Exercises in functional iteration: the function $f(x) = ln(2\text{-}exp(\text{-}x))$

A selfstudy using formal powerseries and operator-matrices

Gottfried Helms 10.12.2010

update 12.02.2011

1. Definition

The function considered is an example taken from a private conversation with D.Geisler and W.Jagy. Here I discuss properties of fractional iteration of

$f(x) = ln(2-exp(-x))$

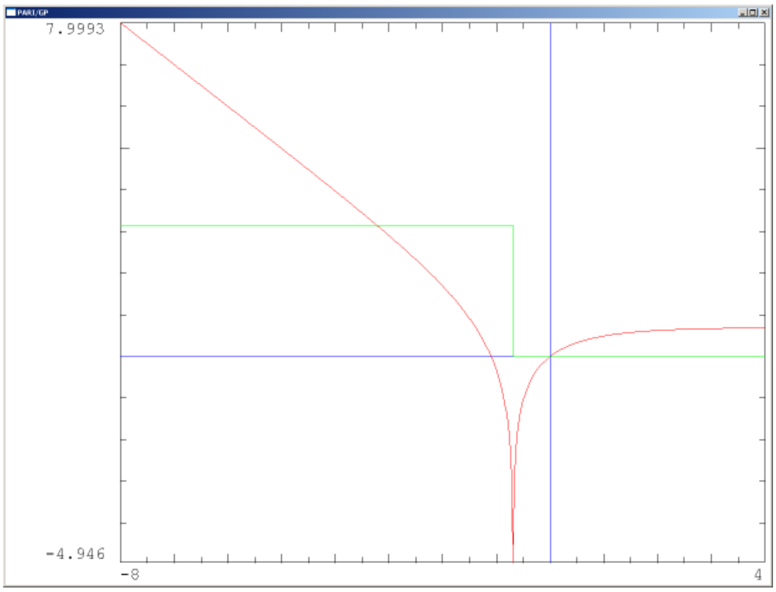
$f^{\circ 1}(x) = f(x)$

$f^{\circ 0}(x) = x$

$f^{\circ h+1}(x) = f^{\circ h}(f(x))$ // *meaning the h'th iterate*

$f^{\circ -1}(x) = -ln(2\text{-}exp(x)) = -f(-x)$ // *mirroring around the origin*

A rough idea can be got by the following plot. The function has real range only for $-ln(2)<x$; this is the right red curve in the following. For $x\text{-}> inf$ it approximates the constant $ln(2)$. For $x<\text{-}ln(2)$ the function has the imaginary component Pi^*I, (green lines) and its real part approaches the function $f(x)=-x$ for $x\text{-}>\text{-}inf$ (left red curve)



Because there is the fixpoint $f(0)=0$ we can construct formal powerseries for arbitrary continuous iterates and because $f'(x)<1$ for $x>0$ it is an attracting fixpoint and iteration to positive heights is a very well converging process. Here the coefficients $c_{k,h}$ for the formal powerseries of a certain iteration-height h can be taken by a set of polynomials in h which shall be determined later:

$f^{\circ h}(x) = c_{1,h}{}^*x + c_{2,h}{}^*x^2 + c_{3,h}{}^*x^3 + ...$

We shall find, that each coefficient c is a polynomial in the h-parameter which can be explicitely be determined (without need of recursion)

2. Procedere:

2.1. Construct the matrix-operator for the function

$f(x)=ln(2 - exp(-x))$

The powerseries for this can be given in rational coefficients:

$$f(x) = x - x^2 + x^3 - 13/12*x^4 + 5/4*x^5 - 541/360*x^6 + 223/120*x^7 + O(x^8)$$

The associated matrix-operator ***M*** (of infinite size) begins with

$$M= \begin{bmatrix} 1 & . & . & . & . & . & . & . \\ 0 & 1 & . & . & . & . & . & . \\ 0 & -1 & 1 & . & . & . & . & . \\ 0 & 1 & -2 & 1 & . & . & . & . \\ 0 & -13/12 & 3 & -3 & 1 & . & . & . \\ 0 & 5/4 & -25/6 & 6 & -4 & 1 & . & . \\ 0 & -541/360 & 17/3 & -41/4 & 10 & -5 & 1 & . \\ 0 & 223/120 & -1381/180 & 65/4 & -61/3 & 15 & -6 & 1 \end{bmatrix}$$

We see, that the coefficients for $f(x)^0$ are in the first column (columnindex $c=0$), that for $f(x)^1$ in the second column $(c=1)$, that for $f(x)^2$ in the third column $(c=2)$ and so on.

2.2. The use of the matrix-operator for expression of the required powerseries

With a "Vandermonde"-like vector-type

$$V(x)= row(1 , x , x^2 , x^3, ...)$$

we can write

$$V(x) * M = V(f(x)) = rowvector(1, f(x), f(x)^2 , f(x)^3, ...)$$

which immediately allows generalization for iterates of $f(x)$.

Let's write the h'th iterate $f^{\circ h}(x)$. Then we have, first for integer heights:

$$V(x) * M^h = V(f^{\circ h}(x))$$

where integer powers of ***M*** can exactly be determined by matrix-muliplication of the triangular matrix with itself. This allows still exact rational arithmetic up to arbitrary truncation size.

The question is now: can also fractional iterates be determined. The answer is yes; and this is a known procedure.

2.3. fracional powers of a matrix-operator via log/exp

Since $f(x)$ is a function[1] having $f(0) = 0,\ f'(0) = 1$ we can determine a matrix-logarithm which still provides exact arithmetic:

$$\text{define } M1 = M - ID \qquad \text{(where ID is the identity-matrix)}$$
$$Log(M) = M1 - M1^2/2 + M1^3/3 - \ldots + \ldots \quad \text{(the mercator–series for log)}$$

Then

$$L = Log(M)$$
$$M^h = Exp(h * L)$$

provides the h'th-power of M, again in rational arithmetic.

The logarithm-matrix L has an interesting format:

$$L = \begin{bmatrix} 0 & . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & . & . & . \\ 0 & -1 & 0 & . & . & . & . & . \\ 0 & 0 & -2 & 0 & . & . & . & . \\ 0 & -1/12 & 0 & -3 & 0 & . & . & . \\ 0 & 0 & -1/6 & 0 & -4 & 0 & . & . \\ 0 & -1/360 & 0 & -1/4 & 0 & -5 & 0 & . \\ 0 & 0 & -1/180 & 0 & -1/3 & 0 & -6 & 0 \end{bmatrix}$$

The logarithm of an operator-matrix has always the structure that all columns are smply shifted multiples of the second column; we see, that the coefficients $L_{r,c}$ in some col c and row r are just $c*L_{r+1-c,1}$.. (Thus the matrix L is not a operator-matrix!)

And the second column $(c=1)$ gives the coefficients for a function, which we may call the "iterative-logarithm-function of $f(x)$", as a function we get:

$$lf(x) = -2(x^2/2! + x^4/4! + x^6/6! + \ldots.)$$

which is also

$$lf(x) = -2\,(\cosh(x) - 1) = 2 - (\exp(x) + \exp(-x))$$

A factorially scaled verson of matrix L is

$${}^dF * L * {}^dF^{-1} = FLf$$

and begins with

$$\begin{bmatrix} 0 & . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & . & . & . \\ 0 & -2 & 0 & . & . & . & . & . \\ 0 & 0 & -6 & 0 & . & . & . & . \\ 0 & -2 & 0 & -12 & 0 & . & . & . \\ 0 & 0 & -10 & 0 & -20 & 0 & . & . \\ 0 & -2 & 0 & -30 & 0 & -30 & 0 & . \\ 0 & 0 & -14 & 0 & -70 & 0 & -42 & 0 \end{bmatrix}$$

If we want to compute some general power h of the matrix M we have to evaluate the exponential:

[1] I've seen the term "*schlicht*"-function for this in older literature (german for "simple function")

$M^h = Exp(h * Log (M))$

2.4. The half-power of *M* (using *h=1/2*) give the half-iterate of *f(x)*

We can do this for some actual value of h, say $h=1/2$ to get the powerseries for the half-iterate. The top left segment of the matrix is

$$M^{1/2} = Exp(1/2 * L) = \begin{bmatrix} 1 & . & . & . & . & . & . & . \\ 0 & 1 & . & . & . & . & . & . \\ 0 & -1/2 & 1 & . & . & . & . & . \\ 0 & 1/4 & -1 & 1 & . & . & . & . \\ 0 & -1/6 & 3/4 & -3/2 & 1 & . & . & . \\ 0 & 1/8 & -7/12 & 3/2 & -2 & 1 & . & . \\ 0 & -137/1440 & 23/48 & -11/8 & 5/2 & -5/2 & 1 & . \\ 0 & 71/960 & -287/720 & 5/4 & -8/3 & 15/4 & -3 & 1 \end{bmatrix}$$

and in the second column we find the coefficients of the formal powerseries for $f^{\circ 0.5}(x)$:

$$f^{\circ 0.5}(x) = x - 1/2*x^2 + 1/4*x^3 - 1/6*x^4 + 1/8*x^5 - 137/1440*x^6 + 71/960*x^7 + O(x^8)$$

such that

$$V(x) * M^{0.5} = V(f^{\circ 0.5}(x))$$

or, with a bracket-*[row,col]*-notation for the extraction of the second column of matrix $M^{0.5}$ we write

$$f^{\circ 0.5}(x) = V(x) * M^{0.5} [,1] \qquad \text{//omitting the row-index means the whole column}$$

2.5. The general power of *M* and the general iterate of *f(x)*; symbolically

If we want this in more generality, keeping the iteration-height parameter h as variable we can compute the matrix-exponential symbolically getting the following polynomials in h as entries of the h'th power M^h :

$$M^h = \begin{bmatrix} 1 & . & . & . \\ 0 & 1 & . & . \\ 0 & -h & 1 & . \\ 0 & h\text{^}2 & -2*h & 1 \\ 0 & -h\text{^}3-1/12*h & 3*h\text{^}2 & -3*h \\ 0 & h\text{^}4+1/4*h\text{^}2 & -4*h\text{^}3-1/6*h & 6*h\text{^}2 \\ 0 & -h\text{^}5-1/2*h\text{^}3-1/360*h & 5*h\text{^}4+2/3*h\text{^}2 & -10*h\text{^}3-1/4*h \\ 0 & h\text{^}6+5/6*h\text{^}4+1/40*h\text{^}2 & -6*h\text{^}5-5/3*h\text{^}3-1/180*h & 15*h\text{^}4+5/4*h\text{^}2 \end{bmatrix}$$

where the second column $M^h[,1]$ only is needed to provide the relevant polynomials for the computation of $f^{\circ h}(x)$.

If we insert, for instance, $h=1$, we get the original powerseries for $f(x)$, if we insert $h=1/2$ we get the coefficients of the powerseries for the half-iterate $f^{\circ 0.5}(x)$ and so forth.

2.6. The bivariate coefficients-matrix *POLY*

If the coefficients of *h* in that second column are again represented as a matrix, we can write this as matrix ***POLY*** of coefficients for the bivariate function $f^{\circ h}(x) = V(x) * \mathbf{POLY} * V(h)\sim$

$$POLY = \begin{bmatrix} 0 & . & . & . & . & . & . \\ 1 & 0 & . & . & . & . & . \\ 0 & -1 & 0 & . & . & . & . \\ 0 & 0 & 1 & 0 & . & . & . \\ 0 & -1/12 & 0 & -1 & 0 & . & . \\ 0 & 0 & 1/4 & 0 & 1 & 0 & . \\ 0 & -1/360 & 0 & -1/2 & 0 & -1 & 0 \\ 0 & 0 & 1/40 & 0 & 5/6 & 0 & 1 \end{bmatrix}$$

So ***POLY*** $* V(h)\sim$ gives the second column in $\mathbf{M}^h$ and as a bivariate expression in the matrix-notation we have:

$$\begin{aligned} f^{\circ h}(x) &= V(x) * POLY * V(h)\sim \\ &= x * (1) \\ &+ x^2 * (- 1*h) \\ &+ x^3 * (1*h^2) \\ &+ x^4 * (-1/12*h - 1*h^3) \\ &+ x^5 * (1/4*h^2 + 1*h^4) \\ &+ x^6 * (\ldots) \\ &+ \ldots \end{aligned}$$

2.7. Explicite descriptions of entries in *POLY*

It might be of interest, that ***POLY*** can be rescaled to provide integer entries only (heuristically, no proof yet). This is possible using a factorial (similarity) scaling; here is the top-left segment (assume ^dF as diagonal matrix of factorials *diag([0!,1!,2!,...])*)

$$^dF * POLY * {^dF}^{-1} = FPf = \begin{bmatrix} 0 & . & . & . & . & . & . \\ 1 & 0 & . & . & . & . & . \\ 0 & -2 & 0 & . & . & . & . \\ 0 & 0 & 3 & 0 & . & . & . \\ 0 & -2 & 0 & -4 & 0 & . & . \\ 0 & 0 & 15 & 0 & 5 & 0 & . \\ 0 & -2 & 0 & -60 & 0 & -6 & 0 \\ 0 & 0 & 63 & 0 & 175 & 0 & 7 \end{bmatrix}$$

I succeeded in finding a general expression for the entries in ***FPf*** and thus for that in ***POLY***. Assuming the matrix-indices *r*(ow) and *c*(ol) beginning at zero we have for the elements in ***POLY***:

$$p_{r,c} = \frac{\left(1-(-1)^{r-c}\right)}{r!} \cdot \sum_{k=1}^{c} \left((-1)^k \cdot \binom{c-1}{k-1} \cdot k^{r-1} \right)$$

or with the binomial-coefficient more conveniently adapted:

$$p_{r,c} = \begin{cases} \delta_{r,1} & if \quad c=0 \\ \frac{\left(1-(-1)^{r-c}\right)}{r!} \cdot \frac{1}{c} \cdot \sum_{k=1}^{c} \left((-1)^k \cdot \binom{c}{k} \cdot k^r \right) & if \quad c>0 \end{cases}$$

where d is the Kronecker-symbol

We see that the $p_{r,c}$ are finitely composed polynomials whose number of terms is just equal to the column-index *c*. So to describe the powerseries for the *h*'th iterate of *f(x)* we write:

$$f^{\circ h}(x) = 1x + (p_{2,1}*h)\, x^2 + (p_{3,2} * h^2)\, x^3 + (p_{4,1} * h + p_{4,3} * h^3)\, x^4 + \dots$$

and because we do no more need the matrix-logarithm/matrix-exponential we can determine that coefficients to arbitrarily many terms in exact rational arithmetic and thus the whole function with optimal precision.

2.8. The function $w_x(h)$ (with fixed x) as powerseries in h

It might be of interest to reformulate the function to a fixed parameter x where only the iteration h is a variable argument. This means that we must introduce a family of functions $g(x,c)$ which depend on x and use the coefficients of one column c for its taylorseries. We shall then evaluate the functions $g(x,c)$ first giving the coefficients for the powerseries of $w_x(h)$ in terms of h (which is actually only a rewriting as $w_x(h) = f^{\circ h}(x)$) where

$$w_x(h) = \sum_{c=0}^{\inf} g(x,c) * h^c$$

The functions $g(x,c)$ are first powerseries in x using the entries $p_{r,c}$ along the columns of ***POLY***:

$$g(x,c) = \sum_{r=1}^{\inf} x^r p_{r,c}$$

First we can change order of summation because each $p_{r,c}$ is a finitely composed sum of c terms:

$$\begin{aligned} g(x,c) &= \sum_{r=0}^{\inf}\left(x^r \cdot \frac{\left(1-(-1)^{r-c}\right)}{r!} \cdot \frac{1}{c} \cdot \sum_{k=1}^{c}\left((-1)^k \cdot \binom{c}{k} \cdot k^r \right)\right) \\ &= \sum_{r=0}^{\inf}\left(\frac{1}{c} \cdot \sum_{k=1}^{c}\left((-1)^k \cdot \binom{c}{k} \cdot k^r \cdot x^r \cdot \frac{\left(1-(-1)^{r-c}\right)}{r!}\right)\right) \\ &= \frac{1}{c} \cdot \sum_{k=1}^{c}\left((-1)^k \cdot \binom{c}{k} \cdot \sum_{r=1}^{\inf}\left(\left(1-(-1)^{r-c}\right)\frac{(kx)^r}{r!}\right)\right) \end{aligned}$$

The inner sums can be reformulated into closed forms as exponentials:

$$g(x,c) = \frac{1}{c} \cdot \sum_{k=1}^{c}\left((-1)^k \cdot \binom{c}{k} \cdot \left((e^{kx}-1) - (-1)^c (e^{-kx}-1) \right)\right)$$

This gives at even indexed columns $c=2j>0$:

$$\begin{aligned} g_E(x,c) &= \frac{1}{c} \cdot \sum_{k=1}^{c}\left((-1)^k \cdot \binom{c}{k} \cdot \left((e^{kx}-1) - (e^{-kx}-1) \right)\right) \\ &= \frac{2}{c} \cdot \sum_{k=1}^{c}\left((-1)^k \cdot \binom{c}{k} \cdot \sinh(kx) \right) \end{aligned}$$

and at odd indexed columns $c=2j+1$:

$$\begin{aligned} g_O(x,c) &= \frac{1}{c} \cdot \sum_{k=1}^{c}\left((-1)^k \cdot \binom{c}{k} \cdot \left((e^{kx}-1) + (e^{-kx}-1) \right)\right) \\ &= \frac{2}{c} \cdot \sum_{k=1}^{c}\left((-1)^k \cdot \binom{c}{k} \cdot (\cosh(kx)-1) \right) \end{aligned}$$

Expressed as function in Pari/GP this is

```
\\ user-function g(x,c) = POLY_column_sum
g(x,c) = if(c==0,return(x)); if(c % 2, g_odd(x,c),g_even(x,c))

    \\ internal functions
         g_even(x,c) = 2/c*sum(k=1,c,(-1)^k*binomial(c,k)*sinh(k*x))
         g_odd (x,c) = 2/c*sum(k=1,c,(-1)^k*binomial(c,k)*(cosh(k*x)-1))
```

With this, the function $w_x(h)=f^{\circ h}(x)$ is

$$w_x(h) = \sum_{c=0}^{\inf} g(x,c) * h^c$$

2.9. Numerical examples

At $x=1$ this gives numerically:

$$w_1(h) = 1 - 1.08616126963*h + 1.27645802056*h^2 - 1.60687788465*h^3$$
$$+ 2.13938758000*h^4 - 2.97554022629*h^5 + 4.27895075604*h^6$$
$$- 6.31199338319*h^7 + 9.49553583745*h^8 + O(h^9)$$

The coefficients seem to increase, but alternate in sign. Before further analysis if I apply Euler-summation I find results for fractional heights in the unit-interval $0<=h<=1$

height h	$w_1(h)=f^{\circ h}(1)$
0.00	1.000000000000
0.05	0.948694725422
0.10	0.902729488855
0.15	0.861269948104
0.20	0.823653583114
0.25	0.789346386684
0.30	0.757912297060
0.35	0.728991136470
0.40	0.702282374504
0.45	0.677532970554
0.50	0.654528129922
0.55	0.633084178162
0.60	0.613042999976
0.65	0.594267650303
0.70	0.576638855091
0.75	0.560052195330
0.80	0.544415821523
0.85	0.529648584037
0.90	0.515678492527
0.95	0.502441437932
1.00	0.489880125645

Gottfried Helms, 10.12.2010