A false interpolation paradigm for tetration?

A reconsideration of the vandermonde-interpolation triggered by a difficult problem in the "tetration"-discussion

Motivation:

Fractional iteration of the exponential function (often called "tetration") is still an unsolved problem in complex functional analysis, although promising attempts seem to exist. One common path of the analysis is based on the concept of Abel and/or Schröder-functions. That concepts are designed for analytic functions which have a power series representation, so they should be a valuable tool for the iteration of the exponential-function and relatives to it. While in general such interpolations are not unique we have at least, that for functions whose powerseries have no constant but a linear term we can build a "natural" solution for the power series for any fractional iteration. So for instance for $d(x)=exp(x)-1$ whose linear term has coefficient 1.

For such functions the mechanism of the Schröder-function provides expressions for the coefficients which are depending on the iteration-"height" and provides thus exactly expressible formal power series at any fractional iteration. (That a function, whose integer iterates are even entire, may have formal power series for fractional iterates with convergence-radius zero, is an important result, proven in the 50ies in the 20'th century).

That power series have then coefficients which depend on the iteration "height" (expressible by finite polynomials). So one approach for the application of the Schröder-equations is *a)* to recenter the original power series of the function under consideration around a fixpoint t_k and *b)* to apply then the Schröder-machinery for fractional iteration to the modified power series.

However, if a function has more than one fixpoint, say $t_0,t_1,t_2,...$) then the result of the fractional iteration due to the Schröder-functions is dependent on that fixpoint (while for the integer iterations we can apply the usual evaluation with a unique result and arbitrary accuracy and are not depending on the Schröder-equations and fixpoints). And with iterated exponentiation/tetration we have even infinitely many fixpoints.

Here I try to look deeper at that problem and I compare it with the technique of interpolation using Vandermonde-matrices. The comparable technique for the recentering and Schröder-function can also be expressed by matrix-operators ("Carleman-matrix") and diagonalization (or matrix-logarithm) of infinite-sized (but triangular, and thus rowfinite) matrices. First I recall an earlier heuristical result of mine, that the diagonalization and application of the Schröder-equation using the attracting fixpoint and the Vandermonde-interpolation are essentially the same and lead to the same power series solution. Thus if there is an immanent drawback in the diagonalization-method then this is shared by the Vandermonde based method, and vice versa.

To understand the underlying problems better I apply that type of evaluation to another function, namely the logarithm-function. Then we observe, that a naîve vandermonde-interpolation for the logarithm function leads to a power series with a very similar structure to that for the tetration – but that power series is systematically false for the true logarithm, also does not converge to the true one (even if the order of the interpolation polynomials is increased towards infinity) and finally produces false values at arguments of noninteger exponents.

So this serves as an indicator that possibly that method, applied to tetration, might have the same drawback.

It excited me much, when I found an article in Ed Sandifers monthly column at MAA.org from Dec 2007, where he discussed a (didactical) treatize of Leonard Euler concerning"a false logarithm series" – apparently just that series which I'll discuss here. So that problem of a possible drawback of the vandermonde-interpolation method even for the (ideally) infinite case has already had a wider attention and thus might be a serious entry-point for some reconsideration of the paradigm, based on which we try to interpolate the iterated exponential map to fractional heights.

For the computation of tetration to fractional heights (iterates) I employ the diagonalization of operator matrices (see also "Bell-", "Carlemanmatrix"). This implements well-known manipulations of the coefficients of formal powerseries; in fact if the base *b* for tetration is *b=exp(exp(-1))* this can be done by matrix-logarithm and if *1<b<exp(exp(-1))* we can directly apply diagonalization of the involved Carlemanmatrix.

But because of the notorious difference of solutions for the fractional heights in tetration, when the required recentering of power series are developed simply around different fixpoints, I'm still not confident, that this method is the final/the best solution.

In an earlier *article* I discussed a simple interpolation-approach, intended as a replacement for the diagonalization for difficult (for instance: complex) bases *b* and instead I found, that this agrees with the diagonalization in the limit down to the level of identity of the coefficients of the occuring powerseries, so that interpolation method and diagonalization are essentially the same and no "better" procedure was actually found.

That ansatz of interpolation followed the common idea of polynomial interpolation/Vandermonde interpolation resp. its generalization to the case of infinite order/ of powerseries, or of the use of (finite case) vandermonde-matrix. Here factors like $(x-1),(x^2-1),(x^3-1)$ etc occur typically and essentially in numerators and denominators.

I had already earlier brought the "*false logarithmic series*" to the readers' attention (see note below and *this link*) and this time I tried that interpolation-technique for the tetration to the problem of re-engineering the power series for the logarithm (as found in the Euler-/Sandifer article), and see, whether I would get the correct series or the false series, too.

For the Euler-series let's say more correctly: "***a***" logarithm, because we find, that this interpolation gets correct results at integer arguments but ***systematically wrong*** results at fractional arguments - thus reflecting the very similar observation in tetration, where the different fixpoints give identical results at integer heights but differing results at fractional heights. So in tetration the fractional heights are not reflected optimally with ***any*** such series developed around some fixpoint.

The involved formula using a simple example: a power series for the log-function by vandermonde interpolation

We consider the problem of finding a power series which allows to compute $y=1$ from $x=2^1$, $y=2$ from $x=2^2$, $y=3$ from $x=2^3$ and so on. After we've got the coefficients for that power series in x, we hope, that using other values at x shall allow to get the base-2-logarithm of x in y. This is a classical interpolation problem, which suggests to approached it by the Vandermonde-interpolation. For the Vandermonde-interpolation it is known, that the spacing between the x and the y-values has an impact for the solution but we want to use a naive approach here.

We want to find a powerseries for the logarithm to base b, say $b=2$; such that with this series at argument x we find the *base-2*-logarithm of x. Propose this with the initially unknown coefficients *K,a,b,c,d,...*

(1) $f(x) = K + ax + bx^2 + cx^3 + \dots$ // *unknown coefficients K,a,b,c to be determined*
$=?= \log_2(x)$

and let's approach this problem stepwise from finite polynomials of increasing order to a final generalization to a power series.

Let's skip the approximations by linear and quadratic formula and immediately step to that of cubics.

Approximation using cubic polynomial

So first we set up a set of equations to find the unknown coefficients *K,a,b,c* for a cubic polynomial.

We write (1) inserting the known x and y-values

x	$f_3(x)$	$= y$
$x=2^0$:	$1\,K + 2^0\,a + (2^0)^2\,b + (2^0)^3\,c$	$= 0$
$x=2^1$:	$1\,K + 2^1\,a + (2^1)^2\,b + (2^1)^3\,c$	$= 1$
$x=2^2$:	$1\,K + 2^2\,a + (2^2)^2\,b + (2^2)^3\,c$	$= 2$
$x=2^3$:	$1\,K + 2^3\,a + (2^3)^2\,b + (2^3)^3\,c$	$= 3$

and solve by the vandermonde-method.

Let's write this as matrix-equation.

First we write the matrix of coefficients $\mathbf{VV_3}$ (subscript 3 for the dimension)

$$VV_3 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 2^2 & 2^4 & 2^6 \\ 1 & 2^3 & 2^6 & 2^9 \end{vmatrix}$$

the given y-values and the sought coefficients as vectors:

C_3 = *columnvector[K,a,b,c], holding the* C_3*-oefficients*
Y_3 = *columnvector[0,1,2,3], the given logarithm values to base 2*

Using the four given numerical examples in x and y according to eq (1) we write the matrix-equation

$$VV_3 \cdot C_3 = Y_3$$

Then we solve for $\mathbf{C_3}$

$$C_3 = VV_3^{-1} \cdot Y_3$$

This gives in C_3 the vector

$C_3 = [-31/21 , 7/4 , - 7/24 , 1/56]$

Using that coefficients in C_3 we get a polynomial of order 3 in x :

$f_3(x) = -31/21 + 7/4 x - 7/24 x^2 + 1/56 x^3$

Trivially, for the first four natural exponents we get with this the correct solutions:

$f_3(2^0) = 0 \quad = \log_2(1)$
$f_3(2^1) = 1 \quad = \log_2(2)$
$f_3(2^2) = 2 \quad = \log_2(4)$
$f_3(2^3) = 3 \quad = \log_2(8)$

But the correctness is limited to that four exponents – for the interpolation to some other, for instance fractional exponent between them, we get

$f_3(2^{0.5}) = 0.46585755... \qquad \neq 0.5 = \log_2(2^{0.5})$

and also for integer exponents, which exceed the list of given example-values we need clearly higher-order polynomial approximations.

Higher order polynomials

From the computation-scheme it is obvious how this can be generalized to higher order poylnomials and higher order approximates. But however high order we try: although the coefficients of the polynomials converge to that of a certain powerseries if we increase the order of approximation polynomials we will ***systematically*** not get the approximating procedure to the true logarithms at fractional exponents.

Example for values computed with the series of inreasing orders n: $x = sqrt(2)$, $\log_2(x) = 0.5$ where the approximation-order (order of polynomials for the vandermonde-interpolation) is in the subscript of $f_n(\ )$:

$f_{12}(2^{0.5}) = 0.473784748806...$
$f_{24}(2^{0.5}) = 0.473811031008...$
$f_{48}(2^{0.5}) = 0.473811037422...$
$f_{96}(2^{0.5}) = 0.473811037422...$
...
$f_n(2^{0.5}) = \qquad$ *for n->oo converging to that value*

with a *systematic* deviation from the correct value 0.5 of about

$f_{oo}(2^{0.5}) – 0.5 = – 0.0261889625777$

The (false) limiting power series

$f_3(x) = -31/21 + 7/4 x - 7/24 x^2 + 1/56 x^3$

The polynomial, as the order n goes to infinity, seems to converge to the powerseries:

$\lim_{n->oo} f_n(x) = – c_{EB}$
$\qquad + 2 x - 4/9 x^2 + 8/147 x^3 - 16/4725 x^4$
$\qquad + 32/302715 x^5 + O(x^6)$

$$\lim_{n->\inf} f_n(x) = -c_{EB} + \frac{2}{1}x - \frac{1}{(2^2-1)}\frac{2^2}{2^2-1}x^2 + \frac{1}{(2^2-1)(2^3-1)}\frac{2^3}{2^3-1}x^3 - ...$$

$$\lim_{n->\inf} f_n(x) = -(c_{EB} + \sum_{k=1}^{\inf}\left((-1)^k \frac{1}{\prod_{j=1}^{k}(2^j-1)}\frac{2^k}{2^k-1}x^k \right)$$

where $c_{EB}=\lim_{n\to\infty} c_n = 1.606695152415291763783301523190...$ $= \sum_{k=1}^{n} \frac{1}{2^k - 1}$

(Erdös/Borwein-constant)

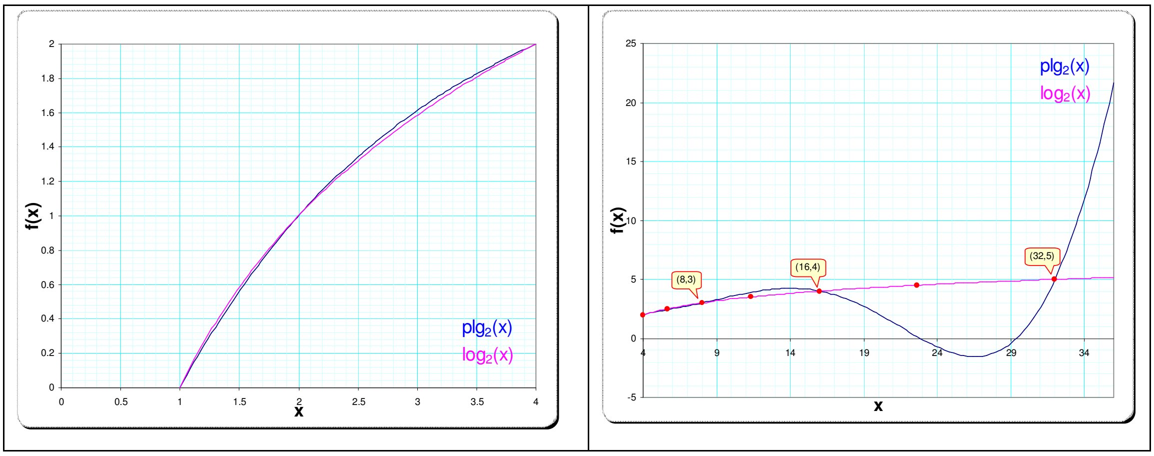
A better representation for the coefficients at x for computation in a programm-loop is

$a_0 = 1.606695152415291763783301523190...$
$a_1 = 2$
$a_k = a_{k-1} \cdot 2\,(2^{k-1}-1)/(2^k-1)^2$

That series gives correct results for powers of *2* with natural exponents but will be false with fractional exponents (or another base).

It is of interest here that this series can exactly be related to the expansion of the series which –according to Ed Sandifer's MAA column in Dec 2007 – has been studied by L. Euler in his paper about the "false logarithm" (that article of year ~1750 is E190 in the Eneström-index). See for an illustration the appendix.

The correct and the false logarithm:



The whole method of interpolation is based on the paradigm of polynomial interpolation, which even if generalized to infinite order of polynomials will remain to give false results for fractional exponents for the logarithm-case and for fractional heights in the tetration-case.

The matrix-method for tetration employs either directly the same interpolation-method (*see my discussion on "exponential polynomial interpolation", an ugly term, but I did not find a better one*) or in an obscured way (*we can express an identity between diagonalization and this interpolation-method*).

So -possibly- the same way as we needed a move from this interpolation-paradigm to arrive at a meaningful series for logarithm, we need a move to arrive at a more meaningful interpolation for fractional tetration.

What do you think?

Gottfried Helms

Note: The original idea of the "false logarithm" was triggered by an article "How Euler did it - a false logarithm series" of Ed Sandifer in MAA-online, where he discussed a similar analysis led by L.Euler for didactical(?) reasons. For the Euler-paper and the Ed Sandifer-article see:

[Euler190] "Consideratio quarumdam serierum quae singularibus proprietatibus sunt praeditae"
("Consideration of some series which are distinguished by special properties")
L. Euler Eneström-index E190.

[ES2007-12] A false logarithm series (Discussion of E190)
Ed. Sandifer in: "How Euler did it" Dec 2007
http://www.maa.org/editorial/euler/How%20Euler%20Did%20It%2050%20false%20log%20series.pdf

The "False logarithmic series" with base *b=3* :

$$\lim_{n\to\inf} f_n(x) = -c + \frac{1}{1-3^1}\frac{3}{1-3^1}x + \frac{1}{(1-3^1)(1-3^2)}\frac{3^2}{1-3^2}x^2 + \frac{1}{(1-3^1)(1-3^2)(1-3^3)}\frac{3^3}{1-3^3}x^3 - \ldots$$

$$\lim_{n\to\inf} f_n(x) = -c + \frac{(1-3)}{1!_3}\frac{3}{1-3^1}x + \frac{1}{(1-3)^2 2!_3}\frac{3^2}{1-3^2}x^2 + \frac{1}{(1-3)^3 3!_3}\frac{3^3}{1-3^3}x^3 - \ldots$$

The constants *c* for the false logarithmic series *plg()* to some other bases:

base	*value*	*value*(base-1)*
1.5	*-3.8971550754879328101*	*-1.9485775377439664051*
2	*-1.6066951524152917637*	*-1.6066951524152917637*
3	*-0.68215350260523806676*	*-1.3643070052104761335*
4	*-0.42109768603342377730*	*-1.2632930581002713319*
5	*-0.30173385359797245795*	*-1.2069354143918898318*
10	*-0.12232424342624452626*	*-1.1009181908362007364*
19	*-0.058487248157017253694*	*-1.0527704668263105665*

Pseudocode for Pari/GP and a bit more discussion

False logarithms: we want the vector of coefficients A such that

$V(2^h)\sim * A = h$ // set out for integer h

$[V(2^0),V(2^1),V(2^2),\ldots]\sim * A = [0,1,2,3,\ldots]\sim$ // set out for integer h

$VV_2\,(0..inf) * A = L = colvector(0,1,2,\ldots)$
A : coefficients,
L : solutions in integer logs
$(M\ D\ M\sim) * A = L$
$D\ M\sim * A = M^{-1}\ L$
$M\sim * A = D^{-1}\ M^{-1}\ L$
$A = M^{-1}\sim * (D^{-1}\ M^{-1}\ L)$ // lim dim-> inf analytical description possible

Tetration : find matrix POLY such that
$V(x)\sim * POLY * V(u^h) = exp_b^{\circ h}(x)$
and
$POLY * V(u^h) = A(h)$
A(h) : coefficients for $exp_b^{\circ h}(x)$
then
$V(x)\sim * A(h) = exp_b^{\circ h}(x)$

$POLY * VV_u(0..inf) = AH(0..inf)$
$POLY * M*D*M\sim = AH(0..inf)$
$POLY * M*D = AH(0..inf) * M^{-1}\sim$
= <this is triangular>
$POLY * M = (AH(0..inf) * M^{-1}\sim) * D^{-1}$
$POLY = (AH(0..inf) * M^{-1}\sim * D^{-1})\ *\ M^{-1}$
here (*) means we have to approximate the asymptotic values,
in that matrix-multiplication. But we can find closed forms
(rational numbers) for the final entries in POLY

However, a better configuration saves the rationale of the idea for the logarithm. We have to adapt two parameters:

1) We expect the result=*0* at ***x=1***, so the powerseries at *(x-1)* should have no constant term. Then the result at *x=0* (powerseries at *–1*) can be arbitrary, especially can be *–infty*. Thus we modify the ***VV***-matrix accordingly.

2) We need to replace the list of integer exponents by such of fractional exponents. For instance, instead of looking at the logs of $2^0,2^1,2^2,\ldots$ we look at that of $2^1,2^{1/2},2^{1/3},\ldots$ Again we have to modify the ***VV***-matrix.

Here is a codesnippet for varying dimension dim

```
{flogtst(dim=4,a=2)=local(A,VV,Logs,X);

  Logs = vectorv(dim,r,1/r);
    \\   Logs = vectorv(dim,r,r-1);  \\ [0,1,2,…] gives "false" logarithms
  X = vectorv(dim,r,a^Logs[r]-1);    \\ using "-1" centers the powerseries
    \\  X = vectorv(dim,r,a^Logs[r]);    \\ gives the eulerian "false logarithms"
  VV = matrix(dim,dim,r,c,X[r]^(c-1));
  A = matsolve(VV,Logs);
 return(A); }

dim=12;A = flogtst(dim)
x1 = 2^0.9 - 1
print( sum(k=0,dim-1, A[1+k]*x1^k)) // modest accurate

dim=48;A = flogtst(dim)
x1 = 2^0.9 - 1
print( sum(k=0,dim-1, A[1+k]*x1^k)) // much accurate
```

The Euler-series is given by Sandifer:

$$s = \frac{1-x}{1-a} + \frac{(1-x)(a-x)}{a-a^3} + \frac{(1-x)(a-x)(a^2-x)}{a^3-a^6} + \ldots$$

If we express the matrix-formula for the above solution we have

$VV \cdot A = Y$
$A = VV^{-1} \cdot Y$

such that we can then say (and compute):

$V(x) \cdot A = V(pseudolog(x))$

However, ***VV*** has no obvious reciprocal in the case of infinite size. So we do the usual path of the ***LDU***-decomposition of ***VV*** and then use the well-defined inverses of the triangular (***L***,***U***) and diagonal (***D***) factors to produce a valid power series for the function:

$VV \cdot A = Y$
  where $VV = L \cdot D \cdot U$
$L \cdot D \cdot U\, A = Y$
$D \cdot U\, A = L^{-1} \cdot Y$
$U\, A = D^{-1} \cdot L^{-1} \cdot Y$

Here we stop, because the left-multiplication with the U^{-1} leads to nonconvergent dot-products on the rhs. Instead we assume this as "change of basis" for our power series and rewrite

$V(x) \cdot A = V(x) \cdot U^{-1} (U\, A)$
$\quad = (V(x) \cdot U^{-1}) \cdot (D^{-1} \cdot L^{-1} \cdot Y)$

Then we find, that $(V(x) \cdot U^{-1})$ provides just the numerators in the Euler-formula, and the $(D^{-1} \cdot L^{-1} \cdot Y)$ - part just the denominators.

So the vandermonde-style interpolation that we used here for the logarithm produces just the "false logarithm series" of L. Euler for the logarithm.

The same vandermonde-style interpolation applied to the iterated exponentiation gives a power series for the interpolation which looks much similar to the "false logarithm" series, so the questionable interpolations by this method is like questionable. Moreover, that vandermonde interpolation leads to the same formula as the diagonalization, so also the diagonalization (using recentering around a fixpoint t_k) is questionable in the same way and might be systematically distorted for the non-integer heights for the same (obscure) reason.