Infinite alternating sums of powertowers of like height - two conjectures

Update version 5 (15.10.2007)
"theorem"-> "conjecture"... ;-)

Note:

The following is an excerpt of a not yet finished complete article, which derives the (new?) general identities concerning infinite series of powertowers of like height. It is meant to allow criticism and correction of errors before compiling the final article with a possible serious error. Comments are welcome!

Relevant references will be supplied in the coming article.

A more introductory version of this is in

http://go.helms-net.de/math/pdf/Tetration_GS.pdf

Comments are invited to *mailto:helms@uni-kassel.de* , subject: tetration-conjecture 0708

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Gottfried Helms, 15.10.2007

1. Tetration (Powertowers) :a conjectured general series-identity

1.1. Intro

Tetration is widely discussed in terms of "continuous tetration", "fractal iteration" and some more. In the shadow of this, the simple version of towers of integral heights, sometimes called "natural tetration", seems to be a bit underrated, although it seems to follow some general properties, which are of basic interest for number theory; one of these is the question of infinite series of towers.

The results here are conjectures concerning alternating series of powertowers, which seem new and may be crosschecked. It is only in short described below; a more detailed description of the approach is done in the earlier version of this manuscript and shall be done separately later again, hoping I can add formal proofs.

Also this is a mainly formal derivation according to matrix-relations; where I omit the description of bounds for the parameters. However everywhere summability of divergent series is assumed, where they are regularly Euler-summable, for instance the series of the (Dirichlet-) eta-function (or "alternating zeta"), like *eta(-1)=1-2+3-4 ...* or *eta(0)=1-1+1-1...*

The conjectures state the following identities for tetration-analogues for the geometric series.

1.2. Conjecture 1

Denote tetration by the following recursive function:

(1.2.1.) $$T_s^{(0)}(x) = x \qquad T_s^{(1)}(x) = s^x \qquad T_s^{(m)}(x) = s^{T_s^{(m-1)}(x)}$$

Conjecture 1:

For any height m>=0 *of the tetration-function* $\mathbf{T_s^{(m)}}(x)$ *for any admissible base* s *we have:*

(1.2.2.) $$\sum_{k=0}^{oo} (-1)^k * T_s^{(m)}(k) + \sum_{k=0}^{oo} (-1)^k * T_s^{(m)}(-k)$$
$$= T_s^{(m)}(0) \qquad = T_s^{(m-1)}(1) \qquad = T_s^{(m-2)}(s)$$

"admissible" means here s of the range where the sums converge or can be summed regularly in the sense of divergent summation technique.

Examples:

(1.2.3.) m=0 $$0 = \left(\sum_{k=0}^{inf} (-1)^k * k\right) + \left(\sum_{k=0}^{inf} (-1)^k * (-k)\right)$$
$$0 = (1-2+3-4+\ldots-\ldots) + (-1+2-3+4+\ldots-\ldots)$$

(1.2.4.) m=1 $$1 = \left(\sum_{k=0}^{inf} (-1)^k * s^k\right) + \left(\sum_{k=0}^{inf} (-1)^k * s^{-k}\right)$$
$$1 = (s^0 - s^1 + s^2 - s^3 + \ldots - \ldots) + (\frac{1}{s^0} - \frac{1}{s^1} + \frac{1}{s^2} - \frac{1}{s^3} + \ldots - \ldots)$$

This may also be rewritten as

$$T_s^{(1)}(0) = 1 = (s^0 + \frac{1}{s^0}) - (s^1 + \frac{1}{s^1}) + (s^2 + \frac{1}{s^2}) - (s^3 + \frac{1}{s^3}) + \ldots -$$

The case m=1 reflects actually the geometric series and is consistent with their direct evaluation.

(1.2.5.) $m=2$

$$s=\left(\sum_{k=0}^{inf}(-1)^k * s^{s^k}\right)+\left(\sum_{k=0}^{inf}(-1)^k * s^{s^{-k}}\right)$$

$$s=(s^{s^0}-s^{s^1}+s^{s^2}-s^{s^3}+\ldots-\ldots)+(\sqrt[s^0]{s}-\sqrt[s^1]{s}+\sqrt[s^2]{s}-\sqrt[s^3]{s}+\ldots-\ldots)$$

This may be rewritten as

$$T_s^{(2)}(0)=s=(s^{s^0}+\sqrt[s^0]{s})-(s^{s^1}+\sqrt[s^1]{s})+(s^{s^2}+\sqrt[s^2]{s})-(s^{s^3}+\sqrt[s^3]{s})+\ldots-\ldots$$

(1.2.6.) $m=3$

$$s^s=\left(\sum_{k=0}^{inf}(-1)^k * s^{s^{s^k}}\right)+\left(\sum_{k=0}^{inf}(-1)^k * s^{s^{s^{-k}}}\right)$$

$$s^s=\left(s^{s^{s^0}}-s^{s^{s^1}}+s^{s^{s^2}}-s^{s^{s^3}}+\ldots-\ldots\right)+\left(s^{\sqrt[s^0]{s}}-s^{\sqrt[s^1]{s}}+s^{\sqrt[s^2]{s}}-s^{\sqrt[s^3]{s}}+\ldots-\ldots\right)$$

This may be rewritten as

$$\begin{aligned}
T_s^{(3)}(0) &=\sum_{k=0}^{inf}(-1)^k *(T_s^{(3)}(k)+T_s^{(3)}(-k))\\
&=\sum_{k=0}^{inf}(-1)^k *(s^{s^{s^k}}+s^{\sqrt[s^k]{s}})\\
s^s &=(s^{s^{s^0}}+s^{\sqrt[s^0]{s}})-(s^{s^{s^1}}+s^{\sqrt[s^1]{s}})+(s^{s^{s^2}}+s^{\sqrt[s^2]{s}})-(s^{s^{s^3}}+s^{\sqrt[s^3]{s}})+\ldots-\ldots
\end{aligned}$$

Note, that the strongly divergent expressions using the positive exponents in the formulae can thus easily be obtained by the summation of the series with negative exponents, whose terms quickly approach *1* in their absolute values, and can thus be Cesaro- or Euler-summed, when the above identities are numerically checked.

Another way to write these is

(1.2.7.) $m=0$

$$\boxed{\begin{array}{lll} & +0 & -1+2-3+4\ldots.\\ \ldots+(-4)-(-3)+(-2)-(-1) & +(-0) & =0\end{array}}$$

(1.2.8.) $m=1$

$$\boxed{\begin{array}{lll} & +s^0 & -s^1+s^2-s^3+s^4\ldots.\\ \ldots+s^{-4}-s^{-3}+s^{-2}-s^{-1} & +s^0 & =s^0\end{array}}$$

(1.2.9.) $m=2$

$$\boxed{\begin{array}{lll} & +s^{s^0} & -s^{s^1}+s^{s^2}-s^{s^3}+s^{s^4}\ldots.\\ \ldots+s^{s^{-4}}-s^{s^{-3}}+s^{s^{-2}}-s^{s^{-1}} & +s^{s^0} & =s^{s^0}\end{array}}$$

for any m, which look much similar to that of Laurent-series-expansions[1] and suggest then a very basic property of such series, namely, that all (alternating) Laurent-series of this form sum up to zero. For s in the range $e^{-e} < s < e^{1/e}$ this can also be verified numerically by Euler-summation.

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[1] http://mathworld.wolfram.com/LaurentSeries.html

Conjecture 2

Denote the variant of tetration as the following recursive function:

(1.3.1.) $$U_s^{(0)}(x) = x \qquad U_s^{(1)}(x) = s^x - 1 \qquad U_s^{(m)}(x) = s^{U_s^{(m-1)}(x)} - 1$$

Conjecture 2:

For any height m>=0 of the tetration-function $\mathbf{U_s^{(m)}}(x)$ *we have:*

(1.3.2.) $$0 = \left(\sum_{k=0}^{oo} (-1)^k * U_s^{(m)}(k)\right) + \left(\sum_{k=0}^{oo} (-1)^k * U_s^{(m)}(-k)\right)$$

Examples:

(1.3.3.) *m=1*

$$0 = \left(\sum_{k=0}^{inf} (-1)^k * (s^k - 1)\right) + \left(\sum_{k=0}^{inf} (-1)^k * (s^{-k} - 1)\right)$$

$$(s^0 - s^1 + s^2 - s^3 + \ldots - \ldots) - \frac{1}{2} = -\left(\frac{1}{s^0} - \frac{1}{s^1} + \frac{1}{s^2} - \frac{1}{s^3} + \ldots - \ldots\right) + \frac{1}{2}$$

$$s^0 - s^1 + s^2 - s^3 + \ldots - \ldots = 1 - \left(\frac{1}{s^0} - \frac{1}{s^1} + \frac{1}{s^2} - \frac{1}{s^3} + \ldots - \ldots\right)$$

The case m=1 reflects actually the geometric series and is consistent with their direct evaluation.

(1.3.4.) *m=2*

$$0 = \left(\sum_{k=0}^{inf} (-1)^k * (s^{s^k - 1} - 1))\right) + \left(\sum_{k=0}^{inf} (-1)^k * (s^{s^{-k} - 1} - 1))\right)$$

$$s^{s^0 - 1} - s^{s^1 - 1} + s^{s^2 - 1} - s^{s^3 - 1} + \ldots - \ldots = 1 - \left(s^{\frac{1}{s^0} - 1} - s^{\frac{1}{s^1} - 1} + s^{\frac{1}{s^2} - 1} - s^{\frac{1}{s^3} - 1} + \ldots - \ldots\right)$$

Note, that this is also equivalent to *conjecture 1, example m=2* by multiplication times *s*:

$$s^{s^0} - s^{s^1} + s^{s^2} - s^{s^3} + \ldots - \ldots = s - \left(s^{\frac{1}{s^0}} - s^{\frac{1}{s^1}} + s^{\frac{1}{s^2}} - s^{\frac{1}{s^3}} + \ldots - \ldots\right)$$

$$s^{s^0} - s^{s^1} + s^{s^2} - s^{s^3} + \ldots - \ldots = s - \left(\sqrt[s^0]{s} - \sqrt[s^1]{s} + \sqrt[s^2]{s} - \sqrt[s^3]{s} + \ldots - \ldots\right)$$

(1.3.5.) *m=3*

$$0 = \left(\sum_{k=0}^{inf} (-1)^k * (s^{s^{s^k - 1} - 1} - 1)\right) + \left(\sum_{k=0}^{inf} (-1)^k * (s^{s^{s^{-k} - 1} - 1} - 1)\right)$$

or $$s^{s^{s^0 - 1} - 1} - s^{s^{s^1 - 1} - 1} + s^{s^{s^2 - 1} - 1} - \ldots + \ldots\ldots = 1 - \left(s^{s^{s^{-0} - 1} - 1} - s^{s^{s^{-1} - 1} - 1} + s^{s^{s^{-2} - 1} - 1} - \ldots + \ldots\right)$$

$$\sqrt[s]{s^{s^{s^1}}} - \sqrt[s]{s^{s^{s^2}}} + \sqrt[s]{s^{s^{s^3}}} - \ldots + \ldots = s - \left(\sqrt[s]{s^{\sqrt[s^1]{s}}} - \sqrt[s]{s^{\sqrt[s^2]{s}}} + \sqrt[s]{s^{\sqrt[s^3]{s}}} - \ldots + \ldots\right)$$

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2. Short derivation of the results.

2.1. Notations

The function ***T*** can be described by a matrix-multiplication

(2.1.1.) $V(x)\sim * {}^dV(\log(s)) * B = V(y)\sim$

where

(2.1.2.) *V(x) is a vector containing powers of x and is assumed as diagonal when prefixed* $^dV(x)$

$V(x) = \text{colvector}(1,x,x^2,x^3,....)$ *V(x)~ denoting its transpose*

*The notation V(x) means a type of vector here - that is, only if the vector in discussion is of the form of consecutive powers of a single argument x in the described form, then it is a V(x)-vector. This is then of importance when a matrix-multiplication V(x)~ * **A** = V(y)~ is discussed: since the output has the same form as the input, the multiplication can be iterated without change of the type of the result-vector .*

This remark seems to be important because of some misreadings recently.

(2.1.3.) ***B** is an infinite square matrix defined by*

$\boldsymbol{B} = \text{matrix}(\ c^r/r!\ )$

where c indicates the columnindex

and r the row-index, both starting at 0

$$\begin{bmatrix} 1/0! & 1/0! & 1/0! & 1/0! & 1/0! & 1/0! \\ 0/1! & 1/1! & 2/1! & 3/1! & 4/1! & 5/1! \\ 0/2! & 1/2! & 4/2! & 9/2! & 16/2! & 25/2! \\ 0/3! & 1/3! & 8/3! & 27/3! & 64/3! & 125/3! \\ 0/4! & 1/4! & 16/4! & 81/4! & 256/4! & 625/4! \\ 0/5! & 1/5! & 32/5! & 243/5! & 1024/5! & 3125/5! \end{bmatrix}$$

2.2. Arguing for conjecture 1

Formula (2.1.1) expresses the obvious evaluation of exponential-series for a parameter s (and an additional co-exponent x), which performs exponentiation of powers of x into powers of s^x .

For convenience I use the short notation

(2.2.1.) $B_s = {}^dV(\log(s)) * B$

First we'll establish the formal lemma:

Lemma:

(2.2.2.) If the input-vector is of the ***V( )***-type, then also the output-vector, and for input ***V(x)*** it is $\boldsymbol{V}(s^x)$:

$V(x)\sim * B_s = V(s^x)\sim$ *// as far as log(s) is defined*

Proof: By the basic property of exponential-series it is obvious, that the left-multiplication of ***B*** as well as of $\boldsymbol{B_s}$ indeed preserves the ***V()***-type of the input vector for the output-vector, since for each column c of the result we have:

(2.2.3.) $V(x)\sim * B\,[c] \ = \Sigma_{r=0..inf}\ x^r * (c^r / r!) = \Sigma_{r=0..inf} (x\,c)^r / r! = e^{xc}$
$= (e^x)^c$

(2.2.4.) $V(x)\sim * B_s\,[c] \ = \Sigma_{r=0..inf}\ x^r * (\log(s)*c)^r / r!) = \Sigma_{r=0..inf} (\log(s^x)\,c)^r / r! = s^{xc}$
$= (s^x)^c$

(---- end of proof)

By (2.1.1) we thus find in the second column of ***V(y)~*** the scalar result $y=s^x$, since ***V(y)*** contains

(2.2.5.) $V(y) \ = \text{colvector}(y^0,y^1,y^2,y^3,...)$ *and*
$V(y)[1] = y = s^x = T_s^{(1)}(x)$

By *preservation of the form of the input-vector* it is obvious, that this matrix-multiplication can be repeated, so we get

(2.2.6.) $V(x)\sim * B_s^m = V(y)\sim$

the result in the second column of $V(y)$ the m-fold iterated exponential:

(2.2.7.) $y = T_s^{(m)}(x)$

Since the function $T_s^{(m)}(x)$ is expressible using a matrix-power (which is independent from x), we may apply linear combinations to $V(x)$ and get the the corresponding linear-combinations in the result.

For the infinite alternating series of vectors $V(0)-V(1)+V(2)$.... we get

(2.2.8.) $(V(0) - V(1) + V(2) - V(3)+\ldots) * B_s = [\ 1/2, (s^0 - s^1 + s^2 - s^3 +\ldots-\ldots), (s^0 - s^2 + s^4 - s^6 +\ldots-\ldots),\ldots]$

where the second column of the result is just the alternating geometric series in s.

The lhs can be rewritten as vector of the *eta( )*-function, with the modification, that it starts at 0, where also $0^0=1$ is assumed:

(2.2.9.) Let $eta_0(-n) = 0^n - 1^n + 2^n - 3^n + \ldots.$

Then let

(2.2.10.)
$$\begin{aligned} H &= (V(0) - V(1) + V(2) - V(3) \ldots) \\ &= colvector(\ (0^0-1^0+2^0-3^0+\ldots),(0^1-1^1+2^1-3^1+\ldots),(0^2-1^2+2^2-3^2+\ldots),\ldots) \\ &= colvector(\ eta_0(0), eta_0(-1), eta_0(-2), eta_0(-3), \ldots\ ) \end{aligned}$$

and write

(2.2.11.) $H\sim * B_s = Y\sim$

where now the second column of $Y\sim$ contains the result

(2.2.12.) $y = \Sigma_{k=0..inf} (-1)^k s^k = \Sigma_{k=0..inf} (-1)^k T_s^{(1)}(k)$

If we write the same for the $V(x)$-vectors of negative parameters:

(2.2.13.) $(V(-0) - V(-1) + V(-2) -\ldots)\sim * B_s = [\ 1/2, (s^{-0} - s^{-1} + s^{-2} - s^{-3} +\ldots-\ldots), \ldots..]$

and write $H^{(-)}$ for the lhs, we have

(2.2.14.) $H^{(-)} = colvector(\ eta_0(0), -eta_0(-1), eta_0(-2), -eta_0(-3), \ldots\ )$

where the *eta*s of even argument don't change their signs, because $(-n)^{2k} = n^{2k}$ and equivalently for sums of that expression for different n.

Now, if we add the two multiplications we have for the rhs:

(2.2.15.)
$$(H + H^{(-)})\sim * B_s = [\ 1/2, (s^0 - s^1 + s^2 - s^3 +\ldots-\ldots),\ldots] + [\ 1/2, (s^{-0} - s^{-1} + s^{-2} - s^{-3} +\ldots-\ldots), \ldots..]$$

and at the lhs:

(2.2.16.)
$$(H + H^{(-)})\sim = rowvector(2*eta_0(0),\ \ 0,\ \ 2*eta_0(-2),\ \ 0\ ,\ \ 2*eta_0(-4),\ \ 0\ , \ldots\ )$$

The important property of all these constructions is now, that all $eta_0(\ )$ of negative even parameter are also zero, so in fact the system remains:

(2.2.17.) $$(H + H^{(-)})\sim = rowvector(\ 1,\ 0,\ 0,\ 0,\ 0,\ 0, \dots\ )$$

and applied to the matrix-formula we have:

(2.2.18.) $$\begin{aligned}(H + H^{(-)})\sim * B_s &= [\ 1/2, (s^0 - s^1 + s^2 - s^3 + \dots - \dots\ ), \dots] \\ &\quad + [\ 1/2, (s^{-0} - s^{-1} + s^{-2} - s^{-3} + \dots - \dots\ ), \dots\dots]\end{aligned}$$

and since we know, that $(H + H^{(-)})\sim = V(0)\sim$ we have also:

(2.2.19.) $$\begin{aligned}(H + H^{(-)})\sim * B_s &= V(0)\sim * B_s \\ &= rowvector(1, s^0, (s^0)^2, (s^0)^3, \dots) \\ &= rowvector(1,1,1,1,1,\dots)\end{aligned}$$

which is already the result for the $T_s^{(m)}(x)$ - case, where $m=1$.

If we iterate this, then it means we discuss the m'th power of $\boldsymbol{B_s}$, and we have

(2.2.20.) $$\begin{aligned}(H + H^{(-)})\sim * B_s^m &= V(0)\sim * B_s^m \\ &= V(1)\sim * B_s^{m-1} \\ &= Y\sim\end{aligned}$$

where the scalar value y in the second column of $\boldsymbol{Y\sim}$ contains the result:

(2.2.21.) $$y = T_s^{(m-1)}(1) = s^{s^{\dots^{s}}} \text{ whith } (m-1)\text{-fold occurence of } s$$

and is also identical to the explicite expression:

(2.2.22.) $$T_s^{(m)}(0) - T_s^{(m)}(1) + T_s^{(m)}(2) - T_s^{(m)}(3) + \dots - \dots$$

This completes the arguing for conjecture 1.

2.3. Arguing for conjecture 2

Completely analoguously this works for conjecture 2; the only difference is here, that we have a triangular matrix variant of $\boldsymbol{B_s}$, call it $\boldsymbol{C_s}$:

(2.3.1.) $C = F^{-1} * S2 * F$

(2.3.2.) $C_s = {}^dV(\log(s)) * C$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 1 & 1 & . & . & . \\ 0 & 1 & 3 & 1 & . & . \\ 0 & 1 & 7 & 6 & 1 & . \\ 0 & 1 & 15 & 25 & 10 & 1 \end{bmatrix}$$

S2

where $\boldsymbol{S2}$ contains the Stirling numbers 2'nd kind, and $\boldsymbol{F}$ is the diagonalmatrix of factorials

(2.3.3.) $F = \mathrm{diag}(0!, 1!, 2!, \ldots)$

Then

(2.3.4.) $V(x)\sim * C_s = \mathrm{rowvector}(\ 1, s^x - 1, (s^x - 1)^2, (s^x - 1)^3, \ldots)$

$= V(s^x - 1)\sim$

$= V(\ U_s^{(1)}(x)\ )\sim$

and since the rhs has again the form of the $\boldsymbol{V(\ )}$-vector type (it contains the consecutive powers of the same argument), $\boldsymbol{C_s}$ can be used the same way as $\boldsymbol{B_s}$ for iterated exponentials. (see the reference to Abramowitz/Stegun's Handbook of mathematical functions)

24.1.4 Stirling Numbers of the Second Kind

I. Definitions

A. $\mathfrak{S}_n^{(m)}$ is the number of ways of partitioning a set of n elements into m non-empty subsets.

B. Generating functions

$$x^n = \sum_{m=0}^{n} \mathfrak{S}_n^{(m)} x(x-1)\ldots(x-m+1)$$

$$(e^x-1)^m = m! \sum_{n=m}^{\infty} \mathfrak{S}_n^{(m)} \frac{x^n}{n!}$$

Abramowitz/Stegun
Handbook of mathematical functions P 824

The arguing is then identical to the previous discussion; only with the difference, that in the iteration from a starting-vector with $V(0)\sim$

(2.3.5.) $V(0)\sim * B_s = V(1)\sim$

(2.3.6.) $V(1)\sim * B_s = V(s)\sim$

and thus

(2.3.7.) $(H + H^{(-)})\sim * B_s = V(0)\sim * B_s = V(1)\sim$

(2.3.8.) $(H + H^{(-)})\sim * B_s^2 = V(1)\sim * B_s = V(s)\sim$

(2.3.9.) $(H + H^{(-)})\sim * B_s^3 = V(s)\sim * B_s = V(s^s)\sim$

...

but using $\boldsymbol{U_s}$ here

(2.3.10.) $V(0)\sim * C_s = V(0)\sim$

and thus

(2.3.11.) $(H + H^{(-)})\sim * C_s = V(0)\sim * C_s = V(0)\sim$

$\boldsymbol{C_s}$ doesn't change the input-vector , and thus our constellation

(2.3.12.) $(H + H^{(-)})\sim * C_s^{\ m} = V(0)\sim$

is constant for all powers of $\boldsymbol{C_s}$. This is obviously due to its triangular shape.

On the other hand, the formal description of the second column of the result in the rhs is the alternating sums of m'th iterates of $U_s^{(m)}(x)$.

We have thus the identity of the second-column of the $\boldsymbol{V(0)}$-vector $(V(0)[1]=0)$ with the sum of the two alternating sums of (the function-values) $U_s^{(m)}(x)$ and $U_s^{(m)}(-x)$, for any integer $m>0$, which completes the arguing for conjecture 2.

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3. References / Hints to further reading:

[Bachmann] Convergence of infinite exponentials
Gennady Bachman
Pacific Journal of Mathematics, Vol. 169, no. 2, 1995
http://projecteuclid.org/euclid.pjm/1102620323

[Berkolaiko] Analysis of Carleman Representation of Analytical Recursions
G. Berkolaiko, S. Rabinovich and S. Havlin
Journal of Mathematical Analysis and Applications 224, 81-90 (1998)
http://www.math.tamu.edu/~berko/papers/pdf/jmaaBRH98.pdf

[Eisenstein] Entwicklung von α ^ (α ^ (α ^ (α ^ ....))).
G. Eisenstein,
J. reine angev. Math. 28, pp 48-52, 1844.
http://www-gdz.sub.uni-goettingen.de/cgi-bin/digbib.cgi?PPN243919689_0028

[EulerE489] De formulis exponentialibus replicatis
L. Euler
Acta Academiae Scientarum Imperialis Petropolitinae 1, 1778, pp. 38-60
http://math.dartmouth.edu/~euler/docs/originals/E489.pdf

[EulerE532] De serie Lambertina plurimisque eius insignibus proprietatibus.
L. Euler
Acta Academiae Scientarum Imperialis Petropolitinae 1779, 1783, pp. 29-51
http://www.math.dartmouth.edu/~euler/pages/E532.html

[Gralewicz] Continuous time evolution from iterated maps and Carleman linearization
P.Gralewicz and K. Kowalski
Department of Theoretical Physics, University of Lódz, Poland
http://arxiv.org/PS_cache/math-ph/pdf/0002/0002044v1.pdf

[Harrell] A Short History of Operator Theory
Evans M. Harrell II
© 2004. Unrestricted use is permitted, with proper attribution, for noncommercial purposes.
http://www.mathphysics.com/opthy/OpHistory.html

[Knoebel] Exponentials Reiterated
R. A. Knoebel
The American Mathematical Monthly, 88, pp. 235-252, 1981.

[Knuth] Mathematics and Computer Science. Coping with Finiteness.
D. E. Knuth
Science 194, pp. 1235-1242, 1976.

[Länger] An elementary proof of the convergence of iterated exponentiations
H. Länger
Elem. Math. 51, pp. 75-77, 1986.

Further links to online-resources

[Galidakis] Ioannis Galidakis, **Tetration, LambertW-function** and more; online at
http://ioannis.virtualcomposer2000.com/math/

[Geiser] Daniel Geiser, **Tetration-pages**; online at
http://www.tetration.org

[McDonnell]
http://www.faculty.fairfield.edu/jmac/ther/tower.htm

[Robbins] Andrew Robbins, **Tetration-pages**, online at:
http://tetration.itgo.com/txt/table-tetnat.txt

[Weissstein] http://mathworld.wolfram.com/LambertW-Function.html

From my own project (06-07 2007):

[PowTowSum] An infinite alternating sum of powertowers of increasing height (the initial heuristic)
http://go.helms-net.de/math/binomial_new/10_4_Powertower.pdf

[PowTowCrit] Critical point for this method of summation at *s=exp(-1)*
http://go.helms-net.de/math/binomial_new/PowertowerproblemDocSummation.htm

[SumLikePow] Summing of like powers (zeta and hurwitz-zeta using matrices)
http://go.helms-net.de/math/binomial_new/04_3_SummingOfLikePowers.pdf