Determining of and finding patterns in n-periodic points of the *exp( )*-function

Abstract: it is known that for any natural number $n>=1$ there are infinitely many n-periodic points [B1990]. To answer some question in the math-forum MSE [He20c] I needed numerical examples of such n-periodic points with some given properties. From my little overview in the literature in that field I didn't find much helpful information, so I developed my own method to easily find n-periodic points [He20a].

The key is not to employ the (iterated) exp()-function itself (and/or its Newton-iteration) but to employ the log()-function instead and explicitely provide a branchindex k, and for iterated log() to provide a fixed vector $\mathbf{K}$ of such branch indexes. With this it is possible to apply only iteration $z_{j+1} \leftarrow \log(z_j, \mathbf{K})$ (not necessarily polished by a following Newton-iteration) to find n-periodic points of any order n. Moreover, it seems, the sets $\mathbb{P}_n$ of the n-periodic points are exactly indexable by the vectors $\mathbf{K}_n$ such that each fixed- or n-periodic point is uniquely defined by the vectors $\mathbf{K}_n=[k_1,k_2,...,k_n]$ (if all $k_i=0$ there are the two conjugate 1-periodic points to be taken, see below).

*Without proof, I **conjecture** that this index adresses all existing n-periodic points, up to conjugacy of the primary fixed point..*

The work is a hobby-exploration, the author is a retired lecturer for statistics in quantitative social research and has been an active amateur in number-theory in various mathematical forums over many years, especially on the problem of fractional iteration of the exponential function ("tetration").

1. The indexing of *n*-periodic points by iterated branched logarithm

1.1. Notations for iteration of *exp()* and *log()*.

We denote $f: z \to exp(z)$, and in the context of iteration we define & write

$$f^{\circ 0}(z) := z$$
$$f^{\circ 1}(z) := f(z)$$
$$f^{\circ h+1}(z) := f^{\circ h}(f^{\circ 1}(z))$$

We denote $g: z \to log(z)$, and for supplying a branchindex $k \in \mathbb{Z}$ we define & write

$$g(z,k) := log(z)+ k\cdot w \qquad \text{where} \qquad w=2\pi \hat{\imath}$$

... for vector of branches

$$g(z,[k_1,k_2]) := log(log(z)+ k_1\cdot w)+ k_2\cdot w$$
$$g(z,[k_1,k_2,...,k_n]) := log(...log(log(z)+ k_1\cdot w)+ k_2\cdot w)...) ... +k_n\cdot w$$

... for iteration given one full branch-vector $\boldsymbol{K_n} =[k_i]_{i=1..n}$ of length n

$$g^{\circ 1}(z, \boldsymbol{K}_n) := g(z, \boldsymbol{K}_n) \qquad \text{where } \boldsymbol{K}_n = [k_1,k_2,...,k_n]$$
$$g^{\circ h+1}(z, \boldsymbol{K}_n) := g^{\circ h}(g^{\circ 1}(z, \boldsymbol{K}_n), \boldsymbol{K}_n)$$

1.2. "Fixed points", "*1*-periodic points"

a) Notations and some known properties

If $z = f(z)$ then z is called a fixed-point, or better for this treatize a "*1*-periodic point".

For the text here, I denote them as p_1 or more precisely with $p_{1:k}$, where k is the branch-index such that

$$p_{1:k} = f(p_{1:k})$$
$$p_{1:k} = g(p_{1:k}, k)$$

and use

$$\mathbb{P}_1 := \{p_{1:k}\}_{k=-oo..oo}$$

for the set of all *1*-periodic points

Theorems in literature -as I have found it so far- are concerned with the question of existence, cardinality of the set of fixed points $\mathbb{P}_1$, as well as about attraction and repulsion of its elements $p_{1:k}$. It is known (for instance by [B1990],pg 156), that

- f has infinitely many fixed points, and that all of them are complex
- the infinite set of fixed points $\mathbb{P}_1$ is countable
- all fixed points are repelling/"repulsive" over iteration of f.

b) How to find/how to index: the Lambert *W*-function

The standard tool to actually find some fixed point p_1 is nowadays surely the Lambert $\boldsymbol{W}$-function, basically

$$p_1 = exp(-W(-1))$$

Note: the "1" occurs as log of the base e of the exponentiation

Moreover ***all*** p_1-points can be found using the Lambert $\boldsymbol{W}$ providing a branch-index $k^* \in \mathbb{Z}$

$p_{1:k^*} = \exp(-W_{k^*}(-1))$ where $k^* \in \mathbb{Z}$

Note: notation for the branched **W** *taken from Corless et. al. [CK1996]*

The infinitude of the fixed points and relation to the index k^* of the branch in the ***W***-function allows an indexed notation for all fixed points (or *1*-periodic points) in this type in the analoguous way as in ***a)***

$\mathbb{P}_1 := \{ p_{1:k^*} \}_{k^*=-oo..oo}$

Note: unfortunately the indexes k *and* k* *are slightly different by the convention in the Lambert W-notation: the ambiguous* k=0 *is separated in* k*=0 or k*=1*, and for all other values we have for* $k<0$ $k=-k^*$ *and for* $k>0$ $k=-k^*-1$ *. But since we do not deal with the Lambert W in the following this difference is not interesting here.*

c) How to find/how to index : fixpoint-iteration over branched logarithm

Instead of using the ***W***-function one can use simple fixpoint-iteration over the inverse function to f, because a fixed point which is repelling over iteration of f is attracting over iteration of its inverse g.

So we have above defined basically

$g(z) := \log(z)$ which is also $=f^{\circ -1}(z)$

The logarithm in the complex numbers is multivalued; to make $g(z)$ a fully usable inverse to $f(z)$ we need to explicitely denote the index k for the branch used:

$g(z,k) := \log(z) + k\cdot w$ where $w = î\, 2\, \pi$

and can then, for some z and the principal branch $k=0$ write

$g(f(z), 0) = z$

However, for some other $z = z' + k \cdot w$ we have to code the branch-index

$g(f(z), k) = g(f(z'+k\cdot w), k) = g(f(z'),k) = z' + k\cdot w = z$

which fully recovers the underlying value.

Note: perhaps a nicer and full exposition of this in Sykora [Syk16].

It is known, that for the exponential-function f all fixpoints are repelling (again for instance reported in [B1990]), thus fixpoint iteration on an initial value z_0 over the logarithm (principal branch) as well as over the logarithm with explicite branch-index is *attractive*. Moreover, it seems that the *basin of attraction* is the punctured complex plane $\mathbb{C} \setminus \{0\}$ and per branchindex k with some more exceptions. For instance, if $k=0$, then the exceptions are $z_0 \notin \{0,1,e,e^e,...\}$.

So the iteration with some chosen index k

$z_0 = 1+I$ // initializing with some value
$z_{j+1} = g(z_j , k)$ // iterating
$p_{1:k} = \lim_{j\to oo} z_j$

approximates an *1*-periodic point. *.*

Note: *if* k<>0 *it suffices that* z_0<>0*, otherwise* $z_0 \notin \{0,1,e,e\char`^e,...\}$

The infinite set $\mathbb{P}_1$ of *1*-periodic points $p_{1:k}$ can now as well be said to be indexed by the branchindex $k \in \mathbb{Z}$.

Note: *The Lambert* ***W****-indexing shall not be of concern in the following because we don't use the Lambert* ***W*** *anymore, but for completeness it should be mentioned, that the branchindex* k^* *in the Lambert* ***W*** *has an integer shift by some constant compared with that* k *in the previous paragraph. See a table of comparision of the Lambert* ***W****-index and the iterated branched logarithm index* k *in Appendix 4.1.*

1.3. *n*-periodic points

The idea of the "*fixpoint-iteration using the branched logarithm*" for finding and even for indexing of the 1-periodic points shall now be generalized to n-periodic points with $n>=2$.

For the sets of n-periodic points $\mathbb{P}_n$ we know, again for instance by [B1990], that

for each $n>=2$

- the set $\mathbb{P}_n$ is infinite
- all $p_{n:K} \in \mathbb{P}_n$ are repelling (over iteration of f)

We assume in the following: because all periodic points are repelling over iteration on f, we'll have again, that

- all periodic points are attracting over iteration on g.
- the basin of attraction for all n is $\mathbb{C}$ in the same way as with the 1-periodic points.

To express the various branch indexes as iteration goes towards n we expand the definition for $g()$ one more time such that we allow a vector $\boldsymbol{K}=[k_1,k_2,...,k_n]$,$k_j \in \mathbb{Z}$ instead of a single branchindex k:

$$g(z,[k_1,k_2]) := g(\ g(z,k_1),k_2) \qquad\qquad =\log(\log(z)+k_1 \cdot w)+k_2 \cdot w$$
$$g(z,[k_1,k_2,...,k_n]) := g(\ ...g(\ g(z,k_1),k_2),...k_n)$$

Empirically, we have always the observation of attraction:

$$z_0 = \text{<some initial value>}$$
$$z_{j+1} = g(z_j, \boldsymbol{K}_n) \qquad // \text{ iteration towards n-periodic point}$$
$$p_{n:K} = \lim_{j\to oo} z_j$$

This way we can safely approximate the n-periodic point $p_{n:K}$ iteratively $j+1$ times up to z_j. If we append then a Newton-iteration on the function $g(z_j,K_n)$ we get quadratic approximation rate towards $p_{n:K}$ to arbitrary precision.

The set of 2-periodic points for instance is then

$$\mathbb{P}_2 = \{p_{2:[k_1,k_2]}\}_{k_1,k_2 \in Z}$$

By heuristics it seems, that this description of $\mathbb{P}_2$ indeed captures the whole set of 2-periodic points; each index $\boldsymbol{K}_2$ gave one 2-periodic point, and there were none else found when other methods were tried.

Note: *I tried manual screening of a square area in $\mathbb{C}$ around the origin in small steps using Newton-iteration over f and all so found n-periodic points had a valid $\boldsymbol{K}_n$-index.*

Note: *Y. Galidakis in a 2005-article described a method for finding p_2-fixpoints with a function called "HyperW()" or "HW()" essentially searching for roots of the polynomials of the truncated powerseries for $f(f(z))-z$. Again all 2-periodic points found with this method had valid $\boldsymbol{K}_2$-indexes.*

Note: *I worked with other bases for the exponential function instead of $e=\exp(1)$. I found generally results in the same style, but sometimes with few exceptions in small n and small k_i, for instance for base $b=\hat{\imath}$ the vector $\boldsymbol{K}_3=[0,0,0]$ gave an additional primitive 3-periodic cycle; $\boldsymbol{K}_3=[-1,0,0]$ allows two different cycles, both attracting for $g(,K)$. The latter occurs the similar way for base $b=-1+I$; for another base $\boldsymbol{K}_3=[1,0,0]$ didn't define a 3-periodic point, but $\boldsymbol{K}_6=[1,0,0,1,0,0]$ defined a primitive 6-periodic point instead. That examples are rare, but I've not yet even a usable part of a systematic table.*
Anyway, none of such exceptions occured with the base e as I use it in this treatize.

1.4. Conjecture about indexing the n-periodic points

My conjecture by this observations is the following:

For base $e=exp(1)$ we have

1) All n-periodic points can uniquely be approximated by fixpoint iteration over $g(,K_n)$ with branch indexes $\boldsymbol{K_n}$ independently of the initial value z_0.

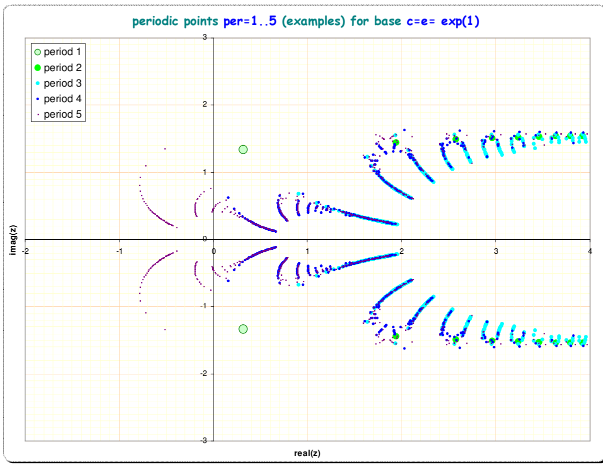
   This has the exception for $\boldsymbol{K_n}=[0],[0,0],...$ where z_0 and $conj(z_0)$ taken as initial value lead to the conjugate primary fixpoints $p_{1:[0]}$ and $conj(p_{1:[0]})$ respectively.

2) the set of n-periodic fixpoints $\mathbb{P}_n$ is in bijection with the set of vectors $\boldsymbol{K_n}$, and may be called "*indexed by $\boldsymbol{K_n}$*" - with the only exception of $\boldsymbol{K_n}$ as mentioned in 1)

There are a lot of more -but rather secondary- observations worth to be made into conjectures. Some examples with graphic visualizations are given in part 2.

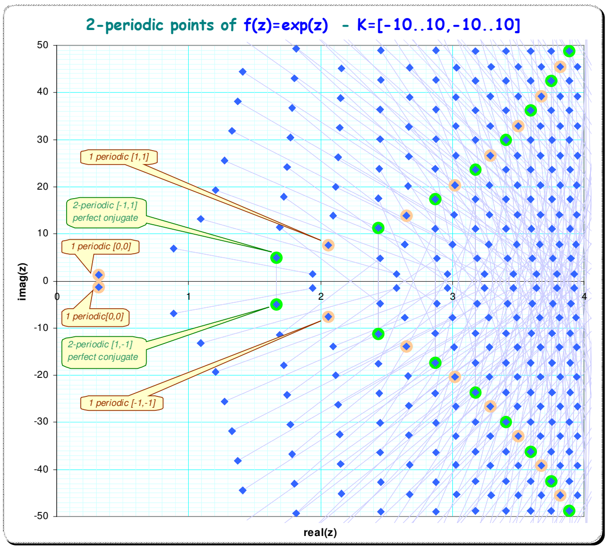
Gottfried Helms, May-Jun'2020

2. Graphical display of some interesting observations



Pic. 0: This is a picture of the periodic points which I found by screening the square around the origin of the complex plane for z in $-4..4 + -4..+4\cdot I$ in steps of $1/20$. The initial values z were used for Newton-iteration over $f^{\circ 1}(z)$, $f^{\circ 2}(z)$, $f^{\circ 3}(z)$, $f^{\circ 4}(z)$ and $f^{\circ 5}(z)$. The usable found values of the n-periodic points are plotted (only that which fit in the given box). The colors in the plot are indicating the period lengthes.

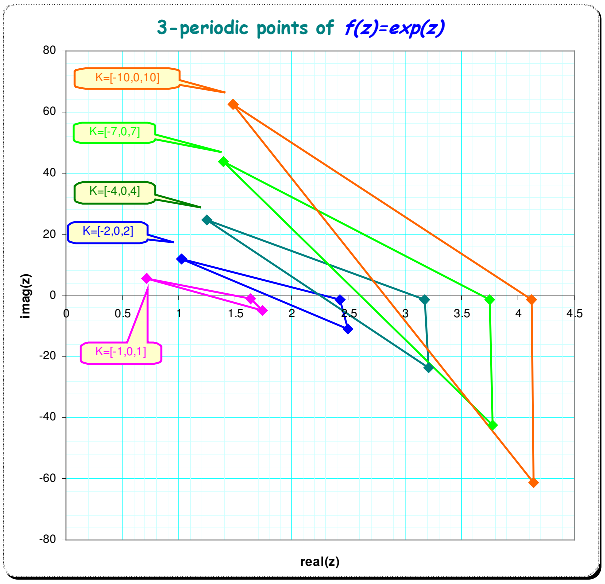
I checked all usable periodic points and found that all had valid $\boldsymbol{K}$-indexes.



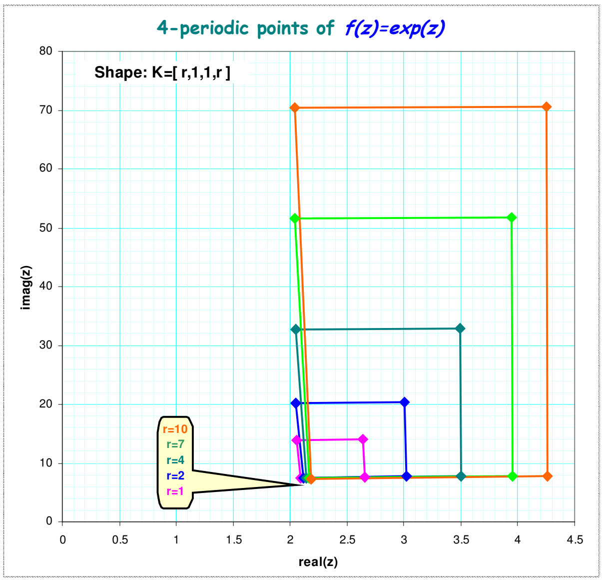
Pic. 1:This are 2-periodic points with indexes $K=[-10..10,-10..10]$ found using the fixpoint iteration as described in 1.1.3 and 1.2. (The picture was cutted to size $real(p_{2:k})=0..5$ and $imag(p_{2:k})=-50..50$ to keep readability).

We see, that for $\boldsymbol{K}=[k_1,k_2]$ with $k_1=k_2$ we have 1-periodic points and for $k_1=-k_2$ we get 2-periodic points with the two perfect conjugate values in one period.

For $[k_1,k_1]$ and $[-k_1,-k_1]$ we get 1-periodic points which are conjugates of each other. If $k_1=0$ then $-k_1=k_1$ and the conjugate of the primary fixed point must be found by using conjugate initial values for the fixpoint-iteration.

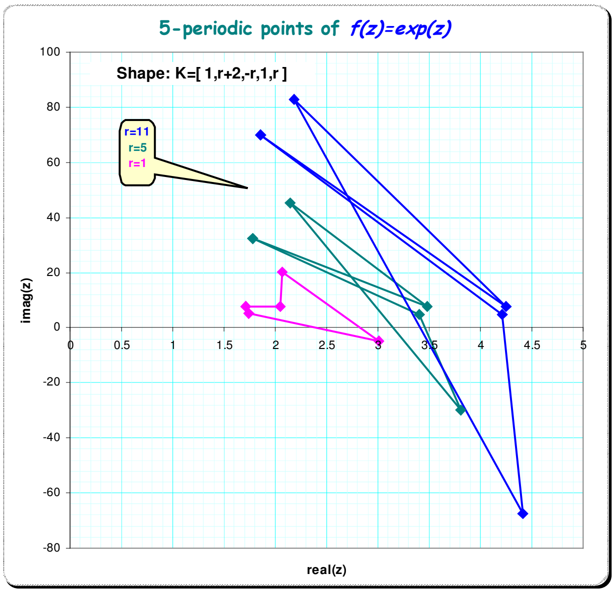


Pic. 2: For display of some examples of 3-periodic points (found according to 1.2) I thought it might be interesting to show various exemplars of a *shape-family*. With this I mean that the index-vector $\boldsymbol{K}$ is modified in a way which keeps its internal structure rather intact, for instance scaling all but one of its entries by a overall scaling. (The opposite: keeping all but one constant can be seen in the picture for the 31-periodic points below)

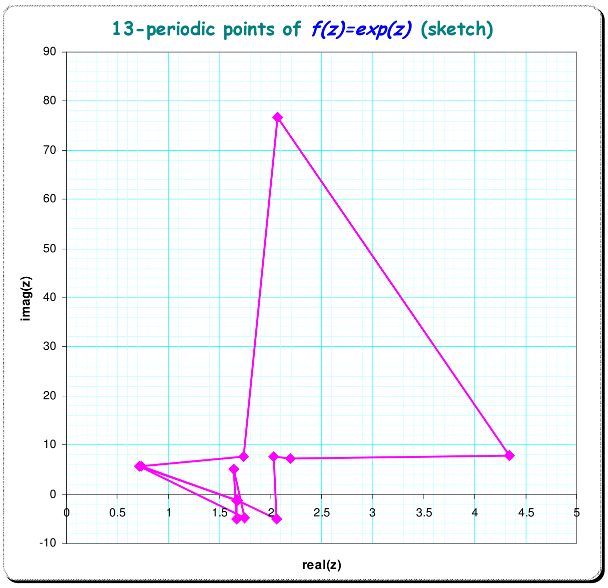


Pic. 3: A similar idea of showing a shape-family with variation in one subset of indexes k .

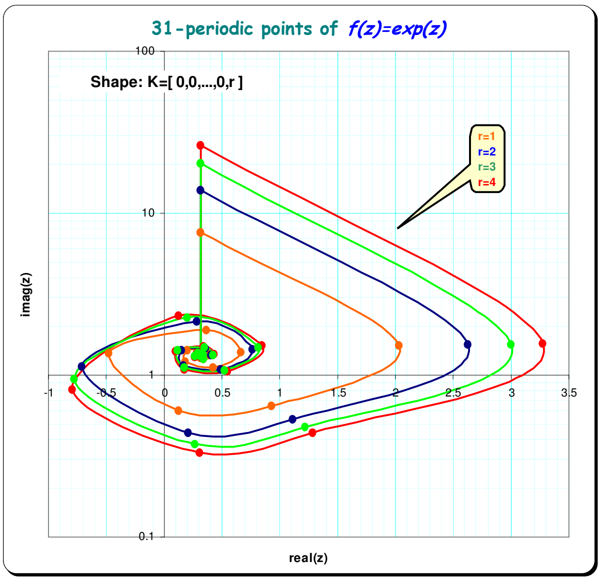
This plot shows *5* exemplars of some *4*-periodic points with a certain shape.



Pic. 4: Similarly for some exemplars of *5*-periodic points.



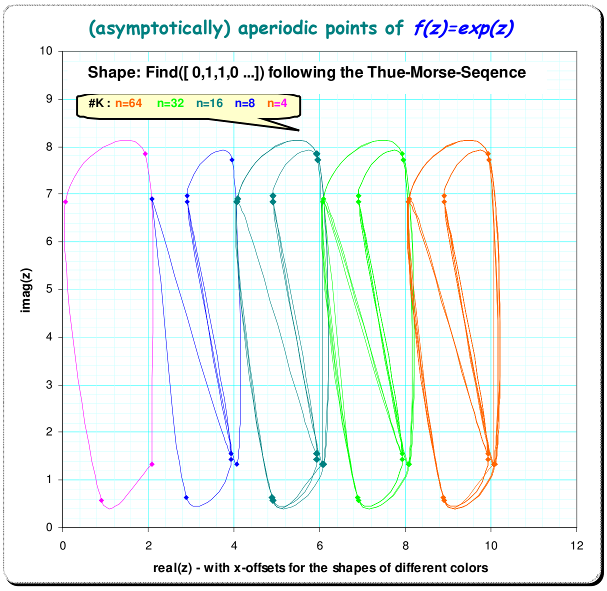
Pic. 5: See that the method can easily find n-periodic points with another n: $n=13$



Pic. 6: An especially interesting shape is the one of $K_{31}=[0,0,....,0,r]$ because this is nearly the fixpoint iteration towards the primary fixed-point. Only that - after the well known attracting orbit has arrived near the fixed point - the iteration jumps by the parameter $k_{31}=r$ back to the eccentric initial point.

Having a period of length *31* and a visible spiral towards the primary fixed point, this picture gives a very instructive view into the internal mechanics of the branched iterated logarithm.

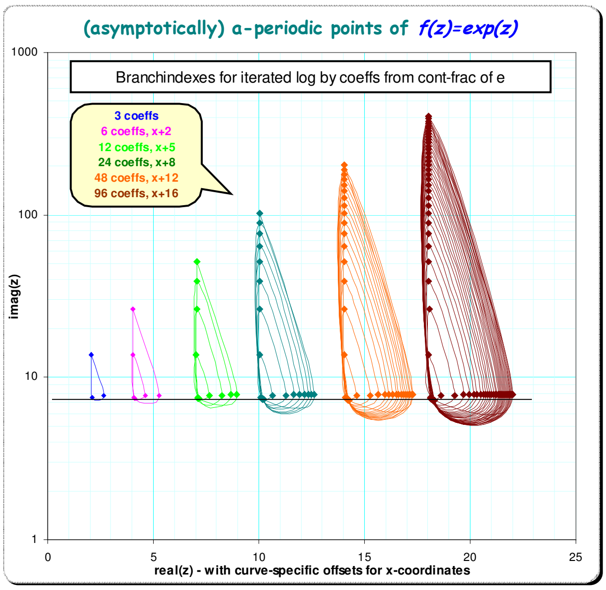
Outlook: A-periodic points?



Pic. 7: Having seen that it is easily possible to find *31*-periodic points (and even *255*-periodic points) simply by blindly increasing the length of the index-vector ***K*** I was interested, how the approximation to some infinite, ***aperiodic*** vector ***K*** would evolve. One very well known infinite sequence - only using numbers *0* and *1*, being very naturally distributed, and being still aperiodic- is the Thue-Morse sequence.

Here are the *n*-periodic points defined by ***K*** of increasing length where ***K*** contains the leading *n* numbers from the Thue-Morse sequence.

The shapes overlay very dense, so I add some *x*-offset to each shape to avoid this overlap. It seems, that increasing the length of ***K*** (by doubling) the basic shape stays constant and only refines at the edges with very small disturbances.



Pic. 8: The indexes k_i in ***K*** in the previous example are only from the set {0,1}. To see one example, where the indexes grow unboundedly (but in a tame pattern) I used the coefficients from the continued fraction of *exp(1)*. Two of three stay constant at *1*, and one of three grows unboundedly.

The members of the shape-family have a very common form, only the size of the shape grows with the length of the ***K***-index.

Closing words:

The latter two examples of approaching a-periodic points is so far only experimental and I don't have further ideas how to make something out of it.

3. Literature and links

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Links to some author's questions and answers in math.stackexchange.com and mathoverflow.net

[He20a] Helms, Gottfried: How to find examples of periodic points of the (complex) exponential-function $z \to exp(z)$? - *presenting the initial idea*
URL (version: 2020-05-14): https://math.stackexchange.com/q/3674391
(https://math.stackexchange.com/users/1714/gottfried-helms)

[He20b] Helms, Gottfried: Is my ansatz for finding n-periodic-points of the exponential-function exhaustive? - *and asking for help for proving*
URL (version: 2020-06-21): https://mathoverflow.net/q/361136
(https://mathoverflow.net/users/7710/gottfried-helms)

[He20c] Helms, Gottfried: Is there a way to calculate the zeros of $f(z,w)= w - z^{z^w}$? - Answer 1
URL (version: 2020-06-08): https://math.stackexchange.com/q/3707910
(https://math.stackexchange.com/users/1714/gottfried-helms)

[He20d] Helms, Gottfried: Is there a way to calculate the zeros of $f(z,w)= w - z^{z^w}$? - Answer 2
URL (version: 2020-06-19): https://math.stackexchange.com/q/3713978
(https://math.stackexchange.com/users/1714/gottfried-helms)

4. Appendix

4.1. Relation between the indexing by Lambert W and by iterated branched logarithm

Table 1: fixed points $p_{w:k^*}=exp(p_{w:k^*})$ according to the branches in the Lambert-W-function and the branchindexes k for the iterated branched logarithm

k*	$p_{w:k^*}$ = $p_{1:k}$	k
...	...	...
-5	3.28777+26.5805*I	4
-4	3.02024+20.2725*I	3
-3	2.65319+13.9492*I	2
-2	2.06228+7.58863*I	1
-1	0.31813+1.33724*I	0
0	0.31813-1.33724*I	0
1	2.06228-7.58863*I	-1
2	2.65319-13.9492*I	-2
3	3.02024-20.2725*I	-3
4	3.28777-26.5805*I	-4
5	3.49852-32.8807*I	-5
...	...	...

Let us use $k^*=-4$ and thus $z_0=p_{w:-4} \sim 3.02+20.27\,\hat{\imath}$. Then we have

$$log(z_0)+k\cdot w = z_0 \qquad \backslash\backslash \text{ by fixed point property}$$
$$k = (z_0 - log(z_0))/w$$
$$k = 3$$
$$==> g(z_0,3) = z_0$$
$$==> \quad z_0 = p_{1:[3]} \text{ is fixed point with index } \boldsymbol{K_1}=[3]$$