Notes on iteration of the exponential with base *b* on the Shell-Thron-region

Let us denote the ***b***ase of an iterated exponentiation with the letter "*b*", such that with some complex number z_k we have the iteration:

(1) f_b: $z_{k+1} = b^{z_k}$ *always using the* ***principal branch of the logarithm***

Let us also assume, that with some number *u* on the circumference of the complex unit-circle and with its exponential *t*, and a real parameter *c* such that

$u = \exp(2 \pi î /c)$ *"u" a number on the circumference of the complex unit disk*
$t = \exp(u)$ *"t" is fixpoint for the iteration using base* $b = t^{1/t}$

we have also a parametrization of the base *b* as

$b = \exp(u/t)$

(Note, that in the thread in MSE what we have called u *here is called* -t there*)*

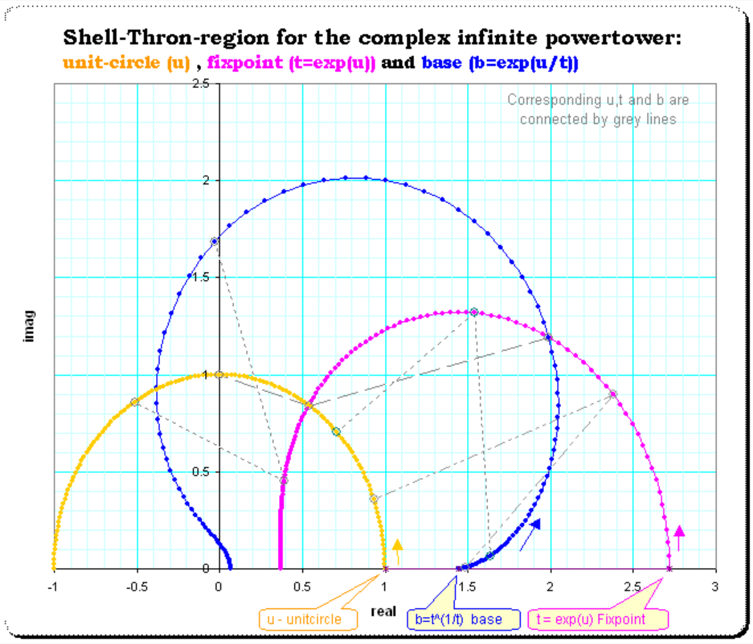
Of course, other way round, it is well known that *u* can be defined by $u = - W(-\log(b))$, where *W* is the Lambert-W-function, and we look only at such *b* , such that $|u|=1$, which means that $u=\exp(2 \pi î /c)$

Shell-Thron-region

For the parameter *u* the yellow circle is relevant: is $t=\exp(u)$ and the base $b = \exp(u/t)$ then the infinite exponential tower to base *b* is convergent if *u* is **inside** the circle and is divergent if *u* is **outside** the circle. *t* (for *u* **on** the yellow circle) is shown with the magenta, and *b* with the blue shape. The interior of the latter is called the "*Shell-Thron-region*"

Significance: If *u* is on the (yellow) circumference of the complex unit-circle ($|u|=1$), then it depends, whether *u* is a) on a rational or b) on an irrational angular position:
in case a) $u^n=1$ for some natural *n* and the exponential tower converges to *t*;
in case b) $u^n=/=1$ for all natural *n* and the exponential tower diverges and the orbit gives some fractal looking shape with excurses arbitrarily "near" to complex infinity

(This is an already known property, see for instance Y. Galidakis' answer. a) has been proven by Baker/ Rippon [BAK83], theorem 1, and have divergence in b) mentioned by reference to Siegel and Rüssmann)



Any *b* of this definition in this essay lies on the border of the so called Shell-Thron-region and we can distinguish two qualitatively different cases:

case 1: *c* is rational. Then also $u^n = 1$ for some natural number *n*

case 2: *c* is irrational. Then $u^n =/=1$ for all natural numbers *n*
subcases: u is algebraic (sqrt), transcendental (like e), Liouville-transcendental

Also, in the following we consider orbits of the iteration on the route
$z_{-1}=0$, $z_0 =1$, $z_1 =b$, $z_2 = b^b$, ...

In an appendix I discuss *case 2* with *c* irrational but also ***non***algebraic and of the "Liouville"-type as of infinite order of transcendence.

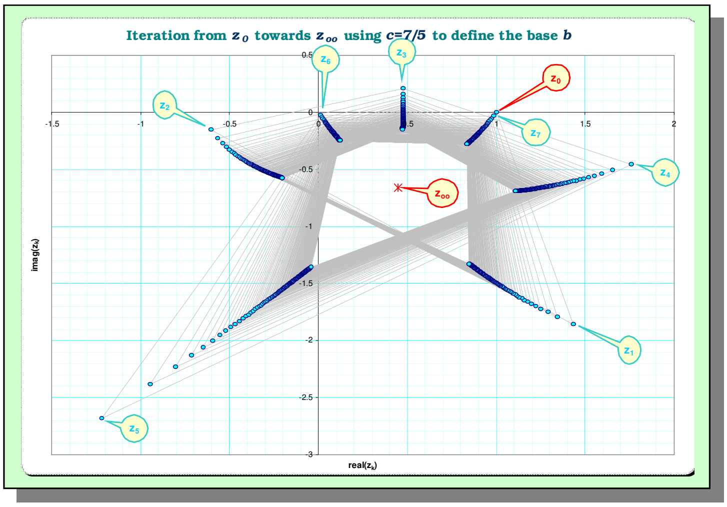
(A couple of references shall be appended)

Gottfried Helms, 4.7.2016 in an evaluation on a question in MSE

http://math.stackexchange.com/q/1820410/1714

Case 1: c is rational. Then u is a complex unit-root of rational order such that $u^n=1$ with some natural number n

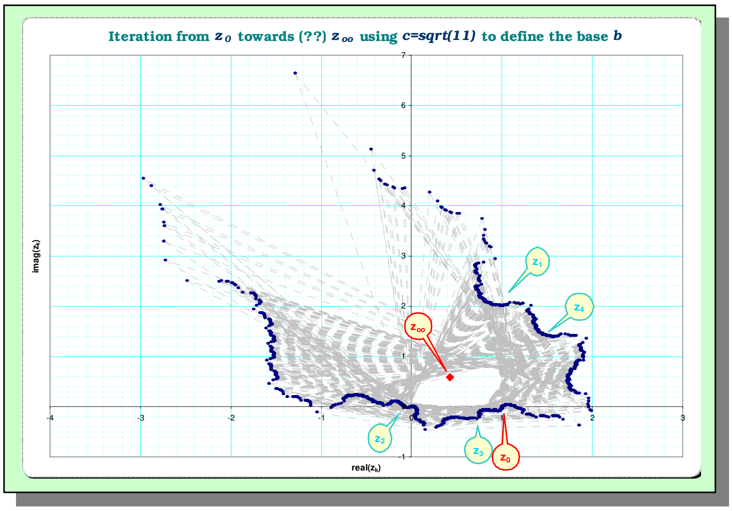
It has already been established [Bak83], that *case 1* enters a contracting orbit which approaches a fixpoint z_{oo} where also $\lim_{n\to oo} z_n=t$ and where the orbit has a clear periodic occurence; in the example shown here, where we use the rational value $c=7/5$ to define first u, then $t=\exp(u)$ and $b=\exp(u/t)$, we have near-periodicity with period 7 and a star-like shape which contracts towards the fixpoint $z_{oo}=t$. The contraction rate here obviously diminishes, but images with exponentially increasing stepwidths still show the ongoing contraction.



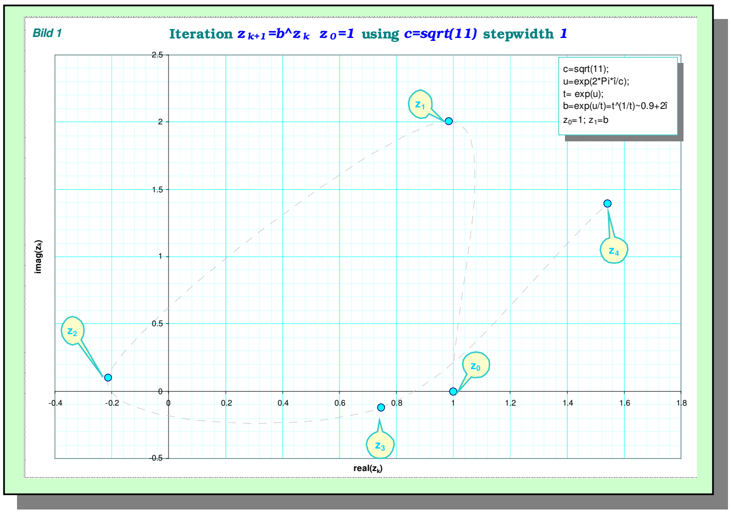
Case 2: *c* is irrational. Then *u* is a complex unit-root of irrational order such that $u^n \neq 1$ for all natural numbers *n*

Using c as algebraic number: A first example should be a simple one - we use $c=\sqrt{11}$. Iterating 1000 times beginning at $z_0=1$ we find some fractal looking shape - and from which we cannot yet guess whether contraction-to or expansion-from the fixpoint t shall occur.

However the fixpoint can numerically be approximated to arbitrary precision when we use the Newton-iteration-method.

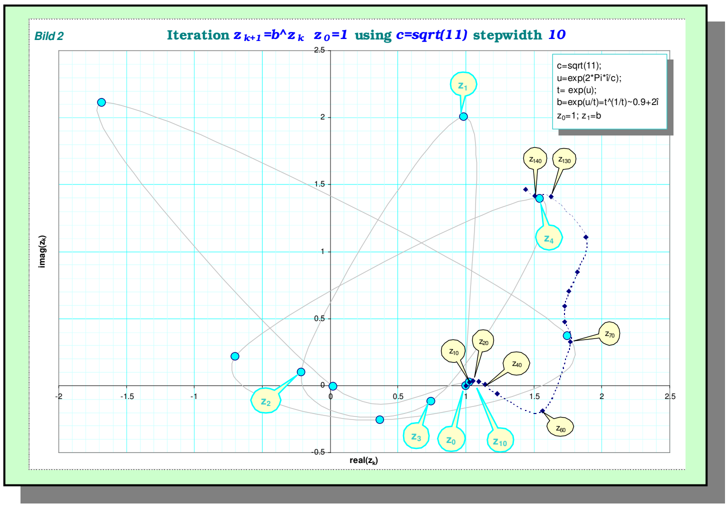


A first mental approximation to see periodicity, results when we look at the iteration in steps of 3 or of 4; all z_k and z_{k+3} resp. z_{k+4} are pairwise neighboured. Here z_{k+3} seem even nearer than z_{k+4}, but unfortunately iterating in steps of 3 would invert the direction of the "multi-step" orbit while iterating in steps of 4 keeps the "forward" direction from z_0 to z_1.



A much better approximation to periodicity occurs already if we iterate in steps of *10* - the distance between z_k and z_{k+10} is much smaller than that of *step-3* or *step-4* iterations.

In the picture at right I've still documented the first *10 1-step*-iterations (big light blue circles connected by a smoothed grey line), and we see, that z_{10} is very near z_0. So if we now use such *10-step* iterations we need a lot of them to "circle" around until we arrive near the start.



Let's introduce the name "multisection-orbit" or "mso" for this *n*-step-iterating, in analogy to the use of this term in context of "multisection of powerseries".

Each of the *mso* shows now an orbit, which is much smoother, looks better interpolatable and is nearly convex, roughly circling around the fixpoint. They are marked with smaller dark-blue diamonds *(and are connected with a dark blue dotted line as cubic spline done by Excel)*.

Of course, this induces an idea how to proceed: what is the next stepwidth *w*, which comes even nearer to the initial value z_0? And then, draw the *mso*'s with *w-step* iterations and so on.

The concept of ***continued fraction*** and the ***convergents*** give an answer for rational approximations to irrational numbers, so maybe we find the same here. Checking for the continued fraction for *c=sqrt(11)* we get *cf(c) = [3, 3, 6, 3, 6, 3, 6, 3, 6, 3,...]* and the list of convergents is

Convergents:	*[1*	*3*	**10*	*63*	**199*	*1257*	**3970*	*25077*	*...]*
	[0	*1*	*3*	*19*	*60*	*379*	*1197*	*7561*	*...]*

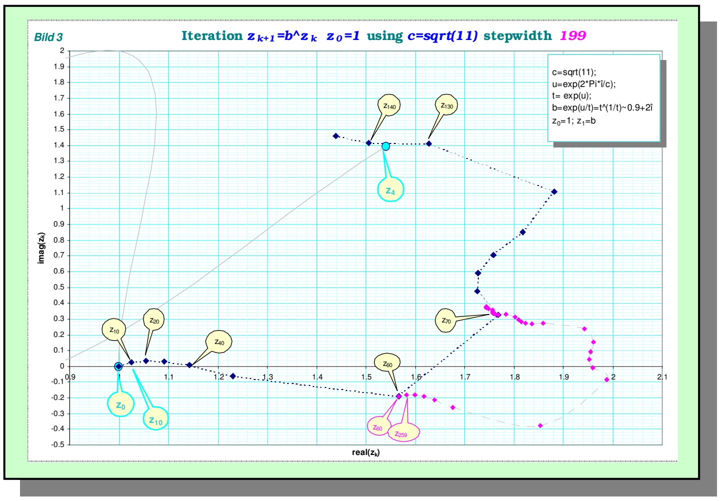
and bingo! - if we look at the sequence of numerators we find stepwidths w_k *(k=0,1,2,...)* with good (and even improving!) approximations to the periodicity, meaning, that

$$|z_3 - z_0| > |z_{10} - z_0| > |z_{63} - z_0| > |z_{199} - z_0| > |z_{1257} - z_0| > |z_{3970} - z_0| > \dots$$
$$\implies \lim_{k \to \infty} |z_{w_k} - z_0| = 0$$

Also we find, that each second of that stepwidthes w_k give "forward" orbiting and the others give "backwards" orbiting (which might be defined by the property, that while the number z_{w_k} is in the near of z_0 we have that $imag(z_{w_k -1})<0$ and $imag(z_{w_k})>0$. (Of course, that definition should be made more precise, but that might be done later). I've marked the "forward"-stepping stepwidths w_k by a star.

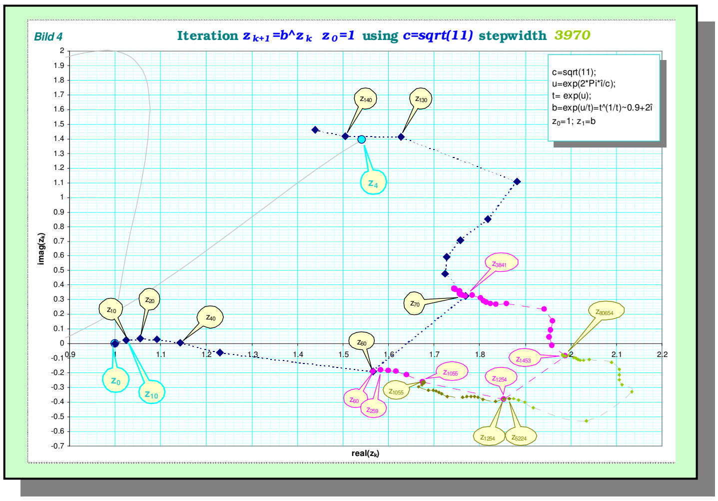
We don't want to draw now all of the full "circle" using stepwidth *w=199* and so on, but want to look at the specific locus between z_{60} and z_{70} in the *10-step*-orbit. There is a relatively large gap there, similarly (but visually smaller) like between z_1 and z_2, and we draw only the orbit from z_{60} near to z_{70} in $w_4=199$-step iterations, and if the fractal shape of the full orbit has some outliers, then some might occur here.

At the right hand is the picture where a handful of *199-step* iterates beginning at z_{60} towards z_{70} are marked (magenta diamonds, magenta dotted interpolation-line) so we have now a finer "interpolation" of the complete orbit from z_{60} via $z_{60}, z_{259}, z_{458}, \ldots, z_{3841}$, $z_{4040} \ldots$ where the latter two encircle the exact value z_{70} very narrow.



Moreover, it seems, that this stepwise refining of periodicity and increasing of stepwidths leads to some repetitive pattern which we might expect between the 7^{th} and 8^{th} magenta dot.

Indeed, refining now to stepwidth $w_6=3970$ (taken from the convergents of the continued fraction), beginning at the 7^{th} magenta dot z_{1254} proceeding in 20 steps of w_6 to z_{80455} (which is very near to the 8^{th} magenta dot z_{1453}) painting with some green color gives an interpretation/interpolation of the gap between that 7^{th} and 8^{th} dot - but which seems to be just a smaller copy of the magenta contour, just stacked besides that at its largest gap.

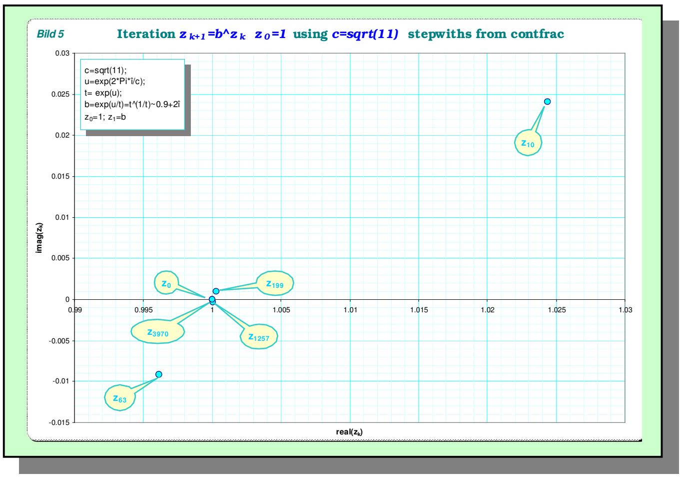


So - without actually proceeding here further - it looks like, that such a repeated stacking of the same pattern occurs infinitely often.

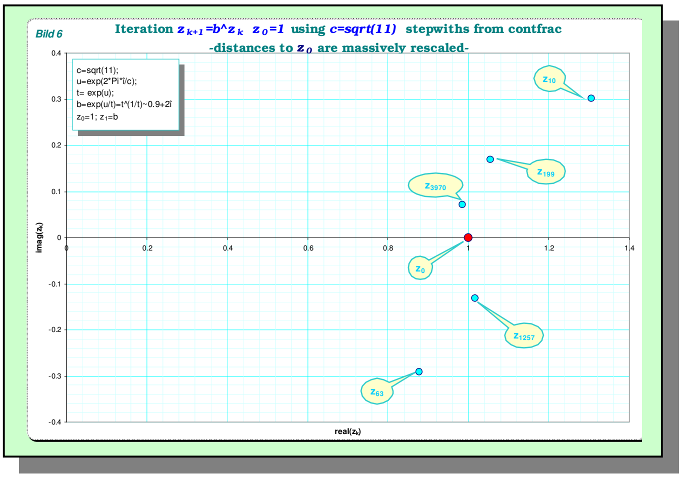
But still we cannot guess, whether this repeated stacking also leads to an infinite ***distance*** from the fixpoint or whether the distance is bounded - because the size of the stacked area seems (by visual inspection) to decrease, and we have no idea yet about the rate of decrease.

A possibly valid argument for the assumption, that the limiting curve with stepwiths $w_k \to \infty$ for $k \to \infty$ should have an excess to infinity: because on the other hand, the orbit taken to $z_{-1} = 0$ should as well be approximated by some z_{-1+w_k} arbitrarily precise, and so its w_k-*step*-precedessor z_{-2+w_k} should "approximate" the complex infinity arbitrarily precise as well - which means: it should escape arbitrarily far from $z_{-1}=0$ resp. from the fixpoint.

Here I've plotted the positions of z_{w_k} in the neighbourhood of z_0 for the first few values w_k of the continued fraction. The distances from z_0 decrease very fast; actually $z_{w_6}=z_{3970}$ is visually no more discernible from z_0. Of course, plotting an interpolation line on the *3970-step* iterations would on most regions of the (fractal) border look continuous - but still there shall be regions, where values z_x and z_{x+3970} are visibly distant.



To improve the visual imagination furtherly here is a version of the previous picture but where the distances to the point z_0 are heavily ***rescaled***. Having the distance between z_{3970} and z_0 now better visible it seems easier to imagine the further progress if we take z_{w_k} for increasing k. Also the alternating character of the imaginary part of the z_{w_k} is now better visible and explains the idea of backwards and forward orbits based on that stepwidths.



Hypothese: likely the orbit is not contracting

From this nice visual presentation, there should then also be an argument against any assumption of contraction of the orbit towards the fixpoint - however by using the point $z_{-1}=0$ instead of $z_0=1$:

> because of the arbitrarily exact approximation of some z_{-1+w_k} to 0 with increasing k and increasing w_k we have an arbitrarily far distance of z_{-2+w_k} from the fixpoint with increasing k and w_k and thus the orbit cannot contract to the fixpoint t.
>
> *Note: this "inserting" of points in the orbit using the indexes given by the continued fraction, which lets the whole curve look more smooth reminds me of the property of the* ***P-adic****-concept, where two numbers with higher exponents are nearer than two numbers with smaller exponent (in the P-adic-metric).*

Excurs in Case 2: c is irrational, but not algebraic (as it was in the example before) and with transcendence degree oo (similar to Liouville-numbers) .

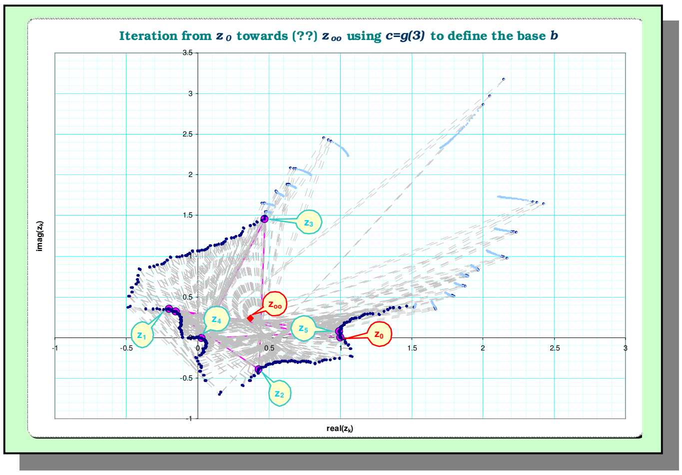
Since such numbers *c* have an extreme well approximation to rational numbers, I expect, that a reason for the difference of the orbits based on rational *c* and orbits based on irrational (algebraic) numbers *c* might become visible.

For the example I tried first the constant *c =g(2)* taken from the function

$$g(x) = \sum_{k=0}^{oo} \frac{1}{2^{x^k}} \qquad c = g(2)$$

which is known to be transcendental and non-algebraic. But the pictures generated using this *c* gave visually no clear new idea. But for *c=g(3)* there are some interesting effects.

The initial picture, where the blue points mark the *1-step* iterations beginning at $z_0=1$ is roughly familiar from the *c=sqrt(11)*-picture. Instead of an approximate *10*-step periodicity (let's call this "pseudo"-periodicity) we have here a *5-step* pseudo- periodicity; the image is nearly rectangular and the right side is wide open with a sharp needle of early points in them. *(The lighter blue points are very high indexed iterates, see below)*



The continued fraction of *c = g(3)~ 0.626953132451* begins like

[0, 1, 1, 1, 2, 7, 1, 1, 1, 1, 1, 511, 2, 1, 1, 1, 7, 2, 1, 1, 1, 134217727, 2,

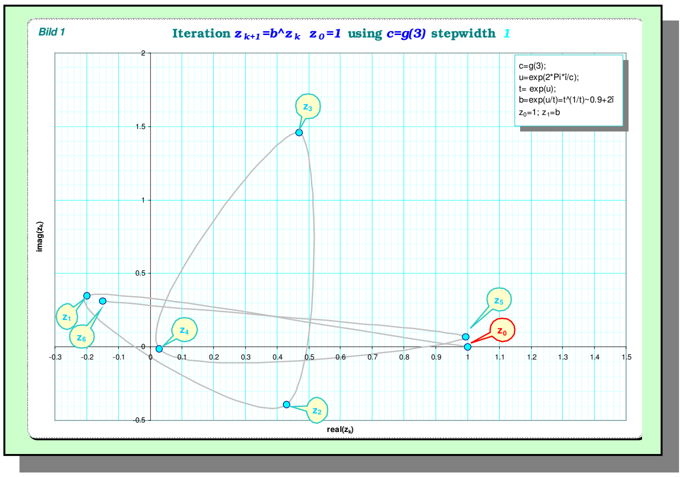
where the high values $7=2^{3^1}-1$, $511=2^{3^2}-1$, $134217727=2^{3^3}-1$ show an easily recognizable progression and the whole pattern up to any index can be generated by some simple recursive rule.

The convergents are

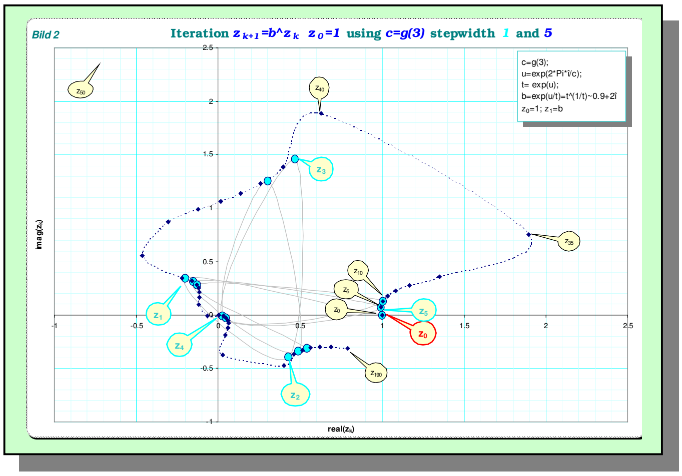
[1	0	1	1	2	*5	37	*42	79	*121	200	*321	164231	...]
[0	1	1	2	3	8	59	67	126	193	319	512	261951	...]

and we expect strong pseudo-periodicity with stepwidths of $w_5=5$, $w_6=37$, $w_7=42$,..., $w_{11}=321$ but then the next new/better pseudo-periodicity occurs only with stepwidth of $w_{12}=164231$!

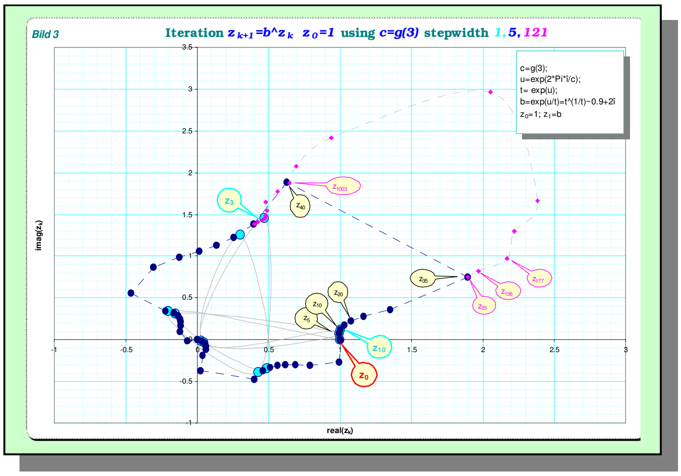
Here we detail the single first few iterates to see the (pseudo)-periodicity of $w_5=5$, such that z_5 is near by z_0. The distinct values z_k are marked by the light blue circles



Because between z_0, z_5 and then z_2 there is a big gap I use now the step-with $w_5=5$ to interpolate the orbit z_0 towards (the near of) z_3 . Proceeding then to one round to arrive in the near of z_0 again needs *40* of that *5-step*-iterates such that z_{200} is very near to z_0 again.

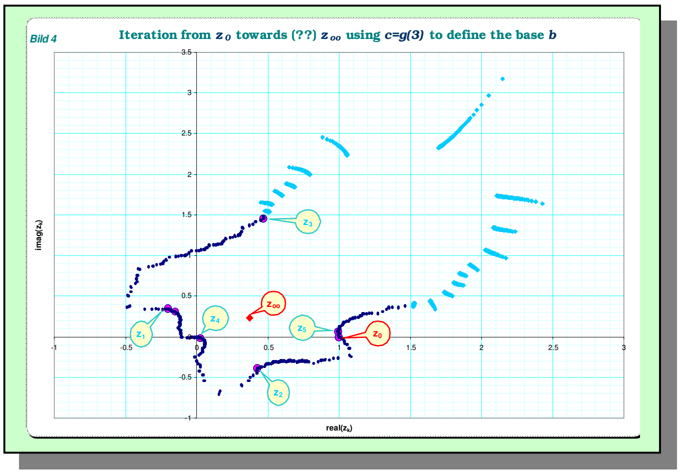


Now we look at the big gap between z_{35} and z_{40} trying to interpolate it by larger stepwidths. But instead of using *200* (which seems much indicated but is w_{10} with backward direction which we want to avoid here) we use now $w_9=121$ and begin at z_{35} to proceed to z_{156} and z_{277} and so on towards the near of z_{40} The partial orbit is marked magenta here.



Interestingly, the dotted magenta line (as Excel's spline-interpolation) has again a big gap, nearly as large as before.

To makes things shorter now, I've not proceeded this way, but simply looked at the region, when iterated 10^8 times and all points with coordinates in the region $x+y\hat{\imath}$ in that quadrant with either x or y or both greater than *1.5* are plotted. This gives the picture with the strange sausage-like looking multisection orbits towards the inner side of the curve, marked with light blue.



I cannot conclude something clear from this, but those sausages reminds me of the contracting star-shape in the pictures for c (being a rational number) in *case 1* - and possibly the pictures can be improved in this or that way.

Perhaps the specific characteristic of the number $c=g(3)$ as "extremely well approximable" to rational numbers forces this familiar visual pattern in some region of the picture.

So far I've to stop my explorations for the moment and can only proceed casually in the next week.

Gottfried Helms, 4.7.2016 (some textual edits 24.9.2018)

References:

The motivating question and dicussion:

[MSE] http://math.stackexchange.com/questions/1757474/convergence-properties-of-zzz-and-is-it-chaotic

Literature:

[BAK83] Baker,I.N.;Rippon,P.J.: Convergence of infinite exponentials
Annales Academiae Scientiarium Fennicae, Vol 8, 1983, 179-186

references in [BAK83]

Siegel, C. L.: Iterations of analytic functions. - Ann. of Math, 43, 1942, 607-612

Rüssmann, H.: Über die Iteration analytischer Funktionen. - J. Math. Mech. 17, 1967,523-532.

Addendum (in context; answer of Yannis Galidakis in [MSE])
(the original "c" and "t" in question&answer are adapted to my conventions here to "b" and "u")

Now you can put the above result into context with your original question. The multiplier of the iterated exponential is given by:

$$u = -W(-\ln(b))$$

As you well found out then, this splits into three cases:

1. $|u|>1$ in which case the iterated exponential diverges (by fixed point iteration)
2. $|u|<1$, in which case it converges (to $W(-\ln(b))/-\ln(b)$ (**Corless**) again by fixed point iteration)
3. $|u|=1$, (boundary of Shell-Thron region) which splits into the two cases below:
   3a. $|u|=1$ and $u^n=1$, i.e. rotating by rational multiples of 2π, which is taken care by **Baker and Rippon** (i.e. converges as in 2.)
   3b. $|u|=1$ and $u^n \neq 1$, i.e. irrational rotation, which gives the analysis above.