Tetration

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Mathematical Miniatures

Pascalmatrix tetrated

"Iterated exponentiation" applied to the Pascalmatrix

Abstract: A short collection of findings for the tetration of the Pascal-matrix

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1. Computation

1.1. Intro

Matrices can be exponentiated[1] and also be logarithmized (if certain conditions are satisfied). With this also a "matrix-power of a matrix" A^B can be defined using this matrix-logarithm and when the exponentiation is resolved into the appropriate powerseries for its argument.

In a recent thread in the *sci.math*-newsgroup we had a discussion about the detail, whether it is more meaningful to define A^B as *Exp(Log(**A**) * **B**)* or *Exp(**B** * Log(**A**))*. For this discussion I studied the whereabouts using the Pascalmatrix as an example and came across the following very astonishing heuristiscs and perspectives, which – concerning the original question – also strongly suggest to use the *Exp(Log(**A**) * **B**)*-version...

Definitions

For the definition of the matrix-exponential the obvious pattern is the exponential-powerseries

(1.1.1.) $exp(x) = 1 + x/1! + x^2/2! + x^3/3! + \dots$

applied to a square matrix-parameter X

(1.1.2.) $Exp(X) = I + X/1! + X^2/2! + X^3/3! + \dots$ // use I for identity-matrix

which, if X is triangular or is nilpotent, gives exact values for each entry of the resulting matrix. Similarly, we can define the matrix-logarithm[2] using the mercator-series with a square matrix-argument

(1.1.3.) $Log(X) = (X-I)/1 - (X-I)^2/2 + (X-I)^3/3 - \dots$

if this series converges, or the better converging series

let $F = (X-I)*(X+I)^{-1}$

(1.1.4.) $Log(X) = 2*(F/1 + F^3/3 + F^5/5 + \dots)$

In the following I'll discuss the lower triangular "Pascalmatrix" P as argument for the powerseries. For the logarithmic series this is a special convenient case, since due to the subtraction $P-I$ we sum powers of a nilpotent lower triangular matrix, whose diagonal is zero and the matrix-logarithm series is then finite for any finite dimension of the matrix. The matrix-logarithm of P is then a nilpotent matrix, too, which in turn reduces the exponential-series to a finite sum giving exact values for any finite matrix size.

The matrix-power of a matrix may then be defined as

(1.1.5.) $A^B = Exp(Log(A)*B) = I + (Log(A) B)/1! + (Log(A)*B)^2/2! + \dots$

where the findings in the following article suggest that the order of the multiplication $Log(A)*B$ might be preferred. This appears because the results are meaningful in and consistent with a wider context of similar scalar functional relations.

[1] For considerations concerning convergence, optimization see for instance [Loan] or [MolerLoan]

[2] see, for instance, [Cardoso]

1.2. The (matrix)-logarithm of *P*

The matrix-logarithm of ***P*** can be defined for any finite size. Example for size *8x8*:

(1.2.1.) $PL = Log(P)$
$= matrix_{r,c=0..inf}(\, r \quad \text{if } r=c+1)$

PL

$$\begin{bmatrix} 0 & . & . & . & . & . & . & . \\ 1 & 0 & . & . & . & . & . & . \\ 0 & 2 & 0 & . & . & . & . & . \\ 0 & 0 & 3 & 0 & . & . & . & . \\ 0 & 0 & 0 & 4 & 0 & . & . & . \\ 0 & 0 & 0 & 0 & 5 & 0 & . & . \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 \end{bmatrix}$$

(see [Helms,PL], [Edelman])

Since the entries are constant if the size is changed, we may also define the infinite sized matrix by this pattern.

1.3. A diagonalization-formula for *P*

For ***P*** of any finite size we cannot find a proper diagonalization satisfying the formula

(1.3.1.) $P\sim = W * diag([\lambda_0, \lambda_1, \lambda_2, ...]) * W^{-1}$

since all eigenvalues are *1* and the sets of eigenvectors are degenerate. (but see Appendix 2.3)

But for the case of infinite size we can find a solution with a non-invertible matrix ***W*** such that

$$P\sim * W = W * diag([\lambda_0, \lambda_1, \lambda_2, \lambda_3, ...])$$

Namely

(1.3.2.) $P\sim * W = W * diag([1, e, e^2, e^3, ...])$
$= W * {}^dV(e)$
$= W * E$ *// give it the shorter name E*

where ***W*** is the factorially scaled vandermondematrix [3]

(1.3.3.) $W = matrix_{r,c=0..inf}(\, c^r / r!\,)$

With this we can calculate parallel with ***P*** and ***E***, according to the diagonalization-rules applied to powers, linear combinations and such functions of ***P*** and ***E***, which are defined by a powerseries-expansion.

Examples:

(1.3.4.)
$P^h \sim * W = W * E^h$
$(P + P^2) \sim * W = W * (E + E^2)$
$Log(P)\sim * W = W * Log(E) = W * diag([0,1,2,3,4,...])$
$Exp(P)\sim * W = W * Exp(E)$

[3] Note, that a rescaled version of ***W*** is known as "Bell-" (or: transposed "Carleman-") matrix for the function *exp(x)* such that

$$V(x)\sim * W = V(e^x)\sim$$

See also [wikipedia]

1.4. Tetrates (iterated exponentials) of *P*

The tetrates of ***P*** may be recursively defined by

(1.4.1.) $P^{\wedge\wedge 2} = P^{P} = Exp(Log(P)*P)$

(1.4.2.) $P^{\wedge\wedge h+1} = P^{P^{\wedge\wedge h}} = Exp(Log(P)*P^{\wedge\wedge h})$

where also

(1.4.3.) $P^{\wedge\wedge 1} = P$

(1.4.4.) $P^{\wedge\wedge 0} = I$

We cannot define the inverse operation since the matrix-logarithm of ***P*** is not invertible.

We get first ***PL * P***:

(1.4.5.) $PL*P$

0	.	.	.	.	.	.	.
1	0	.	.	.	.	.	.
2	2	0	.	.	.	.	.
3	6	3	0	.	.	.	.
4	12	12	4	0	.	.	.
5	20	30	20	5	0	.	.
6	30	60	60	30	6	0	.
7	42	105	140	105	42	7	0

then $\boldsymbol{P}^{\wedge\wedge 2}$

(1.4.6.) $P^{\wedge\wedge 2} = Exp(PL*P) =$

1	.	.	.	.	.	.	.
1	1	.	.	.	.	.	.
3	2	1	.	.	.	.	.
10	9	3	1	.	.	.	.
41	40	18	4	1	.	.	.
196	205	100	30	5	1	.	.
1057	1176	615	200	45	6	1	.
6322	7399	4116	1435	350	63	7	1

then $\boldsymbol{P}^{\wedge\wedge 3}$

(1.4.7.) $P^{\wedge\wedge 3} = Exp(PL*P^{\wedge\wedge 2}) =$

1	.	.	.	.	.	.	.
1	1	.	.	.	.	.	.
3	2	1	.	.	.	.	.
16	9	3	1	.	.	.	.
101	64	18	4	1	.	.	.
756	505	160	30	5	1	.	.
6607	4536	1515	320	45	6	1	.
65794	46249	15876	3535	560	63	7	1

then $\boldsymbol{P}^{\wedge\wedge 4}$

(1.4.8.) $P^{\wedge\wedge 4} = Exp(PL*P^{\wedge\wedge 3}) =$

1	.	.	.	.	.	.	.
1	1	.	.	.	.	.	.
3	2	1	.	.	.	.	.
16	9	3	1	.	.	.	.
125	64	18	4	1	.	.	.
1176	625	160	30	5	1	.	.
12847	7056	1875	320	45	6	1	.
160504	89929	24696	4375	560	63	7	1

The first observation is, that all tetrates so far can be reduced to a simple definition based on the first column only, where the entries of the h'th tetrate are denoted with a small p, the height parameter h and row,col-indexes r,c

(1.4.9.) $$p^{(h)}_{r,c} = p^{(h)}_{r-c,0} * binomial(r,c)$$

that means, only the first column has "significant" values and the subsequent columns are downshifted repetitions, scaled by binomials. See the following representation of $\boldsymbol{P}^{\wedge\wedge 2}$ as hadamard-product (elementwise multiplication)

(1.4.10.) $$P^{\wedge\wedge 2} = \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 3 & 2 & 1 & . & . & . \\ 16 & 9 & 3 & 1 & . & . \\ 125 & 64 & 18 & 4 & 1 & . \\ 1296 & 625 & 160 & 30 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1*1 & . & . & . & . & . \\ 1*1 & 1*1 & . & . & . & . \\ 3*1 & 1*2 & 1*1 & . & . & . \\ 16*1 & 3*3 & 1*3 & 1*1 & . & . \\ 125*1 & 16*4 & 3*6 & 1*4 & 1*1 & . \\ 1296*1 & 125*5 & 16*10 & 3*10 & 1*5 & 1*1 \end{bmatrix}$$

$$= Q_2 ¤ P = \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 3 & 1 & 1 & . & . & . \\ 10 & 3 & 1 & 1 & . & . \\ 41 & 10 & 3 & 1 & 1 & . \\ 196 & 41 & 10 & 3 & 1 & 1 \end{bmatrix} ¤ \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix}$$

This hadamard-structure inherits to all other tetrates of $\boldsymbol{P}$, as well as to that of infinite height (see below). For the interpretation of the numbers along column 0 see the reference to *[OEIS], A000248* and *[Comtet], pg 91, par. 43*, (keywords "*forests*", "*idempotent maps*")

1.5. Use of coefficients for exponential generating function

If we use the factorial scaled tetrates of $\boldsymbol{P}$

$$fPF^{\wedge\wedge h} = F^{-1} * P^{\wedge\wedge h} * F$$

and a further similarity-scaling by ${}^dV(c)$, where $c = log(b)$ then the colsums give

$$V(1)\sim * (\ {}^dV(c)\ fPF^{\wedge\wedge h}\ {}^dV(1/c)) = exp_b^{\circ h}(1)*V(1)\sim$$
$$V(1)\sim * (\ {}^dV(c)\ fPF^{\wedge\wedge h}{}_{[,0]}) = exp_b^{\circ h}(1)$$

for instance $b= 2$, $c = log(2)$,

$$V(log(2))\sim\ fPF^{\wedge\wedge 3}{}_{[,0]} \qquad \text{// "}_{[,0]}\text{" means "first column" using zero-based index}$$
$$= 2^{\wedge\wedge 3}$$
$$= 2^\wedge 2^\wedge 2$$

Unfortunately, this works only for integer iterates.

On the other hand, this seems to give a first, very simple, however limited, access to the change-of-base-problem: for a fixed iteration-height it is easy to change the base by simply providing another vector of logarithms…

See more on this in chap 2.1 and 2.2

1.6. *P* tetrated to infinite height

It is much interesting, that we can define the infinite case, ***P^^oo*** as limit using a simple pattern

(1.6.1.) $PP = \lim_{h\to inf} P^{\wedge\wedge h} :=matrix_{r,c=0..inf} \begin{cases} (r+1)^{r-1} & for \quad c=0 \\ (r+1-c)^{(r-c)-1}\binom{r}{c} & for \quad r\ge c>0 \\ 0 & for \quad c>r \end{cases}$

then ***PP*** looks like

(1.6.2.) $PP = \lim_{h\to oo} P^{\wedge\wedge h} =$

1	.	.	.	.	.	.	.
1	1	.	.	.	.	.	.
3	2	1	.	.	.	.	.
16	9	3	1	.	.	.	.
125	64	18	4	1	.	.	.
1296	625	160	30	5	1	.	.
16807	7776	1875	320	45	6	1	.
262144	117649	27216	4375	560	63	7	1

and also has the "hadamard"-pattern

1*1	.	.	.	.	.
1*1	1*1	.	.	.	.
3*1	1*2	1*1	.	.	.
16*1	3*3	1*3	1*1	.	.
125*1	16*4	3*6	1*4	1*1	.
1296*1	125*5	16*10	3*10	1*5	1*1

Then we have also

(1.6.3.) $P^{PP} = PP$ *// fixedpoint-formula*

$Log(P)*PP = Log(PP)$

which finally leads to the matrix-analogue of the *h( )*-function [4]

(1.6.4.) $Log(P) = Log(PP) * PP^{-1}$

$P = PP^{PP^{-1}}$

(1.6.5.) $PP = h(P)$

and we find, that indeed if evaluated,

(1.6.6.) $PP^{PP^{-1}} = P$

[4] see [Galidakis] or [Knoebel]

(1.6.7.) $\quad Log(P)*PP = Log(PP)$

PP:

$$*\begin{bmatrix} 1 & . & . & . & . & . & . & . \\ 1 & 1 & . & . & . & . & . & . \\ 3 & 2 & 1 & . & . & . & . & . \\ 16 & 9 & 3 & 1 & . & . & . & . \\ 125 & 64 & 18 & 4 & 1 & . & . & . \\ 1296 & 625 & 160 & 30 & 5 & 1 & . & . \\ 16807 & 7776 & 1875 & 320 & 45 & 6 & 1 & . \\ 262144 & 117649 & 27216 & 4375 & 560 & 63 & 7 & 1 \end{bmatrix}$$

Log(P):

$$\begin{bmatrix} 0 & . & . & . & . & . & . & . \\ 1 & 0 & . & . & . & . & . & . \\ 0 & 2 & 0 & . & . & . & . & . \\ 0 & 0 & 3 & 0 & . & . & . & . \\ 0 & 0 & 0 & 4 & 0 & . & . & . \\ 0 & 0 & 0 & 0 & 5 & 0 & . & . \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 \end{bmatrix}$$

Log(PP):

$$=\begin{bmatrix} 0 & . & . & . & . & . & . & . \\ 1 & 0 & . & . & . & . & . & . \\ 2 & 2 & 0 & . & . & . & . & . \\ 9 & 6 & 3 & 0 & . & . & . & . \\ 64 & 36 & 12 & 4 & 0 & . & . & . \\ 625 & 320 & 90 & 20 & 5 & 0 & . & . \\ 7776 & 3750 & 960 & 180 & 30 & 6 & 0 & . \\ 117649 & 54432 & 13125 & 2240 & 315 & 42 & 7 & 0 \end{bmatrix}$$

The possibility to define such a matrix ***PP*** is surprising in the view of the diagonalization representation above. It means, that on the lhs we have a completely harmless looking lower triangular matrix with units in its diagonal, and on the rhs the tetrate of infinite height for parameters which are powers of *e*, and thus all except the first ($e^0=1$) exceed the upper bound for convergent infinite tetration $\eta = e^{1/e}$. The entries of the according tetrates of ***E*** are

$h=1 \qquad E^{\wedge\wedge 1} = diag(1, e, e^2, e^3, \ldots)$

$h=2 \qquad E^{\wedge\wedge 2} = diag(1, e^e, e^{2e^2}, e^{3e^3}, \ldots)$

$h=3 \qquad E^{\wedge\wedge 3} = diag(1, e^{e^e}, e^{2e^{2e^2}}, e^{3e^{3e^3}}, \ldots)$

...

$h \qquad E^{\wedge\wedge h} = diag(1, e^{\wedge\wedge h}, (e^2)^{\wedge\wedge h}, (e^3)^{\wedge\wedge h}, \ldots)$

where, if *h->inf* also all entries diverge to *inf*.

Then, from the fixedpoint-formula (1.4.3), we expect, that ***PP*** resembles the tetration-fixedpoint for the *exp()*-function; the first being about *0.318131 + 1.337235*I* and having an imaginary component!

Numerical computations with powers of ***P***, say $P^{log(m)}$, where *m=sqrt(2)* or (for better convergence) *m=1.1* (reflecting *exp(log(m))=m* as parameter of the ***E***-diagonalmatrix on the rhs of the diagonalization-formula) approximated the expected values surprisingly well.

Using *m=sqrt(2)* I get

$h = 1 \quad E^{\wedge\wedge 1} \approx diag(\ 1.0\ \ 1.414213\ \ 2.0\ \ 2.8284\ \ 3.99999898\ \ 5.6566\ \ 7.98577\ \ 10.912\ldots)$

exact $\qquad = diag(\ 1,\ \ m,\ \ 2,\ \ 2m,\ \ 4,\ \ 4m, \ldots)$

$h = 2 \quad E^{\wedge\wedge 2} \approx diag(\ 1.0\ \ 1.63252\ \ 3.9973\ \ 16.64\ \ 38.6719, \ldots)$

exact $\qquad = diag(\ 1,\ \ m^m,\ \ 2^2 ,\ \ (2m)^{2m},\ \ 4^4 ,\ \ \ldots)$

For the limit-case *h->inf* we need only *h>matrixsize*, since the entries of the matrix are constant in the $h_0 \times h_0$-submatrix for any $h>h_0$. Moreover, the matrix $\mathbf{PP_m}$ to the basis *m* can exactly be computed as similarity-scaling of the original matrix **PP** (the same as $\mathbf{P^m}$ from **P**):

(1.6.8.) using $\mu = \log(m)$

$$P^{\mu} = {}^dV(\mu) * P * {}^dV(1/\mu)$$

$$PP_m = {}^dV(\mu) * PP * {}^dV(1/\mu)$$

Thus

(1.6.9.)

$$PP_m := matrix_{r,c=0..inf} \begin{cases} (r+1)^{r-1}\mu^r & for \quad c=0 \\ ((r-c)+1)^{(r-c)-1}\binom{r}{c}\mu^{r-c} & for \quad r \ge c > 0 \\ 0 & for \quad c > r \end{cases} \quad ;\mu = \log(m)$$

With this for *m=sqrt(2)* and different sizes for the matrices:

(1.6.10.) *m=sqrt(2)* *h>dim* $E_{dim}{}^{\wedge\wedge dim+1}$

	$E_{16}{}^{\wedge\wedge 17} \approx$ diag(1.0	1.9416	6.520 E 1	−4.325 E 3 , ...)
	$E_{32}{}^{\wedge\wedge 33} \approx$ diag(1.0	1.9897	2.992 E 4	−1.722 E 10 , ...)
	$E_{64}{}^{\wedge\wedge 65} \approx$ diag(1.0	1.9993	4.75 E 13	−1.27 E 24 , ...)
exact	$E^{\wedge\wedge inf}$ = diag(1.0	2.0000	<inf>	<inf> , ...)

Using a base nearer at *1* we should get more finite terms for the $\mathbf{E}^{\wedge\wedge inf}$ – expression, according to the number of powers m^k , which are smaller than $e^{1/e}$. Here I use *m=1.1*, whose powers *1, m, ,* m^2 and m^3 are smaller than $e^{1/e}$ but m^4 is greater and thus $(m^4)^{\wedge\wedge inf}$ is an infinite expression:

(1.6.11.) *m = 1.1*

$E_{64}{}^{\wedge\wedge 65} \approx$ diag(1.0, 1.11178201104, 1.27515965772, 1.56385493139, 3.1393, -561.07, ...)

exact $E^{\wedge\wedge inf}$ = diag(1.0, 1.11178201104, 1.27515965772, 1.56385493462, <inf> <inf> ...)

I've currently no further idea, what this does tell me nor what this could be useful for.

1.7. Powers of ***PP*** (=$P^{\wedge\wedge inf}$)

Also we have a very interesting, since simple, pattern with powers of ***PP***:

(1.7.1.) $PP^2 =$

1	.	.	.	.	.	.	.
2	1	.	.	.	.	.	.
8	4	1	.	.	.	.	.
50	24	6	1	.	.	.	.
432	200	48	8	1	.	.	.
4802	2160	500	80	10	1	.	.
65536	28812	6480	1000	120	12	1	.
1062882	458752	100842	15120	1750	168	14	1

(1.7.2.) $PP^3 =$

1	.	.	.	.	.	.	.
3	1	.	.	.	.	.	.
15	6	1	.	.	.	.	.
108	45	9	1	.	.	.	.
1029	432	90	12	1	.	.	.
12288	5145	1080	150	15	1	.	.
177147	73728	15435	2160	225	18	1	.
3000000	1240029	258048	36015	3780	315	21	1

....

(1.7.3.) $PP^{-1} =$

1	.	.	.	.	.	.	.
-1	1	.	.	.	.	.	.
-1	-2	1	.	.	.	.	.
-4	-3	-3	1	.	.	.	.
-27	-16	-6	-4	1	.	.	.
-256	-135	-40	-10	-5	1	.	.
-3125	-1536	-405	-80	-15	-6	1	.
-46656	-21875	-5376	-945	-140	-21	-7	1

The much interesting aspect is, that the sequence of entries in the first column follow a very simple pattern:

(1.7.4.) PP^1 => 1,1,3,16,125,... = $(r+1)^{r-1}$
PP^2 => 1,2,8,50,432,... = $2*(r+2)^{r-1}$
PP^3 => 1,3,15,108,1029,... = $3*(r+3)^{r-1}$
...
(1.7.5.) PP^{-1} => 1,-1,-1,-4,-27,-256,... = $-1*(r-1)^{r-1}$ // where $0^0 = 1$ defined
PP^{-2} => 1,-2,0,-2,-16,-162,... = $-2*(r-2)^{r-1}$

We might formally insert the definition:

(1.7.6.) PP^0 => 1,0,0,0,0,0,... = $0*(r\ \ )^{r-1}$ // where $0*0^{-1} = 1$ defined

With this we have the complete definition

(1.7.7.)
$$PP^k{}_m := matrix_{r,c=0..inf} \begin{cases} k*(r+k)^{r-1}\mu^r & for \quad c=0 \\ k*((r-c)+k)^{(r-c)-1}\binom{r}{c}\mu^{r-c} & for \quad r \ge c > 0 \\ 0 & for \quad c > r \end{cases} \qquad ;\mu = \log(m)$$

where we define $0^0=1$, $0*0^{-1}=1$

1.8. Exponential of ***PP***

One more step might be of interest: what does the exponential of ***PP*** look like?

We get, by rational arithmetic on *Exp(**PP – I**)* ,

(1.8.1.) *Exp(PP – I)* =

1	.	.	.	.	.	.	.
1	1	.	.	.	.	.	.
4	2	1	.	.	.	.	.
26	12	3	1	.	.	.	.
235	104	24	4	1	.	.	.
2727	1175	260	40	5	1	.	.
38699	16362	3525	520	60	6	1	.
649931	270893	57267	8225	910	84	7	1

where also

*Exp(PP) = exp(1)*Exp(PP-I)*

The matrix has still the structure of the hadamard-product of a basic triangle which is determined by its first column only and the binomial-matrix ***P***.

The numbers in the (only significant) first column are again known to OEIS, they are the coefficients of the exponential-generating-function of *exp(-LW(-x)/x – 1)* . (see Appendix 3.4).

That means, if we use

$EP = exp(1) * Exp(PP - I)$
$fEPF = {}^{d}F^{-1} * EP * {}^{d}F$

We get

$V(x)\sim * fEPF = exp(-LW(-x)/(-x)) * V(x)\sim$

or differently written

$Vf(x)\sim * EP = exp(LW(-x)/(-x)) * Vf(x)\sim$

2. Application for tetration

2.1. General

The observation mentioned in chap 1.5 leads to another set of powerseries for tetration.

For the case of infinite height it occurs empirically, that we have a range of admissible bases in the usual sense $1 < b \le e^{1/e}$, but for the case of finite height this might look differently. Assume a base *b* and denote *c=log(b)*, then the following table gives the beginning of the powerseries **in *log(b)*** (!) for few small heights;

	h=0	h=1	h=2	h=3	h=4	h=5	h=6	h=7	..	h->inf
$c^0/0!$ *	1	1	1	1	1	1	1	1	...	1^{-1}
$c^1/1!$ *	0	1	1	1	1	1	1	1	...	2^0
$c^2/2!$ *	0	1	3	3	3	3	3	3	...	3^1
$c^3/3!$ *	0	1	10	16	16	16	16	16	...	4^2
$c^4/4!$ *	0	1	41	101	125	125	125	125	...	5^3
$c^5/5!$ *	0	1	196	756	1176	1296	1296	1296	...	6^4
$c^6/6!$ *	0	1	1057	6607	12847	16087	16807	16807	...	7^5
$c^7/7!$ *	0	1	6322	65794	160504	229384	257104	262144	...	8^6
$c^8/8!$ *	0	1	41393	733833	2261289	3687609	4480569	4742649	...	9^7
$c^9/9!$ *	0	1	293608	9046648	35464816	66025360	87238720	96915520	...	10^8
...	...	...	...	...	...	...	...	...	...	...

Here we see, how the sequence of series converges (in a completely unusually and unexpected way) to a limit-series when *h* is increased: the coefficients at the beginning remain constant for heights *h->oo*. This is a special property which I've not seen with other series in the context of tetration.

We see in the column for the limit *h->inf* the known powerseries for $b^{1/b}$ with the known range of convergence $|b| \le e^{1/e}$ which agrees with all expectation.

The series at finite heights, however, show "defects" from a current indexposition *k=h* on (marked orange); and for all higher indexes this defect increases. So possibly we can even show, that this family of series has infinite radius of convergence if *h* is finite.

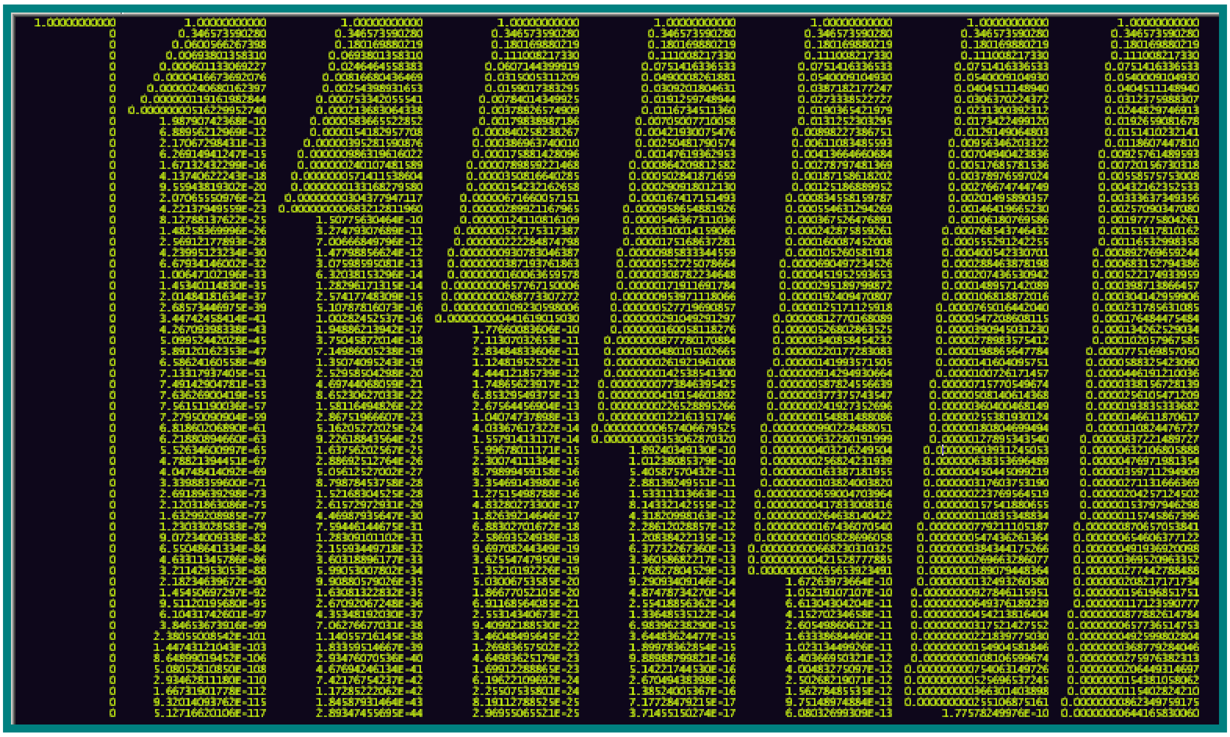
To get another impression, here is the sequence of series for base *b=2*, *c=log(2)~0.693...*

	h=0	h=1	h=2	h=3	h=4	h=5	h=6	h=7	..	->inf
1/0! *	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	...	1^{-1}
1/1! *	0	0.69314718	0.69314718	0.69314718	0.69314718	0.69314718	0.69314718	0.69314718	...	$c*2^0$
1/2! *	0	0.24022651	0.72067952	0.72067952	0.72067952	0.72067952	0.72067952	0.72067952	...	c^2*3^1
1/3! *	0	0.055504109	0.55504109	0.88806574	0.88806574	0.88806574	0.88806574	0.88806574	...	c^3*4^2
1/4! *	0	0.0096181291	0.39434329	0.97143104	1.2022661	1.2022661	1.2022661	1.2022661	...	c^4*5^3
1/5! *	0	0.0013333558	0.26133774	1.0080170	1.5680264	1.7280291	1.7280291	1.7280291	...	c^5*6^4
1/6! *	0	0.00015403530	0.16281532	1.0177113	1.9788915	2.4779659	2.5888714	2.5888714	...	c^6*7^5
1/7! *	0	0.00001525273	0.096427783	1.0035384	2.4481248	3.4987331	3.9215389	3.9984127	...	c^7*8^6
1/8! *	0	0.00000132154	0.054702864	0.96979603	2.9884035	4.8733548	5.9212900	6.2676415	...	c^8*9^7
1/9! *	0	0.00000010178	0.029883675	0.92077561	3.6096395	6.7201179	8.8792320	9.8641450	...	c^9*10^8
...	...	...	...	...	...	...	...	...	...	...

For *h->inf* the resulting series is known to be divergent; for *h<4* we see that the coefficients decrease after a certain maximum; and so the series for finite heights should be convergent if *log(b)=c < 1*. and possibly even for higher *c*. I assume it can be shown, that this occurs at some index in the series for any finite height – but I could not yet study this detail.

Here is an image to illustrate the progressive index, from which on a the coefficients of a powerseries for a certain *h* (read along the *h*'th column) begin to decrease. See the "nose" which nicely mark the position at which the absolute value is smaller than *1e-9* For illustration I used the convergent base *b=sqrt(2)*; but in principle, for higher bases there should only a scaling (or: shifting of the related indexes), occur:

Image



One remarkable difference to the type of series, which I defined in ***[ExactEntries]***[5]**,** is already, that the constant remains the same for all heights, while in the series discussed in ***[ExactEntries]*** the constant in the series for height *h* is always *b^^(h-1)*. This must have an impoact also for the fractional iterates.

For *h=2* the series, found by the two methods begin with the coefficients, shown in the columns, where I inserted *x=1* for the top-parameter in the ***[ExactEntries]***-version, which does not occur in the version of this article

	[exact entries]	[this article]
1*	b	1
1/1!*	b log(b) (log(b))	1 log(b)
1/2!*	b $\log(b)^2$ ($\log(b)+\log(b)^2$)	3 $\log(b)^2$
1/3!*	b $\log(b)^3$ ($1\log(b)+3\log(b)^2+1\log(b)^3$)	10 $\log(b)^3$
1/4!*	b $\log(b)^4$ ($1\log(b)+7\log(b)^2+6\log(b)^3+1\log(b)^4$)	41 $\log(b)^4$
	…	…

[5] see refernece in chap 3

2.2. Fractional iteration

Using the binomial-method for interpolation we can even establish a method for the full fractional tetration; however I didn't study the convergence-conditions yet. Let's call the shown tables of coefficients as matrix ***M***; the list of first columns of tetrated pascalmatrixes:

$$M = concat(P^{\wedge\wedge 0}{}_{[,0]}, P^{\wedge\wedge 1}{}_{[,0]}, P^{\wedge\wedge 2}{}_{[,0]}, ...]$$

such that

$$V(log(b))\sim * M = [1, b, b^{\wedge\wedge 2}, b^{\wedge\wedge 3}, ...]$$

Then the binomial-formula for fractional tetration requires two binomial compositions of the columns of ***M***:

$$M1 = M * P^{-1} \sim$$

reflecting the part

$$a_k = \left(\sum_{j=0}^{k} \left((-1)^{k-j} \binom{k}{j} b^{\wedge\wedge j} \right) \right)$$

in the known binomial formula for fractional tetration; and the coefficients for the powerseries of $b^{\wedge\wedge h}$ for fractional h is then the first column of

$$T_h = M1 * Bin(h)$$

where $Bin(h)$ is the column-vector of binomials

$$Bin(h) = colvector_{k=0..inf}(binomial(h,k))$$

reflecting the composition of $b^{\wedge\wedge h}$ by

$$b^{\wedge\wedge h} = \sum_{k=0}^{\inf} \binom{h}{k} a_k$$

in the binomial-formula.

Note, that until now $\boldsymbol{T_h}$ is a vector of coefficients which is still independent of the base for tetration!

We will have $b^{\wedge\wedge h}$ now by

$$b^{\wedge\wedge h} = V(log(b))\sim * T_h$$

Then not only the powerseries for integer heights look different from my previous approaches, but also the fractional iterates have a completely different shape.

2.3. Conclusion

If I got things right so far, then this is another very remarkable result: in all formulas concerning tetration, which I came across, the base-parameter was nearly intractably involved in the coefficients of the powerseries. If all the above is correct (at least for a certain range of parameter b), then we have a much independent family of powerseries for (and thus also a much different access into) the problem of tetration.

3. Appendix

3.1. Standard matrices

The involved standard-vectors and matrices are as follows. All matrices/vectors are meant to have infinite size.

V(x) := column$_{r=0..inf}$$[1,x,x^2,x^3,...x^r,...]$
an infinite "vandermonde" (column-) vector of a general x
V(x)~ := *the transpose ; the symbol is taken from the convention in Pari/GP*
$^dV(x)$:= *the diagonal arrangement of V(x)*

The column- and rowindices are beginning at zero

F := diag(0!,1!,2!,...)
f := F^{-1} = diag(1, 1, 1/2!, 1/3!,1/4!,...)
allows to define the exponential-series
$Vf(x) = f*V(x)$ = column $_{r=0..inf}[1 , x , x^2/2! , x^3/3! , ... x^r/r! , ...]$

$$dF = \begin{bmatrix} 1 & . & . & . & . \\ . & 1 & . & . & . \\ . & . & 2 & . & . \\ . & . & . & 6 & . \\ . & . & . & . & 24 \end{bmatrix}$$

Matrices

VZ := matrix$_{r=0..inf,c=0..inf}[c^r]$
:= [V(0),V(1),V(2),...]
the collection of V()-vectors of consecutive parameters

$$VZ = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 9 & 16 \\ 0 & 1 & 8 & 27 & 64 \\ 0 & 1 & 16 & 81 & 256 \end{bmatrix}$$

P := matrix$_{r=0..inf,c=0..inf}$[binomial(r,c)]
the lower-triangular Pascal-matrix

$$P = \begin{bmatrix} 1 & . & . & . & . \\ 1 & 1 & . & . & . \\ 1 & 2 & 1 & . & . \\ 1 & 3 & 3 & 1 & . \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}$$

S2 := matrix$_{r=0..inf,c=0..inf}[s2_{r,c}]$
// s2 Stirling-numbers 2nd kind

$$S2 = \begin{bmatrix} 1 & . & . & . & . \\ 0 & 1 & . & . & . \\ 0 & 1 & 1 & . & . \\ 0 & 1 & 3 & 1 & . \\ 0 & 1 & 7 & 6 & 1 \end{bmatrix}$$

S1 := matrix$_{r=0..inf,c=0..inf}[s1_{r,c}]$
// s1 Stirling-numbers 1st kind

$$S1 = \begin{bmatrix} 1 & . & . & . & . \\ 0 & 1 & . & . & . \\ 0 & -1 & 1 & . & . \\ 0 & 2 & -3 & 1 & . \\ 0 & -6 & 11 & -6 & 1 \end{bmatrix}$$

fS2F := *f * S2 * F* *// in other texts denoted as "**U**" for "U-tetration"*
fS1F := *f * S1 * F*
// similarity-scalings for operation with vandermonde vectors

$S1 = S2^{-1}$ *// see, for instance [A&S]*

W .= *f * VZ = fS2F * P~* *// in other texts denotet as "**B**" for "T-tetration"*

3.2. Addendum concerning diagonalization of *P*

Though I didn't formulate the complete proof, it may be worth to notice here, that a diagonalization of ***P*** by a left-eigenmatrix (or of ***P~*** by a right one) for the case of infinite dimension can be defended.

We can exploit the fact, that the matrix ***W*** can be factored into two triangular matrix-factors, which are separately invertible; in my notation:

(3.2.1.) $W = fS2F * P\sim$

(3.2.2.) $= (F^{-1} * S2 * F) * P\sim$

The factors are invertible, and for ***S2*** we use ***S1*** as inverse[6]:

(3.2.3.)
$$\begin{aligned} fS2F*fS1F &= fS1F*fS2F \\ &= F^{-1}\ S1\ F\ F^{-1}\ S2\ F \\ &= F^{-1}\ S1\ S2\ F \\ &= F^{-1}\ \ I\ \ F \\ &= I \end{aligned}$$

(3.2.4.) $\Rightarrow \quad fS2F^{-1} = fS1F$

and for the possibility of inversion of ***P*** we don't need a long derivation.

Then the initial setup

$$P\sim * W = W * E$$

can be expanded and rewritten to

(3.2.5.) $P\sim * fS2F * P\sim = fS2F * P\sim * E$ *// expand W*

(3.2.6.) $F * P\sim * fS2F * P\sim = S2*F* P\sim * E$ *// premultiply by F*

$H * P\sim = ...$

Here we look first at the product ***H = F * P~ fS2F*** and find that the entries of this product are all derivatives of the function *exp(x)-1* and of all its powers evaluated at *x=1*. Thus, although ***P~*** is not rowfinite and ***fS2F*** not columnfinite we have analytically usable (known) entries in the result ***H*** of the matrix-product.

Next, the left-multiplication with ***S1*** is possible without a convergence-problem because ***S1*** is rowfinite, so we can write:

(3.2.7.) $S1 * H * P\sim = F* P\sim * E$ *// 2.2.6 premultiplied by S1*

But moreover, the crucial point is, that also the product

$$S1 * H$$

is (upper) triangular and thus column-finite (heuristic, not yet proven). This allows to proceed with the reciprocals of the remaining second and the third factors on the rhs : the premultiplication by F^{-1} (which is diagonal and can always be used) and finally $P^{-1}\sim$ is thus possible, gives again exact values and provides the sought diagonalization of ***P***. In short we did:

(3.2.8.) $P^{-1}\sim F^{-1}\ S1 * H * P\sim = E$

$P^{-1}\sim fS1F * P\sim * fS2F\ P\sim = E$

(3.2.9.) $P^{-1}\sim (fS1F * P\sim * W) = E$

We cannot write W^{-1} for ($P^{-1}\sim * fS1F$) as first factor in the above (because of an occuring infinite sum (*zeta(1)*)), but must indicate the order of computation by

[6] concerning the reciprocal relation between Stirlingnumbers 2nd and 1st kind see for instance [A&S]

parentheses; and must stay with that we cannot use associativity to change order of computation.

3.3. Combinatorical interpretations in tetrates of *P*

Interestingly, the tetrates of ***P*** are related to some combinatorical properties. We find the entries of the first column of $P^{\wedge\wedge h}$ for *h=2..5* in the OEIS

P^^2 *A000248* *1, 1, 3, 10, 41, 196, ...*

Number of forests with n nodes and height at most 1.

P^^3 *A000949* *1, 3, 16, 101, 756, 6607, 65794, 733833, 9046648, ...*

Number of forests with n nodes and height at most 2.

P^^4 *A000950* *1, 3, 16, 125, 1176, 12847, 160504, ...*

Number of forests with n nodes and height at most 3.

P^^5 *A000951* *1, 3, 16, 125, 1296, 16087, 229384, 3687609, ...*

Number of forests with n nodes and height at most 4.

P^^inf *A000272* *1, 1, 1, 3, 16, 125, 1296, 16807, 262144,*

Number of trees on n labeled nodes: n^(n-2).

See: "Online encyclopedia of Integer sequences, N.J.A. Sloane"

http://www.research.att.com/~njas/sequences/A000248
http://www.research.att.com/~njas/sequences/A000949
http://www.research.att.com/~njas/sequences/A000950
http://www.research.att.com/~njas/sequences/A000951
http://www.research.att.com/~njas/sequences/A000272

3.4. Entry in OEIS for *Exp(PP – I)* :

A124824 LambertW analog of the Bell numbers:
a(n) = (1/e)*Sum_{k>=0} k*(n+k)^(n-1)/k! for n>0 with a(0)=1.

1, 1, 4, 26, 235, 2727, 38699, 649931, 12616132, 278054700, 6861571205, 187474460527,

E.g.f.: A(x) = exp(L(x) - 1), where L(x) = -LambertW(-x)/x. - Vladeta Jovovic, Nov 10 2006
E.g.f.: A(x) = exp(Sum_{n>=1} (n+1)^(n-1)*x^n/n!).
a(n) = Sum_{k=0..n} C(n-1,k-1)*n^(n-k)*Bell(k).

More generally: e.g.f. B(x,m) = exp(L(x)^m - 1) generates the sequence:
a(n) = Sum_{k=0..n}m^k* C(n-1,k-1)*n^(n-k)*Bell(k)
and also
a(n) = (1/e)*Sum_{k>=0} m*k*(n+m*k)^(n-1)/k! for n>0 with a(0)=1.
- Vladeta Jovovic and Paul Hanna, Nov 10 2006

EXAMPLE
A(x) = 1 + x + 4*x^2/2! + 26*x^3/3! + 235*x^4/4! + 2727*x^5/5! +...
E.g.f.: log(A(x)) = L(x) - 1, where L(x) = -LambertW(-x)/x, or,
L(x) = 1 + x + 3*x^2/2! + 16*x^3/3! +...+ (n+1)^(n-1)*x^n/n! +...

Since L(x)^k = Sum_{n>=0} k*(n+k)^(n-1)*x^n/n!, for all k, then the series representation of the g.f. is derived from:
A(x) = (1/e)*Sum_{k>=0} Sum_{n>=0} k*(n+k)^(n-1)/k!*x^n/n!
so that a(n) = (1/e)*Sum_{k>=0} k*(n+k)^(n-1)/k! with a(0)=1.

```
PROGRAM  a(n)=n!*polcoeff(exp(sum(m=0, n, (m+1)^(m-1)*x^m/m!)-1), n)
(PARI)   {a(n)=if(n==0, 1, round(exp(-1)*sum(k=0, 3*n, k*(k+n)^(n-1)/k!)))}
(PARI)   {a(n)=if(n==0, 1, sum(k=0, n, binomial(n-1, k-1)*n^(n-k)*k!* polcoeff(exp(exp(x+x*O(x^k))-1), k)))}
```

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