1. Fixpoint by construction (for "Tetration-Forum")

1.1. Intro

Denote tetration by the following recursive function:

(1.1.1.) $$T_s^{(0)}(x) = x \qquad T_s^{(1)}(x) = s^x \qquad T_s^{(m)}(x) = s^{T_s^{(m-1)}(x)}$$

Note, that I begin to use the notation

$T_x^{(oo)}(1) = {}_{\dots x}x^x$

instead of the common

$T_x^{(oo)}(1) = x^{x^{x^{\dots}}}$

because
* this seems more consistent with my matrix-approach to tetration and
* allows to define a starting value as top-exponent even for infinite towers (which seems also more consistent with the idea of an initial value for iterations (including infinite repetitions).
* this seems also to be more consistent with the notion of several fixpoints
* and with the problem of consistency of partial evaluation of an expression which is meant as being infinitely iterated.

Fixpoints

Since I don't have the Productlog-function ready in Pari/Gp, I used a handwaved recursive tracer to approximate complex "fixpoints" t for real bases s (and currently have to relate to this), such that

(1.1.2.) $$s^t = t \quad or \qquad s = t^{1/t}$$

and

(1.1.3.) $${}_{\dots s}s^{s^t} = t$$

Here I want to have a deeper look into this problem. My approach is here, to assume an arbitrary complex t and see, whether we can construct all real $s>0$ from this assumtion. In fact, I actually assume a parameter u, where $u=log(t)$ first, compute the unique t from this and then s. I omit the periodicity for u for a start.

But the first problem is, can *(1.1.2)* actually be translated into *(1.1.3)*, if t is complex?

1.2. Question: is ${}_{\dots x}x^x = y$ one-to-one translatable into $y^x=x$?

compare: Galidakis: http://ioannis.virtualcomposer2000.com/math/exponents.html (a bit edited):

(2) $[x^{(1/x)}]^{[x^{(1/x)}]^{[x^{(1/x)}]}} =$	{	x	if x is in [1/e,e]
		y, where y is in [1/e, e] and y satisfies: $y^{(1/y)}=x^{(1/x)}$	if x is in (e,+oo)

In a more graphical shape:

$$a = x^{\frac{1}{x}} \Rightarrow {}_{\dots a}a^a = \begin{cases} x & if\ -1 \le log(x) \le 1 \\ y & if\ 1 < log(x) \qquad where -1 < log(y) < 1 \\ & \qquad\qquad\qquad and\ y^{\frac{1}{y}} = x^{\frac{1}{x}} = a \end{cases}$$

In my understanding this first states the ambiguity of representations for a. In the following the LambertW-function is discussed to establish the identity of *(1.1.2)* and *(1.1.3)*. And then, working further through Ionannis' article, I got to the following consideration on my sketchpad.

1.3. Derivation

Assume the formal relation (in this example I use only the principal branch of logarithm)

(1.3.1.) $s = t^{1/t}$ and $u = \log(t)$

Assume the u as a free parameter and t and s depending on u. Denote the components of t and u

(1.3.2.) $u = \alpha + \beta i$ and $t = \exp(u) = a + b i$

then first

$$\begin{aligned} s &= t^{1/t} \\ &= \exp((\alpha + \beta i)/(a + bi)) \\ &= \exp((\alpha + \beta i)(a - bi)/|t|^2) \\ &= \exp(a\alpha + b\beta) * \exp((a\beta - b\alpha) i)^{1/|t|^2} \qquad (1.3.3.) \end{aligned}$$

Then to have s real, given the parameters u and t it is necessary that

(1.3.4.) either $\beta = 0 \implies b = 0$ (the "real-only-case")
(1.3.5.) or $\beta =/= 0$ but $(a\beta - b\alpha) = 0$ (the "complex-to-real" case)

By definition, a and b are functions of α and β, so we need only choose some α and β.

(1.3.6.) $t = a + bi = \exp(\alpha + \beta i) = \exp(\alpha) * (\cos(\beta) + i \sin(\beta))$
(1.3.7.) $a = \exp(\alpha) * \cos(\beta)$ $b = \exp(\alpha) * \sin(\beta)$

Then, to have s purely real it is required by *(1.3.3)*, that

(1.3.8.) $(a\beta - b\alpha) = 0 \implies$
(1.3.9.) $\exp(\alpha) * \cos(\beta)\beta - \exp(\alpha) * \sin(\beta)\alpha = 0$

and since β is an argument of $\cos()$ and $\sin()$, it seems best to choose β as free parameter and α as the dependent:

(1.3.10.) $\alpha = \beta \cos(\beta) / \sin(\beta)$

so

(1.3.11.) $u = \beta \cos(\beta) / \sin(\beta) + \beta i$
$= \beta / \sin(\beta) * (\cos(\beta) + \sin(\beta) i)$

and a certain selection for β defines then the whole formula for s.

We have then

(1.3.12.) $$u = \frac{\beta}{\sin(\beta)} * \exp(\beta i) = \frac{\beta}{\sin(\beta)} * (\cos(\beta) + \sin(\beta) i)$$

(1.3.13.) $$\begin{aligned} t &= \exp(u) \\ &= \exp\left(\beta \frac{\cos(\beta)}{\sin(\beta)}\right) * \exp(\beta i) = \exp\left(\beta \frac{\cos(\beta)}{\sin(\beta)}\right) * (\cos(\beta) + \sin(\beta) i) \end{aligned}$$

(1.3.14.) $$s = t^{\frac{1}{t}} = \exp(\tfrac{u}{t}) = \exp\left(\frac{\frac{\beta}{\sin(\beta)}}{\exp\left(\beta \frac{\cos(\beta)}{\sin(\beta)}\right)}\right)$$

with singularities where β is a integer multiple of π, with the one exception: the singularity at $\beta = 0 * \pi$ is removed and α can assume any value in this case. If $\beta = 0$ then α is a free real parameter , u and t are then real, too, and we get the known form:

(1.3.15.) $$s = t^{\frac{1}{t}} = \exp(\tfrac{u}{t}) = \exp\left(\frac{\alpha}{\exp(\alpha)}\right) = \exp\left(\frac{\alpha}{a}\right) = a^{\frac{1}{a}}$$

1.4. The "real-only" case (real u,t,(ß=0, b=0); real s)

If $\beta=0$ (then also $b=0$), then we may freely choose α having $u=\alpha$ and then $t = exp(u) = exp(\alpha)$ we have due to Euler, the special ranges for

(1.4.1.) s given by $1/e^e < s < e^{1/e}$

(1.4.2.) $1/e < a=t < e$

(1.4.3.) $-1 < \alpha=u < 1$

Where $\alpha=1$ marks also the upper limit for s in the above formula (1.3.15) (when setting $\beta=0$):

(1.4.4.) $$s = t^{\frac{1}{t}} = exp(\tfrac{u}{t}) = exp\left(\frac{1}{exp(1)}\right) = e^{\frac{1}{e}}$$

for $\beta=0$, $u=\alpha$	then $t=a$	and s evaluates to (limit)
α -->-oo	a --> 0	$s = e^{\frac{\alpha}{e^{\alpha}}} = e^{\frac{-oo}{0}} \rightarrow e^{-oo} = 0$
α = -1	a = 1/e	$s = e^{\frac{-1}{e^{-1}}} = e^{-e}$
α = 0	a = 1	$s = e^{\frac{0}{1}} = 1$
α = +1	a = e	$s = e^{\frac{1}{e^{1}}} = e^{\frac{1}{e}}$ (maximum)
α --> +oo	a --> oo	$s = e^{\frac{\alpha}{e^{\alpha}}} = e^{\frac{1}{e^{a-log(a)}}} = e^{\frac{1}{e^{oo}}} \rightarrow e^{0} = 1$

1.5. The "complex to real" case ß<>0 (complex u,t, real s)

Here we need not separate special ranges for α and/or a; so I display the relations between the parameters in a graph. The x-axis shows the sole independent parameter β; the imaginary part of u. From here the parameter α (the real part of u) must satisfy a functional condition dependent on β; in the graph this is the blue line. The symmetry wrt the y-axis shows, that the conjugate solution of u applies with the same parameter α. and gives the same result for s.

Functional (and uniquely dependent) on the initial choice of β are then also the real and imaginary parts of $t = exp(u)$. The real and imaginary parts are displayed in magenta color.

And depending on t also $s = t^{1/t}$ is functionally defined. It is displayed in green color, and rescaled here as $log(log(s))$; this scale is indicated at the right border of the graph. The minimum of the symmetric curve for $log(log(s))$ is

$$log(log(s))=-1,\ log(s) = 1/e\ ,\ s = e^{1/e}\ .$$

this means, this function in β covers all $s > e^{1/e}$ - just the region above the range of the "real-only" case.

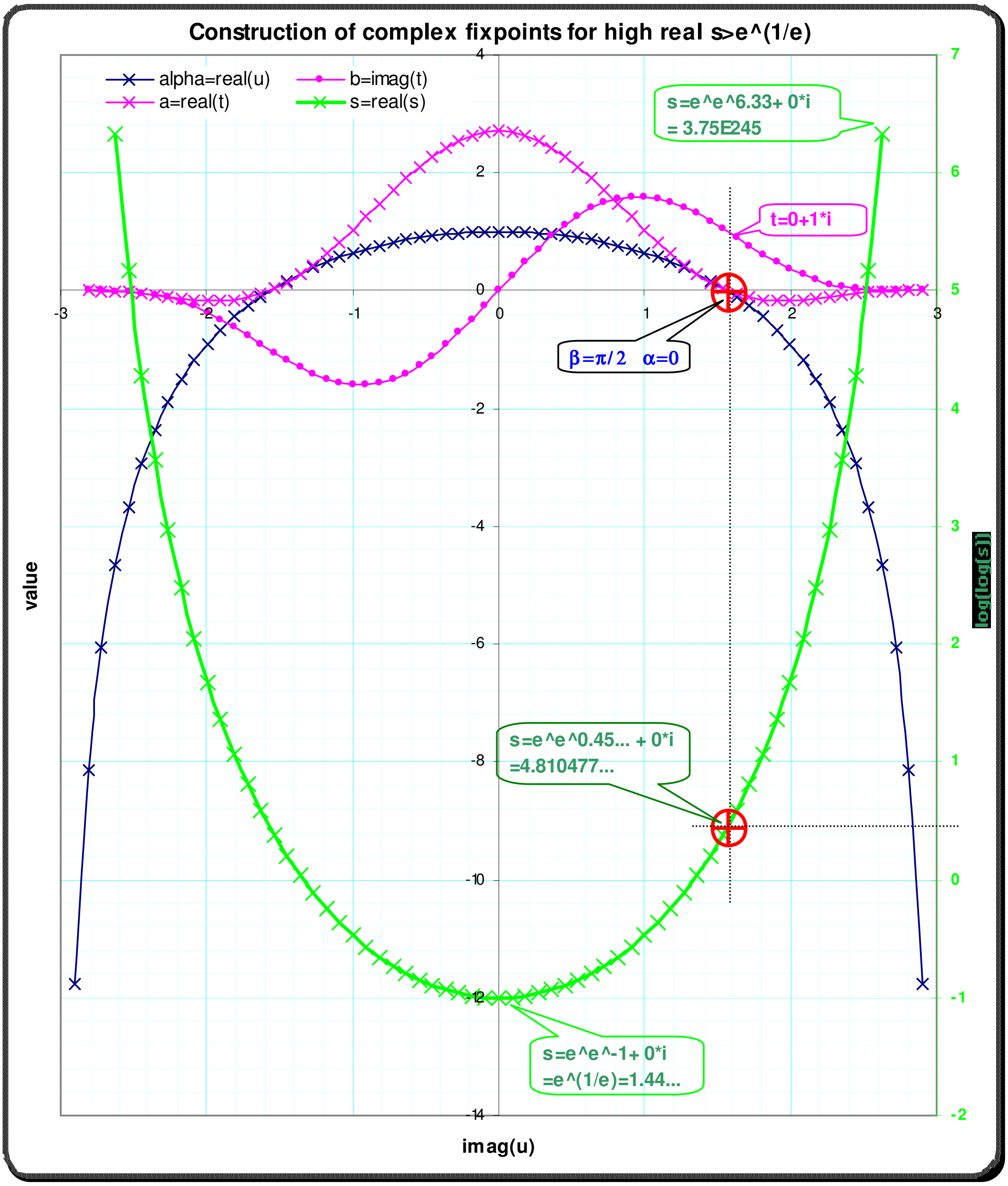
In the following graph the red-marked points have coordinates

$$\begin{aligned} u &= \alpha + \beta i &&= 0 + \pi/2\ i\ , \\ t &= a + b\ i &&= 0 + i \\ s &= &&= e^{\pi/2} \end{aligned}$$

Obviously for the two range-definitions dependend on β (where, if $\beta=0$, α can freely be selected), we get the full set of real values $s>0$ for s.

$$\begin{array}{lll} \beta = 0, & -inf < \alpha < +inf & 0 < s < e^{1/e} \\ 0 < \beta < \pi, & 0 < \alpha < 3 & e^{1/e} <= s < +inf \end{array}$$

Gottfried



A reference:

See also a related description for ranges of the multivalued Lambert-W-function by Corless et al. where η appears as the parameter β in my formula and $\eta \cot \eta = \beta / \sin(\beta) * \cos(\beta)$

http://www.cs.uwaterloo.ca/research/tr/1993/03/W.pdf

> The curve which separates the principal branch, W_0, from the branches W_1 and W_{-1} is
>
> $$\{-\eta \cot \eta + \eta i : -\pi < \eta < \pi\} \tag{4.4}$$
>
> together with -1 (which is the limiting value at $\eta = 0$). The curve separating W_1 and W_{-1} is simply $(-\infty, -1]$. Finally, the curves separating the remaining branches are
>
> $$\{-\eta \cot \eta + \eta i : 2k\pi < \pm\eta < (2k+1)\pi\} \quad \text{for } k = 1, 2, \ldots \tag{4.5}$$