Mathematical Miniatures

Tetration

Gottfried Helms - Univ Kassel 12 - 2007

Polynomial interpolation for fractional iteration

An interpolation-approach to fractional iteration of a powerseries-function

Abstract: In this short article I discuss one way to approach a fractional iterate of a function f(x), which is defined by a powerseries/exponential-series (this concept is not only applicable to the tetration (T-tetration, U-tetration)-problem). The fractional iterate is here understood as interpolation of the coefficients of the set of powerseries, which represent the consecutive integer iterates of the given function, to build a new powerseries using these interpolated coefficients. Two simple examples are given.

The article is simply an extended version of a discussion in the sci.math-newsgroup.

Update 4

Gottfired Helms, D-Kassel *12'2007,*

Project-Index	*http://go.helms-net.de/math/binomial/index*
Intro/notation	*http://go.helms-net.de/math/binomial/intro.pdf*

Assume a function, defined as powerseries in x:

(1) $f(x) = ax + bx^2 + cx^3 + dx^4 + \ldots$

Then the iterations $f(f(x))$, $f(f(f(x)))$, ... are expressible as new powerseries

(2) $f^{(1)}(x) = f(x) \qquad f^{(2)}(x) = f(f(x)) \qquad f^{(h+1)}(x) = f(f^{(h)}(x)) \qquad f^{(0)}(x) = x$

The new coefficients of, for instance of $f^{(2)}(x)$ are finite combinations of the original coefficients $a,b,c,\ldots$ in the following way, which occur, if in (1) for x the function $f(x)$ is inserted, the expressions expanded and like powers of x are collected:

(3)
$$
\begin{aligned}
f^{(2)}(x) &= a\,f(x) + b\,f(x)^2 + c\,f(x)^3 + \ldots \\
&= a(ax + bx^2 + cx^3 + \ldots) \\
&\quad + b(ax + bx^2 + cx^3 + \ldots)(ax + bx^2 + cx^3 + \ldots) \\
&\quad + c(ax + bx^2 + cx^3 + \ldots)^3 \\
&\quad + \ldots \\
&= a^2x + ab\,x^2 + ac\,x^3 + \ldots \\
&\quad + a^2bx^2 + 2ab^2\,x^3 + \ldots \\
&\quad + a^3c\,x^3 + \ldots \\
&= a^2\,x + (ab+a^2b)x^2 + (ac + 2ab^2 + a^3c)\,x^3 + \ldots
\end{aligned}
$$

This can be iterated. One may then observe the progression of the coefficients at each power of x along the iterations like

(4)
$$
\begin{aligned}
f^{(0)}(x) &= 1\ x \\
f^{(1)}(x) &= a\ x \quad + \ b\ \ x^2 \quad + \ c\,x^3 \quad + \ldots \\
f^{(2)}(x) &= a^2x \quad + (ba^2+ba)\,x^2 \quad + (ac + 2ab^2 + a^3c)\,x^3 \quad + \ldots
\end{aligned}
$$

when read column-wise.

If in our powerseries for $f(x)$ the term $a=1$ this simplifies significantly:

(5)
$$
\begin{aligned}
f^{(0)}(x) &= 1\,x \\
f^{(1)}(x) &= 1\,x \quad + b\,x^2 \quad + \ c\,x^3 \quad + \ldots \\
f^{(2)}(x) &= 1\,x \quad + 2b\,x^2 \quad + (2b^2 + 2c)\,x^3 \quad + \ldots
\end{aligned}
$$

and when this is written in a table we see, that – based on the progressions in the coefficients of each power of x – we can build polynomials in h to express the coefficients for each iterate of $f^{(h)}(x)$ and to define continuous iterates as interpolations based on these coefficients-polynomials.

(6) Table of differences of coefficients along iterates of $f^{(h)}(x)$, using $a=1$

diff. index	$f^{(0)}(x)$	$f^{(1)}(x)$	$f^{(2)}(x)$	$f^{(3)}(x)$
Δ^0	$1\,x$ 0 0 0 0	$1\ {*}x$ $1b * x^2$ $1c * x^3$ $1d * x^4$ $1e * x^5$ ...	$1\ {*}x$ $(2b) * x^2$ $(2c+2bb) * x^3$ $(2d+5bc+b^3) * x^4$ $(2e+3(2bd+c^2)+5b^2c) * x^5$ ...	$1\,x$ $(3b) * x^2$ $(3c + 6b^2) * x^3$ $(3d + 15bc + 9b^3) * x^4$ $(3e + 9(2bd + c^2) + 41b^2c + 10b^4) * x^5$ . ...
Δ^1		$0\,x$ $1b\,x^2$ $1c\,x^3$ $1d\,x^4$ $1e\,x^5$	$0\,x$ $(1b)\,x^2$ $(1c + 2bb) * x^3$ $(1d + 5bc + b^3) * x^4$ $(1e + 3(2bd + c^2) + 5b^2c) * x^5$	$0\,x$ $(1b)\,x^2$ $(1c + 4b^2) * x^3$ $(1d + 10bc + 8b^3) * x^4$ $(1e + 6(2bd + c^2) + 36b^2c + 10b^4) * x^5$
Δ^2			$0\,x$ $0\,x^2$ $(2bb)\,x^3$ $(5bc + b^3) * x^4$ $(3(2bd+c^2)+5b^2c) * x^5$	$0\,x$ $0\,x^2$ $(2bb) * x^3$ $(5bc + 7b^3) * x^4$ $(3(2bd + c^2) + 31b^2c + 10b^4) * x^5$
Δ^3				$0\,x$ $0\,x^2$ $0 * x^3$ $(6b^3) * x^4$ $(26b^2c + 10b^4) * x^5$

Legend: linear progression quadratic progression cubic progression biquadratic progression of coefficients

Example 1:

If all coefficients $a,b,c,d,... = 1$ we have the powerseries

(7) $f(x) = f^{(1)}(x) = 1x + 1x^2 + 1x^3 + ...$

the iterates

(8)
$$\begin{aligned} f^{(1)}(x) &= 1x + 1x^2 + 1x^3 + ... \\ f^{(2)}(x) &= 1x + 2x^2 + 4x^3 + ... \\ f^{(3)}(x) &= 1x + 3x^2 + 9x^3 + ... \\ ... &= ... \\ f^{(h)}(x) &= 1x + hx^2 + h^2x^3 + ... \end{aligned}$$

with the according ranges of convergence for x (not discussed here). The coefficients of the powerseries for the fractional iterates h are then simply the powers of h. (which, btw, is another access to describe the first column of the h'th (continuous) powers of the pascalmatrix P).

Example 2:

If the coefficients $a,b,c,d,...$ follow the sequence of reciprocals of factorials, as it is with the second column in the matrix $S2$ (factorial scaled Stirling-numbers 2'nd kind, see appendix) we get

$(a=1,b=1/2!,c=1/3!,....)$

(9)
$$\begin{aligned} f^{(1)}(x) &= 1\,x + 1/2\,x^2 + 1/6\,x^3 + ... \\ f^{(2)}(x) &= 1\,x + 1\,x^2 + 5/6\,x^3 + 5/8\,x^4 + 13/30\,x^5 + ... \\ f^{(3)}(x) &= 1\,x + 3/2\,x^2 + 2\,x^3 + 5/2\,x^4 + 179/60\,x^5 + ... \\ ... &= ... \end{aligned}$$

which agrees with the matrix-computation of the powers of $S2$. The coefficients for the powerseries, which reflects the h'th iterate (which may also be fractional) are then expressible as polynomials in h:

(10) *Polynomials for coefficients at k'th power of x*

x^k	coefficients as functions of h
x^1*	$(1\ )$
x^2*	$(0\ + 1/2\ h\ )$
x^3*	$(0\ - 1/12\,h\ + 1/4\ h^2\ )$
x^4*	$(0\ \ 1/48\,h\ - 5/48\,h^2\ + 1/8\,h^3\ )$
...	...

So we get the powerseries for an arbitrary (fractional or complex) h'th iterate:

(11) $f^{(h)}(x) = 1\,x + (1/2\,h)\,x^2 + (1/4\,h^2 - 1/12\,h)\,x^3 + (1/8\,h^3 - 5/48\,h^2 + 1/48\,h)\,x^4 + ...$

which defines an interpolated (continuous) h'th iterate of the function $f(x) = exp(x) - 1$ (or: "U-tetration" $U_e(x,h)$ of height h)

Extension to infinite series

1) alternating series of increasing height ($h=0,1,2,3,...$ or $h=0,-1,-2-,3,...$)

We may assume **U**-tetration – series of consecutive heights, $h=0,1,2,3,...$, where $U(x,1)=exp(x)-1$

$$AU(x) = x - U(x,1) + U(x,2) - U(x,3) + ...$$

This is reflected by the infinite alternating series of h in formula ***(11)***, expressed by the (modified) Dirichlet-eta-function $\eta(-k) = 0^k - 1^k + 2^k - ... + ...$(*start at index 0*)

$$\begin{aligned}(12)\qquad AU(x) &= 1x \\ &- (1x + (1/2*1) x^2 + (1/4*1^2 - 1/12*1) x^3 + (1/8*1^3 - 5/48*1^2 + 1/48*1) x^4 + ...) \\ &+ (1x + (1/2*2) x^2 + (1/4*2^2 - 1/12*2) x^3 + (1/8*2^3 - 5/48*2^2 + 1/48*2) x^4 + ...) \\ &- (1x + (1/2*3) x^2 + (1/4*3^2 - 1/12*3) x^3 + (1/8*3^3 - 5/48*3^2 + 1/48*3) x^4 + ...) \\ &... \qquad\qquad ... \\ &= \eta(0) x + (1/2*\eta(-1)) x^2 \\ &\quad + (1/4*\eta(-2) - 1/12*\eta(-1)) x^3 \\ &\quad + (1/8*\eta(-3) - 5/48*\eta(-2) + 1/48*\eta(-1)) x^4 + ... \\ AU(x) &= 1/2 x - 1/8 x^2 + 1/48 x^3 + 1/96 x^4 - 19/1920 x^5 - 13/7680 x^6 + 2623/322560 x^7 ...\end{aligned}$$

These coefficients occur similarly, if the geometric-series of $S2$ is computed by the formula

$$AS2 = (I + S2)^{-1}$$

Unfortunately, the absolute values of coefficients grow after they approach a local minimum; the first 32 terms of the powerseries are

$$\begin{aligned}(13)\qquad AU(x) &= 0.5*x - 0.125*x^2 + 0.0208333*x^3 + 0.0104167*x^4 - 0.00989583*x^5 - 0.00169271*x^6 \\ &+ 0.00813182*x^7 - 0.00113157*x^8 - 0.0111201*x^9 + 0.00600352*x^{10} + 0.0232243*x^{11} \\ &- 0.0248572*x^{12} - 0.0689663*x^{13} + 0.122171*x^{14} + 0.275591*x^{15} - 0.745013*x^{16} \\ &- 1.41855*x^{17} + 5.62062*x^{18} + 9.06553*x^{19} - 51.7967*x^{20} - 69.5330*x^{21} + 574.804*x^{22} \\ &+ 617.490*x^{23} - 7577.81*x^{24} - 6052.46*x^{25} + 117228.*x^{26} - 59786.4*x^{27} - 2.10483E6*x^{28} \\ &- 430171.*x^{29} + 4.34384E7*x^{30} - 5.25132E6*x^{31} - 1.02150E9*x^{32} + O(x^{33})\end{aligned}$$

The growthrate of the absolute values of the coefficients of this powerseries is hypergeometric, so the radius of convergence is zero, and also cannot simply be summed via Euler-summation.

If we construct the series of negative heights instead of positive heights, we get the same coefficients, only of inverse sign, except at the first term, and this defines then

$$(14)\qquad AL(x) = x - log(1+x) + log(1+log(1+x)) - ... + ...$$

The sum of $AU(x)$ and $AL(x)$, both expressed by the powerseries, is then

$$(15)\qquad AU(x) + AL(x) = 0.5x + 0.5x = x$$

because all coefficients vanish; but as we have seen in a discussion in sci.math, this result disagrees with numerical computations, which were done by evaluation of the partial sums in some convergent cases (with appropriate bases b). This discrepancy can still not be explained.

2) alternating series of increasing x ($x=0,1,2,3,...$ or $x=0,-1,-2,-3$) with fixed height h

We may assume **U**-tetration – series with consecutive parameters $x=0,1,2,3,...$ and a fixed height, say $h=1$, where again $U(x,1)=exp(x)-1$

$$(16)\qquad GU(h) = GU^+(h) = U(0,h) - U(1,h) + U(2,h) - U(3,h) + ...$$
$$GU^-(h) = U(0,h) - U(-1,h) + U(-2,h) - U(-3,h) + ...$$

This is reflected by the infinite alternating series of x in formula ***(11)***, again expressed by the (modified) Dirichlet-eta-function $\eta(-k) = 0^k - 1^k + 2^k - ... + ...$(*start at index 0*) but inserted for x :

recall (11)

$$f^{(h)}(x) = 1\,x + (1/2\,h)\,x^2 + (1/4\,h^2 - 1/12\,h)\,x^3 + (1/8\,h^3 - 5/48\,h^2 + 1/48\,h)\,x^4 + ...$$

$$(17)\qquad \begin{aligned} GU^+(h) = \;& 1{*}0 + (1/2\,h){*}0^2 + (1/4\,h^2 - 1/12\,h){*}0^3 + (1/8\,h^3 - 5/48\,h^2 + 1/48\,h){*}0^4 + ... \\ & -(1{*}1 + (1/2\,h){*}1^2 + (1/4\,h^2 - 1/12\,h){*}1^3 + (1/8\,h^3 - 5/48\,h^2 + 1/48\,h){*}1^4 + ...) \\ & +(1{*}2 + (1/2\,h){*}2^2 + (1/4\,h^2 - 1/12\,h){*}2^3 + (1/8\,h^3 - 5/48\,h^2 + 1/48\,h){*}2^4 + ...) \\ & -(1{*}3 + (1/2\,h){*}3^2 + (1/4\,h^2 - 1/12\,h){*}3^3 + (1/8\,h^3 - 5/48\,h^2 + 1/48\,h){*}3^4 + ...) \\ & ... \qquad\qquad ... \end{aligned}$$

$$(18)\qquad GU^+(h) = 1\,\eta(-1) + (1/2\,h)\,\eta(-2) + (1/4\,h^2 - 1/12\,h)\,\eta(-3) + (1/8\,h^3 - 5/48\,h^2 + 1/48\,h)\,\eta(-4) + ...$$

$$(19)\qquad GU^+(1) = \eta(-1) + 1/2\,\eta(-2) + 1/6\,\eta(-3) + 1/24\,\eta(-4) + 1/120\,\eta(-5)\,...$$

In $GU^-(h)$ only the terms, where x has odd exponent, and thus the according eta's, change their signs, so we have

$$(20)\qquad GU^-(h) = -1\,\eta(-1) + (1/2\,h)\,\eta(-2) - (1/4\,h^2 - 1/12\,h)\,\eta(-3) + (1/8\,h^3 - 5/48\,h^2 + 1/48\,h)\,\eta(-4) + ...$$

$$(21)\qquad GU^-(1) = -\eta(-1) + 1/2\,\eta(-2) - 1/6\,\eta(-3) + 1/24\,\eta(-4) - 1/120\,\eta(-5)\,...$$

Since all etas of even index are zero, this reduces remarkably:

$$(21)\qquad GU^+(1) = \eta(-1) + 1/6\,\eta(-3) + 1/120\,\eta(-5)\,...$$
$$GU^-(1) = -\eta(-1) - 1/6\,\eta(-3) - 1/120\,\eta(-5)\,...$$

and the formal sum and difference are

$$(22)\qquad GU^+(1) + GU^-(1) = 0$$
$$(23)\qquad GU^+(1) - GU^-(1) = 2\,(\eta(-1) + 1/3!\,\eta(-3) + 1/5!\,\eta(-5)\,...\,) = 2*(e/(1+e) - \eta(0)) \sim 0.46212$$

Note, that we can have *(22)* more general, if we add *(18)* and *(21)* formally, also recalling that the negative even-indexed eta-values are zero:

(24)

$GU^+(h)$	$GU^-(h)$
$1\ \eta(-1)$	$-\ 1\,\eta(-1)$
$+ (1/2\,h)\,\eta(-2)$	$+ (1/2\,h)\,\eta(-2)$
$+ (1/4\,h^2 - 1/12\,h)\,\eta(-3)$	$- (1/4\,h^2 - 1/12\,h)\,\eta(-3)$
$+ (1/8\,h^3 - 5/48\,h^2 + 1/48\,h)\ \eta(-4)$	$+ (1/8\,h^3 - 5/48\,h^2 - 1/48\,h)\ \eta(-4)$
+ ...	....

which is zero not only for height 1 but for all heights h:

$$(25)\qquad GU^+(h) + GU^-(h) = 0$$

(see my related article [tetra-geometric-series])

Appendix:

Pari/GP (%-commands are Pari-TTY meta-tags redirecting matrix-output into separate GUI-windows)

```
tmpser1=Ser([a,b,c,d,e,f])*x
tmpser2 =subst(tmpser1,x,tmpser1)
tmpser3 =subst(tmpser2,x,tmpser1)
tmpser4 =subst(tmpser3,x,tmpser1)
tmpser5 =subst(tmpser4,x,tmpser1)

%box >tst subst(Mat(polcoeffs(tmpser2))~,a,1)
```

Powers of S2:

S2

1	.	.	.	.	.
0	1	.	.	.	.
0	1/2	1	.	.	.
0	1/6	1	1	.	.
0	1/24	7/12	3/2	1	.
0	1/120	1/4	5/4	2	1

$S2^2$

1	.	.	.	.	.
0	1	.	.	.	.
0	1	1	.	.	.
0	5/6	2	1	.	.
0	5/8	8/3	3	1	.
0	13/30	35/12	11/2	4	1

$S2^3$

1	.	.	.	.	.
0	1	.	.	.	.
0	3/2	1	.	.	.
0	2	3	1	.	.
0	5/2	25/4	9/2	1	.
0	179/60	11	51/4	6	1

Links:

[tetration-Index] http://go.helms-net.de/math/tetdocs/index.htm

[tetra-geometric-series] http://go.helms-net.de/math/tetdocs/Tetration_GS_short.pdf

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Gottfried Helms, 23.12.2007