"Wexzal"/"coupled exponent" , "Superroot" - a generalized Lambert-W

Intro:

What is x*, if in* $x^x = y$ *only the* y*-value is given? Or, what is* x *if in* $x^{x^x} = y$ *the* y*-value is given? Of course, the first problem is classically solved by reduction of the expression to some instance of the* Lambert-W*-function*[Corless,1996][WP]*, namely* $x = \exp(W(\log(y)))$*. The term "*Wexzal*" in the title comes from a curious and very involved paper, where the authors elaborated the whereabouts of the first problem mathematically and for some physical application of it, and called that solution* x *the "*Wexzal*" and* "coupled exponent" [Fantini,1998].

But for the second case and for even more generalized cases of higher iterative order the Lambert-W *is useless, and we have not yet a suitable common function analoguous to the* Lambert-W*.*

For the Lambert-W *we can find a power series from the formal inverse of the power series for* $x \cdot \exp(x)$ *, and kindly its range of convergence is nonzero. The generalization to higher orders is straightforward from this idea: I introduce here* $^2W(x)$ *(which is actually* $W(x)$ *),* $^3W(x)$ *and the higher orders by generating formal power series.*

We have then the solutions

- *for* $x^x = y$ *by* $x = \exp(^2W(\log(y)))$ *and*
- *for the problem* $x^{x^x} = y$ *the solution by* $x = \exp(^3W(\log(y)))$ *, and*
- *for the general case* $^nx = y$ *the solution by* $x = \exp(^nW(\log(y)))$ *.*

For the coefficients of the power series for $^nW(x)$*,* $n>2$ *I did not yet find a simple general term, so I can so far only derive the coefficients for any finite order* n *to some leading index* k *(so the problem is not yet finally solved).*

The convergence-radii of the power series are surely small but likely not zero; because of alternating signs it seems possible to extend the radii of convergence by some amount by applicating Euler-summation of appropriate orders. However, I don't have knowledge about a full featured analytic continuation so far.

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[Fantini,1998] *Jay Fantini, Gilbert Kloepfer: Wexzal/The coupled Exponent*
web-article, 1998
online available copy at the tetration-forum:
http://eretrandre.org/rb/files/JayFantini1998_203.pdf

[Corless,1996] *Robert Corless (et.al.): The Lambert-W function*
Advances in Computational Mathematics, volume 5, 1996, pp. 329--359
online available at
http://www.apmaths.uwo.ca/~rcorless/frames/PAPERS/LambertW/LambertW.ps

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[WP] *(Wikipedia, anonymous): Lambert-W function*
https://en.wikipedia.org/wiki/Lambert_W_function
(accessed 11'2015)

1. Solution for $y = x^x$

We can derive a solution by the following scheme where we introduce the letter "*u*" for the *log* of *x* and the letter "*v*" for the *log* of *y* and rewrite

(1.1) $x^x = y$

as

$exp(u\ exp(u)) = y$

and then, logarithmizing both sides,

(1.2) $u\ exp(u) = v$

This gives

(1.3) $u = W(v)$ "W" is the "Lambert-W"-function

With this, the expression for *u* has a Puisieux-series in $v = log(y)$, and finally we have

(1.4) $x = exp(W (log(y)))$

The Taylor-coefficients of the *Lambert-W* can be taken by simple series-reversion of the power series for $x \cdot exp(x)$, and because of the generalization in the next section I write also $^2W(v)$:

$^2W(v) = serreverse(v \cdot exp(v))$ // on Taylor series

We get the powerseries

(1.5) $$^2W(v) = \sum_{k=1}^{oo} \frac{(-k)^{k-1}}{k!} v^k$$

giving

(1.6) $u = {}^2W(v)$ convergence for $|v| < 1/e$

$x = exp({}^2W(log(y)))$ convergence for $e^{-e^{-1}} < |y| < e^{e^{-1}}$

The explicit representation of the coefficients ^2w_k of the power series for the function $^2W(v)$ is simple, exemplarically in the column in the following table:

0			0	
1	$\cdot v^1/1!$		1^0	$\cdot v^1/1!$
-2	$\cdot v^2/2!$		-2^1	$\cdot v^2/2!$
+9	$\cdot v^3/3!$		$+3^2$	$\cdot v^3/3!$
-64	$\cdot v^4/4!$	=	-4^3	$\cdot v^4/4!$
+625	$\cdot v^5/5!$		$+5^4$	$\cdot v^5/5!$
-7776	$\cdot v^6/6!$		-6^5	$\cdot v^6/6!$
+117649	$\cdot v^7/7!$		$+7^6$	$\cdot v^7/7!$
...	...		...	...

If we define the function $^2H(v) = exp({}^2W(v))$ by exponentiation of the formal powerseries, then we have $x = {}^2H(v)$ and we have the definition

(1.7) $$^2H(v) = 1 + \sum_{k=1}^{oo} \frac{(-(k-1))^{k-1}}{k!} v^k$$

The explicite representation of the coefficients ${}^2h_k$ for the function ${}^2H(v)$ is still simple, exemplarically in the column of the table:

1	$\cdot 1$		1	$\cdot 1$
1	$\cdot v^1/1!$		0^0	$\cdot v^1/1!$
-1	$\cdot v^2/2!$		1^1	$\cdot v^2/2!$
4	$\cdot v^3/3!$		2^2	$\cdot v^3/3!$
-27	$\cdot v^4/4!$	=	3^3	$\cdot v^4/4!$
256	$\cdot v^5/5!$		4^4	$\cdot v^5/5!$
-3125	$\cdot v^6/6!$		5^5	$\cdot v^6/6!$
46656	$\cdot v^7/7!$		6^6	$\cdot v^7/7!$
...	...		...	...

Because *exp(x)* is entire/has infinite range of convergence, ${}^2H(v)$ has the same range of convergence as ${}^2W(v)$ (and again, that range can be extended for positive *v* by Eulersummation).

Examples:

(1.8.a) For some simple example using the software *Pari/GP* I took the value *y=1.2*, computed the *x*-value and checked:

$y = 1.2$ $\qquad\qquad$ $v = \log(y)$

Computing $u = {}^2W(v)$ using Eulersummation of small order

$u = 0.155988653442$ $\qquad\qquad$ $x = \exp(u) = 1.16881294101$

and indeed, checking the result I got

$x^x = 1.2000000$

as expected.

(1.8.b) Using a higher value for *y* needs higher order for the Euler-summation and more terms of the power series. I get for *y=5* with *128* series-terms *22* correct digits (Pari/GP-code):

```
    dim = 128       \\ set global variable "dim" for the following matrix-operations
default(seriesprecision,128)
W_2 = serreverse (x · exp(x))

    y = 5
             ESum( 2.2 ) ·  dV(log(y) )·Mat( polcoeffs(W_2 )~)
             u = %[dim,1]

                    \\ u = 0.755827327283
    x = exp(u)      \\    = 2.12937248276

    x^x - y         \\ = -5.45058044643 E-22
```

(1.8.c) and for *y=10* with the same partial series still *14* correct digits:

```
    y = 10
             ESum( 3.0 ) ·  dV(log(y) )·Mat(polcoeffs(W_2)~)
             u = %[dim,1]

                    \\ u = 0.918761335653
    x = exp(u)      \\    = 2.50618414559

    x^x - y         \\ = 2.15329198818 E-14
```

---

2. Solution for $y=x^{x^x}$

Similarly we can derive a solution by the following:

(2.1) $x^{x^x} = y$
and

(2.2) $\exp(u \exp(u \exp(u))) = y$
$u \exp(u \exp(u)) = v$.

Here we cannot proceed with the *Lambert-W*, but need the generalization by the series-reverse of the iterated expression on the lhs, which I call "${}^3W(\cdot)$" here:

(2.3) ${}^3W(v) = \text{serreverse} (v \cdot \exp(v \cdot \exp(v)))$ *// on formal power series*

With this, the expression for u and also x have power series in $v = \log(y)$:

(2.5) $${}^3W(v) = \sum_{k=1}^{oo} \frac{{}^3w_k}{k!} v^k$$

(I do not yet know a simple general term for the ${}^3w$)

Then we have analogusously as in the section above

(2.6) $u = {}^3W(v)$
$x = \exp({}^3W (\log(y)))$

Here are the first few coefficients ${}^3w_k$ for the function ${}^3W(v)$:

0	·1	
1	$\cdot v^1/1!$	
-2	$\cdot v^2/2!$	
3	$\cdot v^3/3!$	
20	$\cdot v^4/4!$	
-295	$\cdot v^5/5!$	
1554	$\cdot v^6/6!$	
16177	$\cdot v^7/7!$	
...	...	...

If as before we define the function ${}^3H(v) = \exp({}^3W(v))$ by exponentiation of the formal power series, then we'll get $x = {}^3H(v)$ by the definition

(2.7) $${}^3H(v) = \sum_{k=1}^{oo} \frac{{}^3h_k}{k!} v^k$$

Here is the table of first few coefficients ${}^3h_k$ for the function ${}^3H(v)$:

1	·1
1	$\cdot v^1/1!$
-1	$\cdot v^2/2!$
-2	$\cdot v^3/3!$
33	$\cdot v^4/4!$
-184	$\cdot v^5/5!$
-695	$\cdot v^6/6!$
32124	$\cdot v^7/7!$
...	...

Unfortunately I could not yet find a short explicite description of the coefficients ${}^3h_k$.

Examples:

(2.8.a) For some simple example I took again the value $y=1.2$, computed the x-value and checked:

$y = 1.2$ $v = \log(y)$

I got computing $u = {}^3W(v)$ using Eulersummation of small order

$u = 0.152624613728$ $x = \exp(u) = 1.16488761404$

and indeed, checking the result I got

$x^{x^x} = 1.2000000$

as expected.

(2.8.b) Using a higher value for y requires higher order for the Euler-summation (even higher than in the ${}^2W(v)$-case) and more terms of the power series. I get for $y=5$ with *128* terms of the partial series *15* correct digits: (Pari/GP-code):

```
        dim = 128       \\ set global variable "dim" for the following matrix-operations
    default(seriesprecision,128)
    W_3 = serreverse( x·exp(x·exp(x)))

        y = 5
                 ESum( 3.1 ) ·  dV(log(y) )·Mat(polcoeffs(W_3 )~)
                 u = %[dim,1]

                        \\ u = 0.576634621850
        x = exp(u)      \\    = 1.78003783883

        x^x^x - y       \\ = -1.34105581617 E-15
```

(2.8.c) and for $y=10$ with the same partial series I got still *11* correct digits:

```
        y = 10
                 ESum(5.0 ) ·  dV(log(y) )·Mat(polcoeffs(W_3 )~)
                 u = %[dim,1]
                        \\ u = 0.654190131594
        x = exp(u)      \\   = 1.92358403644

        x^x^x - y       \\ = -3.97098400734 E-11
```

(2.8.d) Even for $y=-1$ and $y = i$ I can find some approximation; however the best solution with *128* terms and Euler-order which I could only manually optimize so far was:

```
        y = -1          \\ ESum with order o =( 1.3+8.2·I )
        u               \\  = 0.762831989634  + 0.321812259776·I
        x = exp(u)      \\  = 2.03425805694   + 0.678225493699·I
        x^x^x           \\ =-0.998626839391   + 0.0000476837419237·I
        x^x^x - y       \\ = 0.00137316060888 + 0.0000476837419237·I
        y = I           \\  ESum with order o = (0.8+4.7·I)
        u               \\ = 0.606170527439      + 0.366878451450·I
        x = exp(u)      \\ = 1.71138736118       + 0.657645680270·I
        x^x^x           \\ = 0.0000219781965550 + 1.00002585060·I
        x^x^x - y       \\ = 0.0000219781965550 + 0.0000258506028879·I
```

3. Generalization to $^nW(v)$ and $^nH(v)$

Table for $^nW(v)$

Here is the table of the coefficients of the $^nW(v)$-functions. The coefficients in the columns must be multiplied by the cofactors in the last column to make a power series in v.

Example: in the second column we recognize the coefficients for the *Lambert-W (which is* $^2W(v)$*)*:

Table 3.1:

$^1W(v)$	$^2W(v)$	$^3W(v)$	$^4W(v)$	$^5W(v)$	$^6W(v)$	$^7W(v)$	$^8W(v)$	$^9W(v)$	$^{10}W(v)$		
0	0	0	0	0	0	0	0	0	0	...	$\cdot 1$
1	1	1	1	1	1	1	1	1	1		$\cdot v^1/1!$
0	−2	−2	−2	−2	−2	−2	−2	−2	−2		$\cdot v^2/2!$
0	9	3	3	3	3	3	3	3	3		$\cdot v^3/3!$
0	−64	20	−4	−4	−4	−4	−4	−4	−4		$\cdot v^4/4!$
0	625	−295	125	5	5	5	5	5	5		$\cdot v^5/5!$
0	−7776	1554	−1806	714	−6	−6	−6	−6	−6		$\cdot v^6/6!$
0	117649	16177	10927	−12593	5047	7	7	7	7		$\cdot v^7/7!$
0	−2097152	−523832	−31928	87352	−100808	40312	−8	−8	−8		$\cdot v^8/8!$
0	43046721	5347953	1817433	−287271	786249	−907191	362889	9	9		$\cdot v^9/9!$
0	−1000000000	58464710	−56896570	30230	−2872810	7862390	−9072010	3628790	−10		$\cdot v^{10}/10!$
...	...									...	...

It is extremely surprising, that the n leading coefficients of $^nW(v)$ tend to a very simple pattern; it suggests, that for *n --> oo* we get the expression

(3.1) conjecture:

$$\lim_{n \to oo} {}^nW(v) = -v \exp(-v)$$

Table for $^nH(v)$

Let $^nH(x) = \exp({}^nW(x))$ by exponentiation of the formal power series; then the coefficients of the n'th power series give the entries of the n'th column. The coefficients in the columns must be multiplied by the cofactors in the last column to make a power series in v.

Table 3.2:

$^1H(v)$	$^2H(v)$	$^3H(v)$	$^4H(v)$	$^5H(v)$	$^6H(v)$	$^7H(v)$	$^8H(v)$	$^9H(v)$		
1	1	1	1	1	1	1	1	1	...	$\cdot 1$
1	1	1	1	1	1	1	1	1		$\cdot v^1/1!$
1	−1	−1	−1	−1	−1	−1	−1	−1		$\cdot v^2/2!$
1	4	−2	−2	−2	−2	−2	−2	−2		$\cdot v^3/3!$
1	−27	33	9	9	9	9	9	9		$\cdot v^4/4!$
1	256	−184	116	−4	−4	−4	−4	−4		$\cdot v^5/5!$
1	−3125	−695	−1175	625	−95	−95	−95	−95		$\cdot v^6/6!$
1	46656	32124	−3786	−7146	5454	414	414	414		$\cdot v^7/7!$
1	−823543	−369215	92449	−33551	−60431	40369	49	49		$\cdot v^8/8!$
1	16777216	−1298816	1565416	821512	−312488	−554408	352792	−10088		$\cdot v^9/9!$
1	−387420489	143686161	−41559759	−2333439	8371521	−2968479	−5387679	3684321		$\cdot v^{10}/10!$
1	10000000000	−2700449740	−84940030	200845250	−26292430	91462130	−33277870	−59889070		$v^{11}/11!$
...	...								...	...

--

4. An older article in tetration-forum (2009)

file:"I:\Mathe_BasisZahlentheorie\Tetration\Tetrationforum\A short note on superroots.txt"
Recently I found Andrew's remark in "designing a tetration library", that the ***superroots*** *were not yet well developed.*

Facts on superroots seem to be spread over various threads; so to have some collection under an expressive title I've put together some details, tending to compile more information from time to time as they appear. (@admin: maybe that msg is better located in some related thread, for instance the tetration library thread)

Ok, I've put the text in plain text and am lazy to MimeTex it today, perhaps I'll rework it next days.

4.1) A short collection concerning superroots

Starting point is the nice powerseries for

$$g(x) \quad = (1+x)^{(1+x)} - 1$$

Using the exponential-/logarithm-series for this we write first

$$g(x) \quad = \exp(\log(1+x)\cdot(1+x)) - 1$$

and get

$$\begin{aligned} g(x) \quad = \; & 1\cdot x + 1\cdot x^2 + 1/2\cdot x^3 + 1/3\cdot x^4 + 1/12\cdot x^5 + 3/40\cdot x^6 \\ & - 1/120\cdot x^7 + 59/2520\cdot x^8 - 71/5040\cdot x^9 + 131/10080\cdot x^{10} \\ & - 53/5040\cdot x^{11} + O(x^{12}) \end{aligned}$$

This series has the nice property, that the constant term vanishes and also, that the linear term has coefficient 1, so $g(0)=0$ and $g'(0)=1$, so we can do some common operations with it: inversion, iteration, ... always getting exact coefficients.

Now we define higher orders by something like chaining, which is not exactly iteration of g().

The sequence of functions

$$\begin{aligned} & g(x,1) = (1+x)-1 \\ & g(x,2) = (1+x)\verb|^|(1+x) - 1 \\ & g(x,3) = (1+x)\verb|^|(1+x)\verb|^|(1+x) - 1 \\ & \ldots \quad = \ldots \end{aligned}$$

gives similarly nice shaped powerseries, for instance

$$\begin{aligned} g(x,3) \quad & = (1+x)\verb|^|(1+x)\verb|^|(1+x) \; -1 \\ & = 1\cdot x + 1\cdot x^2 + 3/2\cdot x^3 + 4/3\cdot x^4 + 3/2\cdot x^5 + 53/40\cdot x^6 \\ & + 233/180\cdot x^7 + 5627/5040\cdot x^8 + 2501/2520\cdot x^9 + 8399/10080\cdot x^{10} \\ & + 34871/50400\cdot x^{11} + O(x^{12}) \end{aligned}$$

From this it is easy to define a sequence of functions for exponentialtowers of integer heights:

$$f(x,h) = g(x-1,h) + 1 \; = x\verb|^|x\verb|^|x\verb|^|\ldots\verb|^|x \quad // \text{ h-occurences of x}$$

Note, that this is in principle all well known and is merely a restatement of known results.

The unusual aspect with that sequence of powerseries is, that the leading coefficients stabilize when the height increases, and thus we have a "strange" behave when the height increases to infinity.

Example: we get the following table of coefficients (where the rows contain the coefficients for one height and each column is associated with one power of *x*):

```
[code]
´ 0    1    0    0       0     0       0          0               0          ...
  0    1    1    1/2     1/3   1/12    3/40      -1/120          59/2520     ...
  0    1    1    3/2     4/3   3/2    53/40      233/180        5627/5040    ...
  0    1    1    3/2     7/3   3     163/40     1861/360       33641/5040    ...
  0    1    1    3/2     7/3   4     243/40     3421/360       71861/5040    ...
  0    1    1    3/2     7/3   4     283/40     4321/360      102941/5040    ...
  0    1    1    3/2     7/3   4     283/40     4681/360      118061/5040    ...
  0    1    1    3/2     7/3   4     283/40     4681/360      123101/5040    ...
     ...
[/code]
```

where the first column (containing zeros) represent the placeholders for the nonexistent constant terms. (The first row was appended to get a meaningfully interpretation for the "once"-iterate; it represents just *g(x,1) = (1+x) -1* . The limit case for *h->oo* begins with the same coefficients as the last row of the table above)

4.2) Inversion

Since the *g(x,h)*-series have no constant term but a linear term with unit-coefficent, we can invert each of that *g*-series. Expressed by the appropriate *f*-function we get the superroot-powerseries for each integer height. Let's denote the inverse functions as *gi()* and *fi()*, then

for *gi(x,2)* we get

$$gi(x,2) = x - x^2 + 3/2\cdot x^3 - 17/6\cdot x^4 + 37/6\cdot x^5 - 1759/120\cdot x^6 + 13279/360\cdot x^7 - 97283/1008\cdot x^8 + 654583/2520\cdot x^9 - 10800299/15120\cdot x^{10} + 75519317/37800\cdot x^{11} + O(x^{12})$$

and a higher *h*, for instance

$$gi(x,12) = x - x^2 + 1/2\cdot x^3 + 1/6\cdot x^4 - 3/4\cdot x^5 + 131/120\cdot x^6 - 9/8\cdot x^7 + 1087/1260\cdot x^8 - 271/720\cdot x^9 - 2291/10080\cdot x^{10} + 523/630\cdot x^{11} + O(x^{12})$$

which again stabilizes for *h->inf*

```
Table of coefficients for gi(x,h), h=0..9
---------------------------------------------------------------------------------
  0  1   0    0      0      0         0          0          0          0    ...
  0  1  -1  3/2  -17/6   37/6  -1759/120  13279/360  -97283/1008  654583/2520  ...
  0  1  -1  1/2    7/6  -17/4    821/120     -25/12  -56269/2520   52079/720   ...
  0  1  -1  1/2    1/6    1/4   -349/120     161/24    -2642/315     677/72    ...
  0  1  -1  1/2    1/6   -3/4    251/120      -45/8   13897/1260  -10891/720   ...
  0  1  -1  1/2    1/6   -3/4    131/120       -1/8   -5213/1260    8909/720   ...
  0  1  -1  1/2    1/6   -3/4    131/120       -9/8    2347/1260   -4231/720   ...
  0  1  -1  1/2    1/6   -3/4    131/120       -9/8    1087/1260     449/720   ...
  0  1  -1  1/2    1/6   -3/4    131/120       -9/8    1087/1260    -271/720   ...
  0  1  -1  1/2    1/6   -3/4    131/120       -9/8    1087/1260    -271/720   ...
  ....
--------------------------------------------------------------------------------
```

4.3) The *h*'th superroot

The shown powerseries, formally seen, give the functions for the *h*'th superroots:

fi(x^x,2) = x = gi(x^x-1,2)+1
fi(x^x^x,3) = x
...
fi(x^^h,h) = x

and the computation of the *h*'th superroot can be implemented by calls of the *gi(x,h)*-function:

ssrt(x,h) = fi(x,h) = gi(x-1,h) + 1

This gives, if convergent, the base *b*, which must be exponentiated *h* times to equal the given value *x*.

4.4) Convergence:

Concerning the range of convergence I don't have an idea yet. For instance for *g(x,2)* we can guess a rate of decrease similar to μ/k^2 where *k* is the index and *μ* some constant, so we should have a range of convergence for *|x|<=1* only. For *f(x,2)* consequently we had then *0<x<=2* .

It seems, that the occuring divergences are not "too strong" so that we can extend the domain for *x* using Euler-summation to get meaningful approximations even if only *64* or *128* terms are known.

4.5) Interpolation to fractional orders:

The special form of the powerseries, where each *k*'th coefficient becomes constant when chaining-height *h>=k* this poses a new challenge for the interpolation to fractional heights.

I have currently no idea how to proceed here...

(to be continued)

Gottfried