then this gives – as denoted in the last column, where we order for like (compositions of) symbolic coefficients – the following:

diff. index	$f^{\circ 0}(x)$	$f^{\circ 1}(x)$	$f^{\circ 2}(x)$	$f^{\circ h}(x)$
Δ^0				$1\,x$ $(hb\,) * x^2$ $(hc + h(h-1)b^2) * x^3$ $(hd + 5/2h(h-1)bc \qquad + h(h-1)(2h-3)b^3) * x^4$ $(he + 3/2h(h-1)(2bd + c^2) + h(h-1)(26h-37)b^2c + h(h-1)(h-2)(3h-4)b^4) * x^5$ . ...

The same, collected for like powers of h instead:

(3.1.1.4.) *Y = (matrix of polynomials for coefficients at powers of x dependend on h):*

	$*h^0$	$*h$	$*h^2$	$*h^3$	$*h^4$	
x^0	.	.	.	.	.	
x *	1					
x^2 *	0	b				/0!
x^3 *	0	$- b^2$ $+ c$	b^2			/1!
x^4 *	0	$+ 3b^3$ $- 5bc$ $+ 2d$	$- 5b^3$ $+ 5bc$	$2b^3$		/2!
x^5 *	0	$- 16b^4$ $+ 37b^2c$ $- 18bd$ $- 9c^2$ $+ 6e$	$+ 36b^4$ $- 63b^2c$ $+ 18bd$ $+ 9c^2$	$-26b^4$ $+26b^2c$	$6b^4$	/3!

The latter table defines the bivariate powerseries of $f^{\circ h}(x)$, when $f(x) = 1\,x + b\,x^2 + c\,x^3 + \dots$

(3.1.1.5.)
$$\begin{aligned} f^{\circ h}(x) = \; & 1 * x \\ & + \mathbf{(}\, b * h \,\mathbf{)} \;\; * \; x^2 \\ & + \mathbf{(}(-b^2 + c) * h + \;\; b^2 * h^2 \,\mathbf{)} \;\; * \; x^3 / 1! \\ & + \mathbf{(}(3b^3 - 5bc + 2d) * h + (-5b^3 + 5bc) * h^2 + 2b^3 * h^3 \,\mathbf{)} * \; x^4 / 2! \\ & + \dots \end{aligned}$$

whose coefficients at powers of x are polynomials in h .

The highest exponent of h at x^{k+1} is k, but may be lower, if some of the coefficients $b,c,d,\dots$ in the basic formal powerseries are zero. Because of this we might expect, that the convergence-radius of the powerseries of $f^{\circ h}(x)$ decreases with increasing h roughly proportionally, so if the convergence-radius μ for $f^{\circ h}(x)$ is $\mu_h = c/h$ then it is $\mu_{h+1} \sim c/(h+1)$. (This is not perfectly true, since we observe also increasing coefficients with the powers of h for higher powers of x, but I didn't investigate these progressions deeper)

I didn't find simpler rules for extrapolation of the polynomials so far, I can construct them only by the according matrix-operations. Their complicated structure needs enormous amounts of memory if only matrices for size 64×64 should be stored. A file for the symbolic representation of coefficients in textfomat was about *200 Mb* diskspace, and since matrices of this size allow only powerseries of *64* terms (which gives good approximations only for a small range of its parameters), it seems better to compute the coefficients numerically for a current function f and possibly also either for a current fixed iteration-height h (keeping only x as variable) or for a fixed x (keeping only h as variable)

Also remember, that this polynomial-interpolation-approach (as well as the matrix-logarithm-approach) is only exact for powerseries with $a=1$ (and $K=0$), which is a small subset of interesting functions. For instance, for (**T**-) tetration this means restriction to base $e^{1/e}$ and for **U**-tetration to base e.

4. Fractional and continuous iteration for Tetration

4.1. U-Tetration

4.1.1. Definition

I call **U**-tetration, for what Andrew Robbins[4] proposed the canonical name "*decremented iterated exponential*", for shortness here. It is, for a base t, defined by the function:

(4.1.1.1.) $f_t(x) = t^x - 1$ (tetration-forum:= $dxp_t(x)$)

and I use the letter ***U*** here for better reading

(4.1.1.2.) $U_t(x) = t^x - 1$

If the base-parameter $t=e = exp(1)$ I abbreviate this to $U(x)$ simply. Also, for the logarithm of the base-parameter t I write usually the small letter u, so

(4.1.1.3.) $u=log(t)$

(Note: there exists a statement of Erdös/Jabotinsky contradictory to the possibility of real iterates for fractional heights, see footnote[5])

4.1.2. Function and matrix-operator for U_e-tetration

For $\mathbf{U_e}$-tetration the function $U(x)$ resp its iteration $U^{\circ h}(x)$ is defined as follows:

(4.1.2.1.) $U(x) = U_e(x) = exp(x)-1$
$U^{\circ h}(x) = U^{\circ h-1}(exp(x))-1$
$U^{\circ 0}(x) = x$

and the powerseries for $U(x)$ is just the exponential-series in x, where the constant is removed

(4.1.2.2.) $U(x) = 1/1!\, x + 1/2!\, x^2 + 1/3!\, x^3 + ...$

so the symbolic coefficients $a,b,c,d,...$ as in ***(1.1)*** are

(4.1.2.3.) $a=1, \quad b=1/2!, \quad c=1/3!, ...$

The matrix ***U*** (for the primary function $U(x)=U^{\circ 1}(x)$) is (I give only the top-left truncation here and in the following):

(4.1.2.4.) $$\mathbf{U} = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 1/2 & 1 & . & . & . \\ 0 & 1/6 & 1 & 1 & . & . \\ 0 & 1/24 & 7/12 & 3/2 & 1 & . \\ 0 & 1/120 & 1/4 & 5/4 & 2 & 1 \end{bmatrix}$$

which –in my usual notation- equals just the matrix ***S2*** (a factorially similarity-scaling of the matrix of Stirling-numbers *2*'nd kind with offset as given for instance in the wikipedia-definition). The third column gives the coefficients for the powerseries in x for $U(x)^2$, the fourth column those for $U(x)^3$ and so on. So we have, in matrix-notation

(4.1.2.5.) $\mathbf{V}(x)\sim * \mathbf{U} = \mathbf{V}(U(x)) \sim$

---

[4] see [RO08]

5 Erdös/Jabotinski state in [EJ61], there are "*no real non-integer iterates*" for $f(x)=e^x-1$ (means: no real value for fractional height for $\mathbf{U_e}$-tetration), attributing this to I.N.Baker in [BA58]. However, Baker only states, that "*the radius of convergence*" of the powerseries in x for noninteger heights and base=e "*is zero*". Here the issue is not nonexistence, but convergence. Moreover, using the well developed concept of divergent summation we may extend the domain for h and x beyond the classical radius of convergence. A heuristic inspection of the coefficients suggest, that the absolute value of terms is asymptotically of order $exp(r^2)$, where r is the index of term (row-index of matrix). This would mean, that the resulting powerseries in x cannot be Euler-, but possibly be Borel-summed (see [KN]).

or more explicitely

$[1,x, x^2, x^3,...] * \mathbf{U} = [1, U(x), U(x)^2, U(x)^3, ...]$

In Abramowitz/Stegun we find exactly this (however without the notion of a matrix) in the formulae for stirlingnumbers 2nd kind.

4.1.3. The polynomial-interpolation approach to U_e-tetration

The list of *2*'nd columns of the consecutive powers of $\mathbf{U}$ is

(4.1.3.1.) $$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1/2 & 1 & 3/2 & 2 \\ 0 & 1/6 & 5/6 & 2 & 11/3 \\ 0 & 1/24 & 5/8 & 5/2 & 77/12 \end{bmatrix}$$

The *2*'nd column (for $h=1$) gives the coefficients for the powerseries in x for $U(x)=U^{\circ 1}(x)$, the column for $h=2$ those for $U^{\circ 2}(x)$ and so on.

Now we build polynomials in h for the interpolation of the entries for each row across the columns.

The matrix $\mathbf{POLY}$ of coefficients of that polynomials in h, which interpolate the terms for the powerseries for iterated $U^{\circ h}(x)$ as given in $\mathbf{L}$ according to its "height" is

(4.1.3.2.) $$POLY = \begin{bmatrix} 0 & . & . & . & . \\ 1 & 0 & . & . & . \\ 0 & 1/2 & 0 & . & . \\ 0 & -1/12 & 1/4 & 0 & . \\ 0 & 1/48 & -5/48 & 1/8 & 0 \end{bmatrix}$$

To get the coefficients a_k for the powerseries in x for the general height h of $U^{\circ h}(x)$ we postmultiply $\mathbf{POLY}$ with the vandermonde-vector of h : $V(h) = [1, h, h^2, h^3, h^4, ...]$

(4.1.3.3.) $\mathbf{A}_1 = \mathbf{POLY} * \mathbf{V}(h) = [a_0, a_1, a_2, a_3,...]\sim$

Inserted into the symbolic description for the iterable version with a given h as iteration(or "height") -parameter this means in matrix-notation:

(4.1.3.4.) $\mathbf{V}(x) \sim *\mathbf{A}_1 = U^{\circ h}(x)$

In serial notation the previous is (the reintroduced index e at U_e shall remind, that this is base e here)

(4.1.3.5.) $U_e^{\circ h}(x) = 1\, x\;\; h/2! * x^2 + (h/3! + h(h-1)/2!^2)* x^3 + (h/4! + 5/2h(h-1)/2!/3! + h(h-1)(2h-3)/2!^3) * x^4 ...$

$= (1) * x$
$+ (0 + 1h)/2 * x^2$
$+ (0 - 1h + 3h^2)/12 * x^3$
$+ (0 + 1h - 5h^2 + 6h^3)/48 * x^4$
$+ (0 - 4h + 30h^2 - 65h^3 + 45h^4)/720 * x^5$
$+ (0 + 22h - 273h^2 + 890h^3 - 1155h^4 + 540h^5)/17280\, x^6$
$+ ....$

For $h=0$ this degenerates to

(4.1.3.6.) $U_e^{\circ 0}(x) = 1*x + 0 + 0 + ...$

for $h=1$ this gives

(4.1.3.7.) $U_e^{\circ 1}(x) = 1*x + 1/2!\, x^2 + 1/3!\, x^3 + ...$

the known exponential-series for *exp(x)-1* , and for *h=-1* this gives

(4.1.3.8.) $U_e^{\circ -1}(x) = 1*x - 1/2\ x^2 + 1/3\ x^3 + \ldots$

the known powerseries for *log(1+x)* which connects then the negative heights *h* with the inverse function to *exp(x)-1* (iterated).

4.1.4. The matrix-logarithm-approach to U_e-tetration

The matrix-logarithm U_L of U_e can exactly be determined, since due to its unit-diagonal, (U_e - I) is nilpotent to the order of its size and the number of (matrix-) terms of the powerseries for logarithm is therefore finite if the final function $U^{\circ h}(x)$ is approximated by finite matrix-size.

(4.1.4.1.) $$U_L = \begin{bmatrix} 0 & . & . & . & . & . \\ 0 & 0 & . & . & . & . \\ 0 & 1/2 & 0 & . & . & . \\ 0 & -1/12 & 1 & 0 & . & . \\ 0 & 1/48 & -1/6 & 3/2 & 0 & . \\ 0 & -1/180 & 1/24 & -1/4 & 2 & 0 \end{bmatrix}$$

If we multiply this with the continuous height parameter *h* (which has no restriction to be integer now) and build the exponential again, we get the formal composition of the general *h*'th power of U_e:

(4.1.4.2.) $$U_e^h = \begin{bmatrix} 1 & . & . & . & . \\ 0 & 1 & . & . & . \\ 0 & 1/2*h & 1 & . & . \\ 0 & 1/4*h\text{^}2-1/12*h & h & 1 & . \\ 0 & 1/8*h\text{^}3-5/48*h\text{^}2+1/48*h & 3/4*h\text{^}2-1/6*h & 3/2*h & 1 \end{bmatrix}$$

and the second column provides the coefficients for the powerseries in *x* for the *h*'th-iterate of $U_e^{\circ h}(x)$:

(4.1.4.3.) $V(x) \sim * U_e^h [,1] = U_e^{\circ h}(x)$

or

(4.1.4.4.) $U_e^{\circ h}(x) = 1 *x + (0 + 1/2\ h) *x^2 + (0 - 1/12\ h + 1/4\ h^2) *x^3 + \ldots$

which is exactly the same result as we got using the polynomial interpolation.

4.2. General U_t-tetration: Eigensystem-approach

For other bases $t \neq e = exp(1)$, $u=log(t) \neq 1$ the polynomial interpolation as well as the matrix-logarithm cannot be done symbolically, since the first parameter a of the formal powerseries for $U_t(x)$ $[0,a,b,c,d,..]$ as discussed in the beginning occurs with its consecutive powers and the expansion of the matrix-logarithm hasn't nilpotent matrices and must be described as an infinite series. Thus one has to employ the symbolic eigensystem-decomposition of $\mathbf{U_t}$. Fortunately, the matrix $\mathbf{U_t}$ is triangular and we obtain exact solutions for truncations of each size, which are also constant in their top-left truncations across that increasing sizes – so we may usem them as template for the case of infinite size as well.

Let $u=log(t)$ then the matrix-operator $\mathbf{U_t}$, which performs the iteration $x \to t^x - 1$ or

(4.2.1.1.) $U_t = {}^dV(log(t))*U = {}^dV(u)*U$

is (the infinite extension of)

(4.2.1.2.) $$\mathbf{U_t} = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & u & . & . & . & . \\ 0 & 1/2*u^2 & u^2 & . & . & . \\ 0 & 1/6*u^3 & u^3 & u^3 & . & . \\ 0 & 1/24*u^4 & 7/12*u^4 & 3/2*u^4 & u^4 & . \\ 0 & 1/120*u^5 & 1/4*u^5 & 5/4*u^5 & 2*u^5 & u^5 \end{bmatrix}$$

The index indicates the base t here. The second column of $\mathbf{U_t}$ provides the coefficients for the powerseries of $U_t(x) = t^x - 1$

The matrix $\mathbf{U_t}$ is triangular and exactly (up to any truncated size) decomposable into an eigensystem even in symbolic notation (where u is kept as variable) [6]:

(4.2.1.3.) $U_t = W_u * D_u * W_u^{-1}$

where $\mathbf{W_u}$ and $\mathbf{W_u^{-1}}$ are also triangular and $\mathbf{D_u}$ is diagonal.

Here $\mathbf{D_u} = {}^dV(u)$ since the eigenvalues of a triangular matrix are just the entries of their diagonal.

I omit the indexes for the matrices$\mathbf{W_u}$ and $\mathbf{D_u}$ in the following for shortness, since they are constants for a given t, u:

(4.2.1.4.) $$\begin{bmatrix} 1 & . & . & . \\ 0 & 1 & . & . \\ 0 & -u/(2*u-2) & 1 & . \\ 0 & (2*u^3+u^2)/(6*u^3-6*u^2-6*u+6) & -u/(u-1) & 1 \end{bmatrix} \begin{bmatrix} 1 & . & . & . \\ . & u & . & . \\ . & . & u^2 & . \\ . & . & . & u^3 \end{bmatrix} \begin{bmatrix} 1 & . & . & . \\ 0 & 1 & . & . \\ 0 & u/(2*u-2) & 1 & . \\ 0 & (u^3+2*u^2)/(6*u^3-6*u^2-6*u+6) & u/(u-1) & 1 \end{bmatrix}$$

Iterates are interpreted as powers of the matrix-operator $\mathbf{U_t}$ and those are –according to the principles of eigensystem-analysis - computable by powers of $\mathbf{D}$ and their composition using the unchanged matrix-constants $\mathbf{W}$ and $\mathbf{W^{-1}}$.

(4.2.1.5.) $U_t^h = W * D^h * W^{-1} = W * {}^dV(u^h) * W^{-1}$

The coefficients for the terms of the according powerseries in x are then in the second column of the result $\mathbf{U_t^h}$.

(4.2.1.6.) $U_t^{\circ h}(x) = \mathbf{V}(x)\sim * \mathbf{U_t^h} [,1]$

[6] Aldrovandi/Freitas state in [AF97] 1997, p.16 "*Bell matrices are not normal, that is, they do not commute with their transposes. Normality is the condition for diagonalizability. This means that Bell matrices cannot be put into diagonal form by a similarity transformation. (...)*" This remark is a bit misleading; the normality-criterion applies only, if a ***orthonormal*** similarity transform is requested, which is usually also called a rotation. But here we are able to do a similarity transform using triangular matrices.

4.3. Result for general U_t-tetration (Eigensystem-based)

Let $U_t^{\circ h}(x)$ denote the h'th iterate of $U_t(x)$, then its powerseries is:

$$(4.3.1.1.)\quad U_t^{\circ h}(x) = a_1 \frac{x}{1!} + a_2 \frac{u}{u-1}\frac{x^2}{2!} + a_3 \frac{u^2}{(u-1)(u^2-1)}\frac{x^3}{3!} + \ldots + a_k \frac{u^{k-1}}{\prod_{j=1}^{k-1}(u^j-1)}\frac{x^k}{k!} + \ldots$$

where

$$(4.3.1.2.)\quad a_1 = 1\,u^h$$

$$a_2 = -(1)\,u^h + (1)\,u^{2h}$$

$$a_3 = (1+2u)\,u^h - (3+3u)\,u^{2h} + (2+1u)\,u^{3h}$$

$$\begin{aligned} a_4 = & -(1+6u+5u^2+6u^3)\,u^h \\ & +(7+18u+18u^2+11u^3)\,u^{2h} \\ & -(12+18u+18u^2+6u^3)\,u^{3h} \\ & +(6+6u+5u^2+1u^3)\,u^{4h} \end{aligned}$$

$$\begin{aligned} a_5 = & (1+14u+24u^2+45u^3+46u^4+26u^5+24u^6)\,u^h \\ & -(15+75u+130u^2+180u^3+165u^4+105u^5+50u^6)\,u^{2h} \\ & +(50+145u+230u^2+275u^3+215u^4+130u^5+35u^6)\,u^{3h} \\ & -(60+120u+170u^2+180u^3+120u^4+60u^5+10u^6)\,u^{4h} \\ & +(24+36u+46u^2+40u^3+24u^4+9u^5+1u^6)\,u^{5h} \end{aligned}$$

4.3.2. Basic observations:

For $h=0$ all terms except a_1 collapse to zero, so $U_t^{\circ 0}(x) = x$, for $h=1$ the a_k-coefficients cancel against the product in the denominator except one factor $u^h = u$, which combines with u^{k-1} to u^k and produces the exponential-series for $U_t(x) = t^x-1$.

For all integer h the a_k-coefficients contain the product of the associated denominator as factor and build integer functions of u when cancelled with the denominators.

For fractional h the denominators do not cancel. So for fractional h it must be that $|u|<>1$

If $u=1, t=e$ we have $0/0$ – expressions, and the function must be evaluated with other methods (as shown in the description of polynomial interpolation and matrix-logarithm above).

The numerical coefficents in each a_k-expression form matrices $\mathbf{A}_k$, which seem to be computable even without the symbolic eigen-decomposition.

4.3.3. Hypotheses:

1) The first column of the matrices $\mathbf{A}_k$ are Stirling-numbers 2'nd kind, scaled by factorials, signed (rows taken, see next page) .

2) The last column of the matrices $\mathbf{A}_k$ are Stirling-numbers 1'nd kind (rows taken, see next page).

3) The shifting of the rows by integer values of the height-parameter h provides polynomials in u, whose sums according to the above scheme are multiples of the denominator of the current term of the powerseries in x.

4) The combination of ***1)*** and ***2)*** gives initial conditions, which in connection with ***3)*** allow to determine the remaining columns in $\mathbf{A}_k$ uniquely.

Example-computation for coefficient a_4 (denoting it here as A), using the hypotheses

We assume the left and right columns as known (hypothesis ***1)*** and ***2)***), and the property, that integer h provide integer multiples of the denominator (hypothese ***3)***). Let us call the a_4-coefficient of the powerseries as A to prevent confusion of notation here.

denominator at A (=a_4) omitting the factorial

(4.3.3.1.) $D = (u^3-1)(u^2-1)(u-1) = 1\,u^6 - 1u^5 - 1\,u^4 + 0u^3 + 1u^2 + 1u - 1$

For convenience of display I also rewrite A in reverse order of powers of u:

```
A =     - (  6u^3  +  5u^2 +  6u+   1)   u^h
        + ( 11u^3  + 18u^2 + 18u+   7)   u^2h
        - (  6u^3  + 18u^2 + 18u+  12)   u^3h
        + (  1u^3  +  5u^2 +  6u+   6)   u^4h
```

Rewritten showing the unknowns

```
A =     - (  6u^3 + a1u^2 + b1u +   1)  u^h
        + ( 11u^3 + a2u^2 + b2u +   7)  u^2h
        - (  6u^3 + a3u^2 + b3u +  12)  u^3h
        + (  1u^3 + a4u^2 + b4u +   6)  u^4h
```

---

setting $h = 0$, rewritten wrt column-sums

```
A =     - (  6u^3 + a1u^2 + b1u +   1)
        + ( 11u^3 + a2u^2 + b2u +   7)
        - (  6u^3 + a3u^2 + b3u +  12)
        + (  1u^3 + a4u^2 + b4u +   6)
        ------------------------------
        =k*D
```

Obviously k=0 and

(4.3.3.2.) $a_1 - a_2 + a_3 = a_4 \qquad b_1 - b_2 + b_3 = b_4$

$h = 1$, irrelevant powers of u removed, yellow marked entries can directly be determined by col-sums:

```
A =     -  (                          6u^3 + a1u^2 + b1u +  1)
        +  (                  11u^4 + a2u^3 + b2u^2 + 7u      )
        -  (           6u^5 + a3u^4 + b3u^3 + 12u^2           )
        +  (  1u^6 + a4u^5 + b4u^4 + 6u^3                     )
        ---------------------------------------------------------
     =k*D
        = 1*(1 u^6  - 1u^5 - 1 u^4 + 0u^3 +  1u^2 +  1u - 1 )
```

1) Because of coefficient at highest and lowest power of u follows k = 1

2) a_4 = 6-1 = 5 $\qquad b_1$=7-1 = 6

(4.3.3.3.) $h = 2$, irrelevant powers of u removed, yellow marked entries can directly be determined:

```
A =     -  (                                       6u^3 + a1u^2 + b1u +  1 )
        +  (                        11u^5 + a2u^4 + b2u^3 + 7u^2            )
        -  (             6u^7 + a3u^6 + b3u^5 + 12u^4                       )
        +  (  1u^9 + a4u^8 + b4u^7 + 6u^6                                   )
        ---------------------------------------------------------------------
     =(k1*u^3 +k2*u^2 + k3*u + k4) * D

        = 1*( 1u^9  - 1u^8 - 1u^7 +  0u^6 +  1u^5 +  1u^4 - 1u^3              )
        +k2*(         1u^8 - 1u^7 -  1u^6 +  0u^5 +  1u^4 + 1u^3 - 1u^2       )
        +k3*(                1u^7 -  1u^6 -  1u^5 +  0u^4 + 1u^3 + 1u^2 - 1u  )
        + 1*(                        1u^6 -  1u^5 -  1u^4 + 0u^3 + 1u^2 + 1u - 1 )
```

1) Because of coefficient at highest and lowest powers of u follows $k_1 = 1$, k_4=1

2) Because a_4 is known, k_2 can be determined by second column-sum; analoguously b_1 and k_3 : k_2=6 , k_3=7

3) Since all k are known, all column-sums are known and all remaining entries can be determined:

$b_4 = 6,\ a_3 = 18\ \ b_3 = 18\ \ a_2 = 18\ \ b_2 = 18\ \ a_1 = 5$

Stirling 2'nd kind (no shift) — St2

1	.	.	.	.	.
1	1	.	.	.	.
1	3	1	.	.	.
1	7	6	1	.	.
1	15	25	10	1	.
1	31	90	65	15	1

*St2 *diag(0!,1!,2!,...) signed* — S2F

1	.	.	.	.	.
-1	1	.	.	.	.
1	-3	2	.	.	.
-1	7	-12	6	.	.
1	-15	50	-60	24	.
-1	31	-180	390	-360	120

Stirling kind 1 (no shift) — St1

1	.	.	.	.	.
-1	1	.	.	.	.
2	-3	1	.	.	.
-6	11	-6	1	.	.
24	-50	35	-10	1	.
-120	274	-225	85	-15	1

4.4. Coefficients as dependent on the height-parameter h in u^h

Let $v = u^h$, then the coefficients c_k at each k'th power of v are ***series*** of the following structure:

$$v\ (x/1! - u/(u-1)*x^2/2! + (1+2u)\,u^2/(u-1)(u^2-1)*x^3/3! + \dots) \quad = u^h * c_1(x,u)$$
$$v^2\ (\quad u/(u-1)*x^2/2! + (3+3u)\,u^2/(u-1)(u^2-1)*x^3/3! + \dots) \quad = u^{2h} * c_2(x,u)$$
$$v^3\ (\quad (2+u)\,u^2/(u-1)(u^2-1)*x^3/3! + \dots) \quad = u^{3h} * c_3(x,u)$$
$$\dots$$

and the value $U_t^{\circ h}(x)$ is then a trivariate function in u,h,x, where we may assume a given u and x:

$$U_t^{\circ h}(x) = f_{u,x}(h) = c_1(x,u) * u^h + c_2(x,u) * u^{2h} + c_3(x,u) * u^{3h}$$

For all $c_k(x,u)$ it is for $x=0$ $c_k(0,u) = 0$ and thus, as expected

$$U_t^{\circ h}(0) = (t^0 - 1)^{\circ h} = 0$$

Let $x=1$ then this is a shorter form for the usual $t\hat{}\hat{}h$ – notation ($u=log(t)$)

$$v\ (1/1! - u/(u-1)/2! + (1+2u)\,u^2/(u-1)(u^2-1)/3! + \dots) \quad = u^h * c_1(u)$$
$$v^2\ (\quad u/(u-1)/2! + (3+3u)\,u^2/(u-1)(u^2-1)/3! + \dots) \quad = u^{2h} * c_2(u)$$
$$v^3\ (\quad + (2+u)\,u^2/(u-1)(u^2-1)/3! + \dots) \quad = u^{3h} * c_3(u)$$
$$\dots$$

For $|u|>1$ this may be rewritten as

$$v\ (1/1! - 1/(1-1/u)/2! + (1/u+2)/(1-1/u)(1-1/u^2)/3! + \dots) \quad = u^h * c_1(u)$$
$$v^2\ (\quad 1/(1-1/u)/2! + (3/u+3)/(1-1/u)(1-1/u^2)/3! + \dots) = u^{2h} * c_2(u)$$
$$v^3\ (\quad (2/u+1)/(1-1/u)(1-1/u^2)/3! + \dots) \quad = u^{3h} * c_3(u)$$
$$\dots$$

If u is a rational complex value on the unit-circle, then we get periodically infinities in this series (because some denominators evalute to zero), but it seems, that if u is an irrational complex unit-root, then the series doen't show this effect and can possibly evaluated.

4.5. Conclusion/perspective for U- and T- tetration if hypotheses hold:

If the hypothese ***4)*** in the previous holds, then each term-matrix $\mathbf{A}_k$ can be uniquely determined individually - without need of the eigensystem-decomposition of the matrix-operator $\mathbf{U}_t$ and even without other terms being involved. It provides a computation scheme for arbitrary many terms for the powerseries for fractional iterates of the function $U_t^{\circ h}(x)$ in sequential or random order.

Since the **U**-tetration $U_t(x) : x \to t^x - 1$ and **T**-tetration $T_b(x) : x \to b^x$ can be converted into each other by shift and rescale of the x-parameter and by relating the bases-parameters b and t

(4.5.1.1.) $T_b^{\circ h}(x) = (U_t^{\circ h}(x/t - 1)) + 1)*t$
where
$$b = t^{1/t}$$

this provides also a systematic access to the powerseries for and the characteristics of the **T**-tetration, which is the commonly understood "tetration" if its parameter x is $x=1$.

Since we had no special restrictions (except the final convergence consideration) on the parameters b, t u and h, and since for all b (with some, at most enumerable infinitely many, exceptions) we can determine fixpoints t (possibly using complex values and then using their principal branches of their logarithms for u), the above describes a very general framework – perhaps the most complete one – for the problem of continuous extension of integer tetration.

The surprising possibility to be able to compute the terms for the powerseries in x, as indicated in the previous paragraphs asks for an iterated gaussian-elimination-procedure, which may come out to be a new, but basic process, which needs description of the recursive algorithm. Also the computed numbers $a_1,a_2,\dots$ $b_1,b_2,\dots$ seem to be basic constants with some flavour of being somehow "eigen-numbers" of the sequences of Stirling-numbers in the related rows of the matrices of Stirling-numbers of 1'st and 2'nd kind. Since they are ultimately derived from the Taylor-series-expansion for the function $f(x) = e^x-1$ iterations of other function of the same type ($f(x) = sin(x)$) may be similarly dependend on such typical "eigen-numbers" accordingly to the coefficients in their series-expansion. But also it may be possible to find another process, which describes these numbers with less effort... This remains open here for further investigation.

Both methods:

* matrix-logarithm for bases t where $|u|=1$, (tetration for base $b=t^{1/t}$)
* eigensystem-decomposition for other bases

together answer some strange properties of tetration, if the above hypotheses hold:

Q: why does tetration converge for $1/e^e < b < e^{1/e}$ but diverge for other b?

A: (Eigensystem): because then $1/e<t<e$ and $|u|<1$. Then the sequence of absolute values of eigenvalues $[1,u,u^2,...]$ is a convergent sequence and thus powers of the diagonalmatrix $D^h = diag(1,u,u^2,...)^h = diag(1,u^h,u^{2h},...)$ provide convergent sequences. For $|u|>1$ the diagonalmatrix of eigenvalues (as well as its positive powers) contain divergent sequences.

Q: why does tetration oscillate if $b<1/e^e$?

A: (Eigensystem): because $u<-1$. Say, $u=-k$, (where k is assumed as positive number>1), then the set of eigenvalues is $[1,-k,k^2,-k^3,k^4,-k^5,....]$ and is used as diagonalmatrix D. Even powers of D, say D^w, where $w=2*n$ give $[1,k^w,k^{2w},k^{3w},...]$ and odd powers D^v, where $v= w+1$ gives $[1,-k^v,k^{2v},-k^{3v},...]$. The signs of each second entry in the resulting diagonal-matrices is alternating between v and w and since k>1 the sequences of powers of eigenvalues are also divergent, this leads to the oscillation of values/bifurcation for even/odd integer heights for $U_t^{\circ h}(x)$ or $T_b^{\circ h}(x)$ (U- and (T-) tetration)

Q: ... (to be continued)

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5. Appendices

5.1. Indeterminacy of reciprocal in case of infinite matrices

It was mentioned, that with infinite matrices the reciprocal may not uniquely determined. This is true even for the case of rowfinite (triangular) matrices, and should be considered in more detail. Here I give a simple heuristic.

We know, that the logarithm-function is multivalued in $\mathbb{C}$, and with tetration we need an apriori consideration of this property.

For the inverse of the ***U***-matrix (=***S2***-matrix in my other articles), which I may denote here as ***S1***, this has the following consequence.

Recall the powerseries for logarithms, (as it occurs as well with ***S1***) as

(5.1.1.1.) $\log(1+x) = 1/x - 1/2x^2 + 1/3x^3 - \dots$

and $\exp(\log(1+x)) - 1 = x$. The multivaluedness can then be described from

(5.1.1.2.) $\exp(\log(1+x) + k*2\pi i) - 1 = x \qquad //k \text{ integer}$

Since the $k*2\pi i$ – term is constant w.r.t. x, we may add this term to the powerseries

(5.1.1.3.) $\log(1+x)+2k\pi i = 2k\pi i + 1x - 1/2\, x^2 + 1/3\, x^3 - \dots$

and formally we have to do the construction of the matrix $S1_k$ for the current branch of logarithm according to **(1.1)** by the series-expansion-process, based on the coefficients of the formal powerseries

(5.1.1.4.) $f(x) = K + a\,x + b\,x^2 + c\,x^3 + d\,x^4 + \dots$

with $K = k*2\pi i =/= 0$, $a=1$, $b=-1/2$, $c = 1/3$ and so on.

Let's denote $f_k(x)$ for the function, where a certain k is selected, in this example $k=-1$

The first four powers of $f_k(x)$ are then

(5.1.1.5.)
$$
\begin{aligned}
f_k(x)^0 &= 1 \\
f_k(x)^1 &= K + 1\,x - 1/2\,x^2 + 1/3\,x^3 - 1/4\,x^4 + \dots \\
f_k(x)^2 &= K^2 + (2K)\,x + (1 - 2K/2)\,x^2 + (2K/3 - 1)\,x^3 + \dots \\
f_k(x)^3 &= K^3 + (3K^2)x + (3K - 3/2K^2)x^2 + (1 - 3K + K^2)\,x^3
\end{aligned}
$$

and the entries of the columns of $S1_k$ must now be determined by evaluation of the parentheses.

Numerically this gives about

$$
S1_1 = \begin{bmatrix}
1.000 & -6.283*I & -39.48 & 248.1*I & 1559. \\
0 & 1.000 & -12.57*I & -118.4 & 992.2*I \\
0 & -0.5000 & 1.000+6.283*I & 59.22-18.85*I & -236.9-496.1*I \\
0 & 0.3333 & -1.000-4.189*I & -38.48+18.85*I & 236.9+305.6*I \\
0 & -0.2500 & 0.9167+3.142*I & 28.11-17.28*I & -216.1-210.4*I
\end{bmatrix}
$$

S1k

To check, whether this is a valid reciprocal for ***U*** (=***S2***) in the sense that first we compute the first branch-logarithm and then exponentiate to get the original value, we need to matrix-multiply the row-vectors of $S1_1$ with the col-vectors of ***S2***.

The first nontrivial column in ***S2*** is the second, which contains the coefficients of the powerseries for $\exp(x)-1$. Using the first row in $S1_1$ this gives

(5.1.1.6.) $1*0 + K/1! + K^2/2! + K^3/3! + \dots = \exp(K) - 1 = \exp(2\pi i) - 1 = 0$

Next row gives

(5.1.1.7.)
$$
\begin{aligned}
&0*0 + 1/1! + 2K/2! + 3K^2/3! + \\
&= 1/0! + K/1! + K^2/2! + \qquad = \exp(K) = \exp(2\pi i) = 1
\end{aligned}
$$

and it needs then be proved by induction, that indeed the equality

(5.1.1.8.) $S1_k * S2 = I \qquad // k=/=0$

holds for the case of infinite sized matrices to verify the correctness for the 2'nd column of $\mathbf{S2}$.

Numerically, size of $64x64$ suffices, to get good approximation in the top left using $k=-1$:

$$(5.1.1.9.)\quad S1_{-1} * S2_{\,64x64} = \begin{bmatrix} 1.00000 & 0 & 0 & -0.0000000302794-0.00000000880113*I \\ . & 1.00000 & 0 & -0.0000000908382+0.000000307680*I \\ . & . & 1.00000 & 0.00000158405+0.000000307680*I \\ . & . & . & 1.00000-0.00000540649*I \\ . & . & . & -0.0000130919+0.00000413178*I \end{bmatrix}$$

Because the generation formula shows, that the entries along the rows in $\mathbf{S1}_k$ grow only geometrically with k, but the entries of the columns in $\mathbf{S2}$ eventually decrease hypergeometrically, all occuring sums are convergent and the precision of the entries can be increased, if convergence-acceleration is used, for instance Euler-summation.

I didn't consider the problem of iterations here; but it should be mentioned, that we have then to deal with divergent summation with complex series, and heuristics indicate, that simple Euler-summation may not suffice for that.

For the second iterate we get by such expansions of powers of $f(x)$:

$$(5.1.1.10.)\quad \begin{aligned} f^{\circ 2}(x) &= K \\ &+ a\,(K + a\,x + b\,x^2 + c\,x^3 + d\,x^4 + \dots) \\ &+ b\,(K^2+2Ka\,x+(a^2+2Kb)\,x^2+(2Kc+2ba)\,x^3+(2ac+b^2+2Kd)\,x^4 +(2bc+2da+2Ke)\,x^5 \quad) \\ &+ c\,(K^3+3K^2a\,x+(3Ka^2+3K^2b)\,x^2+(3K^2c+(a^3+6Kba))\,x^3+\dots) \\ &+ \dots \end{aligned}$$

Collect like powers of x:

$$(5.1.1.11.)\quad \begin{aligned} f^{\circ 2}(x) &= K + (aK + bK^2 + cK^3 + \dots) \\ &+(a\,(1a + 2bK + 3cK^2 + \dots)) * \mathbf{x} \\ &+((a^2\,(1b + 3cK + 6dK^2 + 10eK^3 + \dots) \\ &\qquad + b\,(1a + 2bK + 3cK^2 + 4dK^3 + \dots) * \mathbf{x^2} \\ &+((a^3\,(1c + 4dK + 10eK^2 + \dots\dots) \\ &\qquad + 2ab\,(1b + 3cK + 6dK^2 + 10eK^3 + \dots) \\ &\qquad + c\,(1a + 2bK + 3cK^2 + 4dK^3 + \dots)) * \mathbf{x^3} \\ &+((a^4\,(1d + 5eK + 10fK^2 + \dots\dots) \\ &\qquad + 3a^2b\,(1c + 4dK + 10eK^2 + 20fK^3 + \dots) \\ &\qquad + (2ac + b^2)(1b + 3cK + 6dK^2 + 10eK^3 + \dots) \\ &\qquad + d\,(1a + 2bK + 3cK^2 + 4dK^3 + \dots)) * \mathbf{x^4} \\ &+ \dots \end{aligned}$$

write $g(k) =(K + aK + bK^2 + cK^3 + \dots)$ and the derivative of g at K $g'(K) = dg(K)/dK$ then

$$(5.1.1.12.)\quad \begin{aligned} f^{\circ 2}(x) &= g(K) \\ &+ a\,g'(K) && * \mathbf{x} \\ &+(a^2/2!\,g''(K) + b\,g'(K)) && * \mathbf{x^2} \\ &+(a^3/3!\,g'''(K) + 2ab/2!\,g''(K) + c\,g'(K)) && * \mathbf{x^3} \\ &+(a^4/4!\,g''''(K) + 3a^2b/3!\,g'''(K)) +(2ac+b^2)/2!\,g''(K) + d\,g'(K)) && * \mathbf{x^4} \\ &+ \dots \end{aligned}$$

It may be written more clearly in a sketched matrix-notation (I omit (K) at g here):

$$dV(x)* \begin{bmatrix} 1 & & & & \\ 0 & a & & & \\ 0 & b & 1a^2 & & \\ 0 & c & 2ab & 1\,a^3 & \\ 0 & d & (2ac+1b^2) & 3a^2b & 1\,a^4 \end{bmatrix}$$

$$* \operatorname{diag}(g(K), g'(K)/1!, g''(K)/2!, g'''(K)/3!, g''''(K)/4!, \dots)$$

Inserting values $a=1$, $b=-1/2$, $c=1/3$, $d=-1/4$ gives then

$$
\begin{aligned}
(5.1.1.13.)\quad f^{\circ 2}(x) &= K + g(K) \\
&+ \quad g'(K) && * x \\
&+(1/2!\, g''(K)) \quad -1/2\, g'(K)) && * x^2 \\
&+(1/3!\, g'''(K) \quad -1/2!\, g''(K) \quad +1/3\, g'(K)) && * x^3 \\
&+(1/4!\, g''''(K) \quad -1/4\, g'''(K) \quad +11/24\, g''(K)) \quad -1/4\, g'(K)) && * x^4 \\
&+ \dots
\end{aligned}
$$

Now set $K_1 = K+1$ and set $g(K) = \log(K_1)$, $g'(K)=1/K_1$, $g''(K)=-1/K_1^2$, $g'''(K)=2!/K_1^3$, ...allowing divergent summation, and reorder in the rows, then

$$
\begin{aligned}
(5.1.1.14.)\quad f^{\circ 2}(x) &= K + \log(K_1) \\
&+ \ 1/K_1 && * x \\
&- \ (1/K_1 \ + \ 1 \ /K_1^2) && * x^2 /2 \\
&+ (1/K_1 + 3/2/K_1^2 + 1/K_1^3) && * x^3 /3 \\
&- (\ 1/K_1 + 11/6/K_1^2 + 2/K_1^3 + 1/K_1^4) && * x^4 /4 \dots
\end{aligned}
$$

The view on the columns suggest to use column-wise summation to find the final value.

If indeed the columns follow the Stirling-numbers 1'st kind, we would get

$$
\begin{aligned}
(5.1.1.15.)\quad f^{\circ 2}(x) &= K + \log(K_1) + \log(1+x)/K_1 + (\log(1+x)/K_1)^2 + (\log(1+x)/K_1)^3 + \dots \\
&= K + \log(K_1) + \ \log(1+x)/K_1/(1 - \ \log(1+x)/K_1) \\
&= K + \log(K_1) + 1/(K_1/\log(1+x) - 1)
\end{aligned}
$$

5.2. Identity of tetration, using different fixpoints

In tetration we have the property, that, assuming a base b there exist a number t, which is constant under iteration. This means:

$$b^t = t$$

or $$t = b^t = b^{b^t} = b... b^{b^t} = T_b^{\circ n}(t) \qquad // \text{for all } n$$

The analogue in the operation of multiplication would be, that

$$t = t*b = t*b*b = t*b^n$$

and this is obviously possible for any t and only $b=1$. In multiplication b is called the invariant element (or unit-element), there exists only one ($b=1$ [7]) and it has his invariance-property for all numbers t.

In tetration, this "unit" element is a function of t and can be computed by

$$b_t^t = t \quad => \quad b_t = t^{1/t}$$

since

$$(t^{1/t})^t = t^{1/t*t} = t^1 = t$$

Since L. Euler we know, that b_t has an upper limit and has its maximum for $t=e$ at $b_t=e^{1/e}$; a graph shows the relation between t and b_t.

Even more, since in

$$b_t = t^{1/t} = exp(log(t)/t)$$

the log-function is (infinitely) multivalued, we have infinitely many $b_{t,k}$ steming from the same t.

Say, $u_k = log(t) + k*2 pi i$, then we have a unique t,

$$t = exp(u_k)$$

but the the tetration-expression

$$b_{t,k} = exp(u_k/t)$$

is still infinitely multivalued for one t, according to the k'th branch of complex logarithm u_k of t.

So let us consider u_k as basic independent variable. The domain for u_k in $t=exp(u_k)$ is the whole complex plane, where the horizontally infinite strips $C_{-oo..+oo,\ k*2pi..(k+1)*2pi}$ periodically map to the whole complex plane. This map is also continuous with the exception of the missing value of $t=0$ which can only be seen as limit, as u approaches the negative infinity.

So the domain for t is dense (except $t=0$) and also the inverse map (giving periodically complex u_k) is continuous in each strip, we have, that

$$t^{1/t} = exp(u_k/t) = b_{t,k}$$

is continuous with respect to u of one selected strip, has no pole, but is multivalued with repect to u_k, from different strips with $k*2 pi i$. Different from the equality for integer k of all $exp(u) = exp(u_k)=exp(u_0 + k*2 pi i)$ we have, that

$$exp((u_0 + k*2 pi i)/t) = exp(u_0/t + k/t * 2 pi i)$$

and these are only equal for constant u_0 and t, if t is a divisor of k. This means, that we have the same value for $b_{t,k}$ on different branches $k_0,k_1,...$, for which t (or its numerator, if t is rational) is a common divisor of k_0,k_1,k_2. (this implies also t is rational). In reverse view this means, that if $b_{3,0}=3^{1/3}$ then the values of the $3k$'th branches of the t-invariant elements $b_{3,0},b_{3,3},b_{3,6}$ are all equal.

Now, since $b_t=t^{1/t}$ is also an invariant for t with respect to the exponentiation, we have

* for each t different invariants b_{u_k} ,

[7] which interestingly equals $t*(1/t)$ - but does this really mean anything?

However, multiple t may have the same function-value b_t, for instance $t_1=2$ and $t_2=4$ have the same unit-element $b_2=b_4=sqrt(2)$, or, using the more common notion, $b=sqrt(2)$ has two fixpoints.

5.3. Experiments with $f(x) = x*e^x$ and $f^{-1}(x)$="Lambert-W"

Considering the function $f(x)=x\ e^x$ exhibits some easyness implied by this matrix-method of inversion and another aspect of different behave of a function at integer and fractional arguments.

First note the matrix-operator for $f(x)$. It is generated from

(5.3.1.1.) $f(x) = x*(1+x/1! + x^2/2! + x^3/3! + \dots)$

by the previously decribed method. The occuring matrix-operator has the form:

(5.3.1.2.) $F =$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 1 & 1 & . & . & . \\ 0 & 1/2 & 2 & 1 & . & . \\ 0 & 1/6 & 2 & 3 & 1 & . \\ 0 & 1/24 & 4/3 & 9/2 & 4 & 1 \end{bmatrix}$$

and the coefficients of the powerseries are in the second column, such that

(5.3.1.3.) $V(x)\sim * F = V(x\ e^x)\sim = [1, xe^x, (xe^x)^2, (xe^x)^3,\dots]$

The triangular inverse can easily be determined, we get

(5.3.1.4.) $F^{-1} = W =$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & -1 & 1 & . & . & . \\ 0 & 3/2 & -2 & 1 & . & . \\ 0 & -8/3 & 4 & -3 & 1 & . \\ 0 & 125/24 & -25/3 & 15/2 & -4 & 1 \end{bmatrix}$$

which contains - the coefficients for the ***Lambert-W***-function!

The required matrix-operation is then

(5.3.1.5.) $V(x)\sim * W = V(w(x))\sim$

to determine the Lambert-W-value for x (for x in range of convergence or using divergent summation).

The famous coefficients can better be seen, if the matrix is factorially scaled

(5.3.1.6.) $^dFac(1)*F^{-1}\ {}^d*Fac(-1)= FWf =$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & -2 & 1 & . & . & . \\ 0 & 9 & -6 & 1 & . & . \\ 0 & -64 & 48 & -12 & 1 & . \\ 0 & 625 & -500 & 150 & -20 & 1 \end{bmatrix}$$

and we see the coefficients $1^0,2^1,3^2,4^3,5^4,\dots$ in the second column, and this agrees with the series-description of the Lambert-W:

(5.3.1.7.) $W(x) = 1^0/1!\ x - 2^1/2!\ x^2 + 3^2/3!\ x^3 - 4^3/4!\ x^4 + \dots - \dots$

This is a very easy access to this famous function; and by powers of the matrix we can even iterate the function F and W (while we get even more divergence with the iterates of W, but this doesn't matter here).

The half-iterate of the Lambert-W begins with

(5.3.1.8.)
$W^{\circ 0.5}(x)=1\ x\ -1/2\ x^2\ +1/2\ x^3\ -31/48\ x^4\ +91/96\ x^5\ -2873/1920\ x^6\ +2845/1152\ x^7\ -150327/35840\ x^8 +O(x^9)$

or

(5.3.1.9.)
$W^{\circ 0.5}(x)= 1.0\ x\ -0.5\ x^2\ +0.5\ x^3\ -0.645833\ x^4\ +0.947917\ x^5\ -1.49635\ x^6\ +2.46962\ x^7\ -4.19439\ x^8\ +7.26496\ x^9\ -12.7707\ x^{10}\ +22.7309\ x^{11}\ -40.9236\ x^{12}\ +74.4549\ x^{13}\ -136.697\ x^{14}\ +252.797\ x^{15} + O(x^{16})$

Moreover, there is a curiosity with the function *F* itself.

If we define two variants as

(5.3.1.10.) $F_1(x,r) = x^r e^x$ $F_2(x,r) = x^r e^{-x}$

and the alternating sums

(5.3.1.11.) $AF_1(r) = \sum_{k=0}^{oo} (-1)^k * k^r e^k$ $AF_2(r) = \sum_{k=0}^{oo} (-1)^k * k^r e^{-k}$

then, first, for the exponent $r=1$ we get the surprising result, that

(5.3.1.12.) $AF_1(1) - AF_2(1) = 0$

where the divergent series $AF_1()$ is Euler-summed.

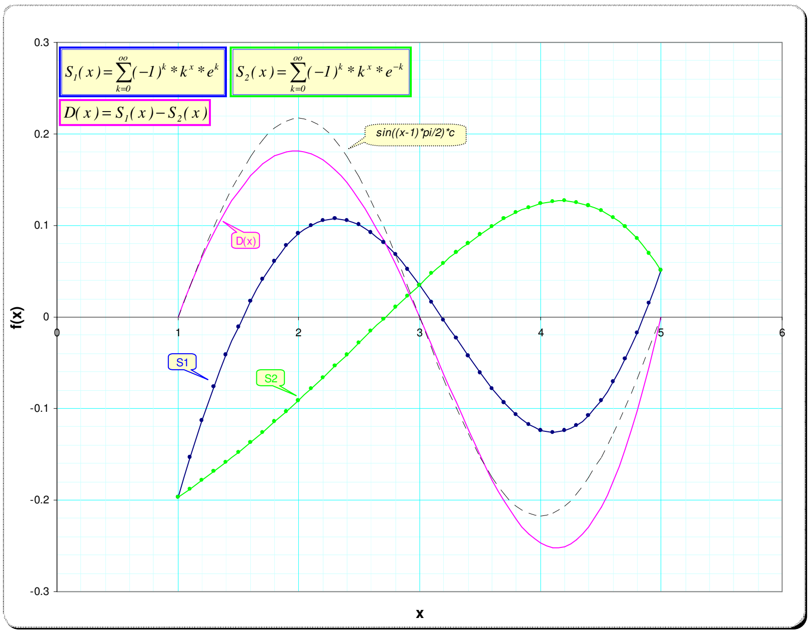
This is much interesting, and it seems, that generally

(5.3.1.13.) $AF_1(k) + (-1)^k AF_2(k) = 0$

for ***integer*** *k* and have a sinusiodal characteristic as function *d(x)* of ***continuous*** *x*.

(5.3.1.14.) $AF_1(x) - AF_2(x) = d(x)$ // *d(x) sinusoidal periodic with period* $\pi/2*x$

Here is a graph, which shows $AF_1(x)$ (blue, "S_1" in the plot), $AF_2(x)$ (green, "S_2" in the plot), $AF_1(x)$-$AF_2(x)$ (magenta "*D*" in the plot) and a scaled sinus-curve (dotted, black):



Although I didn't get into more depth with this yet, it reminds of a similar effect with the alternating Tetra-series, and also of the difference, which occurs, if fractional tetration is computed via different fixpoints. The sinusoidal effect of differences, if fractional arguments are involved, seems to be ubiquitous...

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18.3.2008

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