04 "Gauss"-matrix GS

Abstract: The matrix **GS** *occurs as triangular scheme of coefficients, if the derivatives of the Gauss-function are computed.*

This article is just a minor extension of the main subject (which covers binomial and related matrices) and is added here only because of the intriguing hierarchy of the matrix-logarithms of GS and the binomial-matrix. The problem of the computing of integrals is only mentioned at the end, but it seems to be part of an interesting simple scheme, and is related to the techniques of divergent summation, which will be dealt with in a later chapter.

Contents:

For an intro about the conventions of notation and naming of basic-matrices see [intro]
http://go.helms-net.de/math/binomial/intro.pdf

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1. Definitions/ Identities

1.1. The (normal) Gaussian-function (normal distribution) and derivatives

The matrix ***GS*** occurs, if the coefficients of *z* in the derivatives of the (standardized) Gauss-function *N(z)* are computed.

Let

(1.1.1.) $$N(z) = \frac{1}{\sqrt{2\pi}} exp\left(-\frac{z^2}{2}\right)$$

Rewrite the constant term as *c*, and the exponential-term as *E(z)*. Then define

(1.1.2.) $f := N(z) = c * E(z)$

Then

(1.1.3.)
$$\begin{aligned}
f &= cE(z) *(1) \\
f' &= cE(z) *(-1 z) \\
f'' &= cE(z) *(-1 + 1 z^2) \\
f''' &= cE(z) *(3 z - 1 z^3) \\
f^{(4)} &= cE(z) *(3 - 6 z^2 + 1 z^4) \\
f^{(5)} &= cE(z) *(-15 z +10 z^3 - 1 z^5) \\
f^{(6)} &= cE(z) *(-15 +45 z^2 -15 z^4 +1 z^6)
\end{aligned}$$
... etc

1.2. The matrix GS and GS $^{-1}$

The infinite lower triangular matrix of the cofactors of *z* in *(1.1.3)* is

Example:

(1.2.1.) GS =

1	.	.	.	.	.	.	.
0	-1	.	.	.	.	.	.
-1	0	1	.	.	.	.	.
0	3	0	-1	.	.	.	.
3	0	-6	0	1	.	.	.
0	-15	0	10	0	-1	.	.
-15	0	45	0	-15	0	1	.
0	105	0	-105	0	21	0	-1

For the following definition it is easier to use ***GS***$^{-1}$, the reciprocal of ***GS***, which is the unsigned version[1], also steming from the formal inverse of the *N*-function:

(1.2.2.) $$1 / N(z) = \sqrt{2\pi}\, exp\left(\frac{z^2}{2}\right)$$

Example:

(1.2.3.) GS^{-1} =

1	.	.	.	.	.	.	.
.	1	.	.	.	.	.	.
1	.	1	.	.	.	.	.
.	3	.	1	.	.	.	.
3	.	6	.	1	.	.	.
.	15	.	10	.	1	.	.
15	.	45	.	15	.	1	.
.	105	.	105	.	21	.	1

[1] (for details see chapter "the reciprocal" below).

Defining a factorial function only valid for the even numbers $n=2m$:

(1.2.4.) $$f(n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \dfrac{1}{2^{n/2}} \dfrac{n!}{(n/2)!} & \text{if } n \text{ is even} \end{cases}$$

for instance

$f(0) = f(2) = 1$
$f(4) = 1*3$
$f(6) = 1*3*5$
$f(8) = 1*3*5*7$

then for example, the unsigned entries in row r of the first column of GS^{-1} are:

(1.2.5.)	$GS^{-1}_{r,0} = f(r)$

The whole matrix GS^{-1} can now be seen as the hadamard-product of the binomial-matrix P and the triangular matrix, which occurs, if the first column of GS^{-1} is downshifted by one row for each column:

Example:

$$GS^{-1} = \begin{bmatrix} 1 & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . \\ 1!! & . & 1 & . & . & . & . & . \\ . & 1!! & . & 1 & . & . & . & . \\ 3!! & . & 1!! & . & 1 & . & . & . \\ . & 3!! & . & 1!! & . & 1 & . & . \\ 5!! & . & 3!! & . & 1!! & . & 1 & . \\ . & 5!! & . & 3!! & . & 1!! & . & 1 \end{bmatrix} ¤ \begin{bmatrix} 1 & . & . & . & . & . & . & . \\ 1 & 1 & . & . & . & . & . & . \\ 1 & 2 & 1 & . & . & . & . & . \\ 1 & 3 & 3 & 1 & . & . & . & . \\ 1 & 4 & 6 & 4 & 1 & . & . & . \\ 1 & 5 & 10 & 10 & 5 & 1 & . & . \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & . \\ 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \end{bmatrix}$$

P

where " ¤ " denotes the elementwise (Hadamard)-multiplicator

Description of all entries:

(1.2.6.)	$GS^{-1}_{r,c}$	$= 0$	*(if (r-c) is odd)*
		$= f(r-c) * binomial(r,c)$	*(if (r-c) is even)*

or, using the Hadamard-product-representation

$$\mathbf{GS} := \quad GS_{r,c} = \frac{1}{2^m} * \frac{(2m)!}{m!} * P_{r,c} \qquad \text{where } m = \frac{r-c}{2}, (r-c) \text{ is even}$$

explicitely :

(1.2.7.)	$\mathbf{GS} := \quad GS_{r,c} = \dfrac{1}{2^m} \dfrac{1}{m!} \dfrac{r!}{c!}$	*if (r-c) is even, m=(r-c)/2*

The column-signed version GS_J:

A different column-signed version is

using $J = diag(1,-1,1,-1,....)$

(1.2.8.) $GS_J = GS * J = J * GS$

Example:

$$GS_J = GS * J = J * GS \qquad \begin{bmatrix} 1 & . & . & . & . & . & . & . \\ 0 & 1 & . & . & . & . & . & . \\ -1 & 0 & 1 & . & . & . & . & . \\ 0 & -3 & 0 & 1 & . & . & . & . \\ 3 & 0 & -6 & 0 & 1 & . & . & . \\ 0 & 15 & 0 & -10 & 0 & 1 & . & . \\ -15 & 0 & 45 & 0 & -15 & 0 & 1 & . \\ 0 & -105 & 0 & 105 & 0 & -21 & 0 & 1 \end{bmatrix} \text{GSj}$$

This triangle with this sign-schema is also known as "*Coefficients of unitary Hermite polynomials He_n(x)*" in the "*Online Encyclopedia of Integer Sequences*" (OEIS) [*A066325*].

A066325 Coefficients of unitary Hermite polynomials He_n(x).

1, 0, 1, -1, 0, 1, 0, -3, 0, 1, 3, 0, -6, 0, 1, 0, 15, 0, -10, 0, 1, -15, 0, 45, 0, -15, 0, 1, 0, -105, 0, 105, 0, -21, 0, 1, 105, 0, -420, 0, 210, 0, -28, 0, 1, 0, 945, 0,

COMMENT Also number of involutions on n labeled elements with k fixed points times (-1)^(number of 2-cycles). Also called normalized Hermite polynomials.

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E. WEISSSTEIN in Mathworld [*mw-hermite*] reports this as .

A modified version of the Hermite polynomial is sometimes (but rarely) defined by

$$\mathrm{He}_n(x) \equiv 2^{-n/2} H_n\left(\frac{x}{\sqrt{2}}\right) \tag{59}$$

(Jörgensen 1916; Magnus and Oberhettinger 1948; Slater 1960, p. 99; Abramowitz and Stegun 1972, p. 778). The first few of these polynomials are given by

$$\mathrm{He}_1(x) = x \tag{60}$$
$$\mathrm{He}_2(x) = x^2 - 1 \tag{61}$$
$$\mathrm{He}_3(x) = x^3 - 3x \tag{62}$$
$$\mathrm{He}_4(x) = x^4 - 6x^2 + 3 \tag{63}$$
$$\mathrm{He}_5(x) = x^5 - 10x^3 + 15x. \tag{64}$$

When ordered from smallest to largest powers, the triangle of nonzero coefficients is 1; 1; -1, 1; -3, 1; 3, -6, 1; 15, -10, 1; ... (Sloane's A096713).

1.3. The reciprocal GS^{-1}

Defining a diagonalmatrix, containing alternating signs,

$$J_2 := diag([1 , -1 , -1 , 1 , 1 , -1 , -1 , 1 ,...])$$

(1.3.1.) $$J_{2\ r,r} := (-1)^{binomial(r+1,2)}$$

from which also follows

(1.3.2.) $$J_2^{-1} = J_2$$

then to determine the reciprocal is a simple similarity-transformation/-scaling:

(1.3.3.) $$GS^{-1} = J_2 * GS * J_2^{-1}$$

GS is just the row and column-signed version of its reciprocal, and the reciprocal has only positive values:

Example:

GS^{-1}

GS^-1

1	.	.	.	.	.	.	.
.	1	.	.	.	.	.	.
1	.	1	.	.	.	.	.
.	3	.	1	.	.	.	.
3	.	6	.	1	.	.	.
.	15	.	10	.	1	.	.
15	.	45	.	15	.	1	.
.	105	.	105	.	21	.	1

This similarity-pair ***GS*** and ***GS^{-1}*** is mutually related completely analoguous to the pair of the Pascalmatrices ***P*** and its reciprocal ***P^{-1}*** , which are also similar-transforms according to $P^{-1} = J * P * J^{-1}$

(see more about ***P*** and ***P$_J$*** in the chapter [*binomialmatrix*])

A consequence of this is, that the row **or** column-signed-version

$$jGS = J_2 * GS = jGS^{-1}$$

is its own reciprocal

(1.3.4.) $$jGS * jGS = I$$

since

$$J_2\, GS = GS^{-1} * J_2 = GS^{-1} * J_2^{-1} = (J_2\, GS)^{-1}$$

Example:

$jGS * jGS = I$

cGS

1	.	.	.	.	.	.	.
0	-1	.	.	.	.	.	.
1	0	-1	.	.	.	.	.
0	-3	0	1	.	.	.	.
3	0	-6	0	1	.	.	.
0	-15	0	10	0	-1	.	.
15	0	-45	0	15	0	-1	.
0	-105	0	105	0	-21	0	1

*

cGS

1	.	.	.	.	.	.	.
0	-1	.	.	.	.	.	.
1	0	-1	.	.	.	.	.
0	-3	0	1	.	.	.	.
3	0	-6	0	1	.	.	.
0	-15	0	10	0	-1	.	.
15	0	-45	0	15	0	-1	.
0	-105	0	105	0	-21	0	1

=

I

1	.	.	.	.	.	.	.
.	1	.	.	.	.	.	.
.	.	1	.	.	.	.	.
.	.	.	1	.	.	.	.
.	.	.	.	1	.	.	.
.	.	.	.	.	1	.	.
.	.	.	.	.	.	1	.
.	.	.	.	.	.	.	1

Historical excurs

In his article "*Über eine ausgezeichnete Eigenschaft der Laguerre- und Hermite-Polynomiale*" [2] KURT ENDL reports this property as "involutory"; common and special to those two (rowscaled) sets of polynomials (and different from other common orthogonal polynomials like Legendre-, Tschebyscheff-polynomials).

Since the matrix $\mathbf{L}$ of the coefficients of the Laguerre-polynomials is only a row-scaled version of the $\mathbf{P}_{2J}$-matrix (see binomialmatrix), and this only a similarity-scaled version of $\mathbf{P}_J$,

(1.3.5.) $P_{2\,J} = P_2 * J = {}^dFac(1)*P_J * {}^dFac(1)^{-1}$

(1.3.6.) $L = {}^dFac(1)^{-1} * P_{2\,J}$

and $\mathbf{P}_J$ being its own reciprocal:

(1.3.7.) $P_J = P_J^{-1}$

it follows that

(1.3.8.) $P_{2J} = P_{2J}^{-1}$

as well as

(1.3.9.) $({}^dFac(1)*L) = P_{2J} = P_{2J}^{-1} = ({}^dFac(1)*L)^{-1}$

ENDL observed this self-reciprocity of the rowscaled Laguerre-matrix as well as of the rowscaled Hermitean matrix (here in its normed version),

(1.3.10.) $jGs = (jGS)^{-1}$

and defines a family of orthogonal polynomials P_k, which all share this self-reciprocal property, where P_1 represents the (rowscaled) Laguerre-polynomials and P_2 the (rowscaled) Hermitean polynomials.

In my notation this relates to $\mathbf{P}_2$ and $\mathbf{GS}$, which occur as matrix-exponentials of basic subdiagonal-matrices (see chapter below) which contain

- *binomial(r,1) (=r)* in the first subdiagonal ($\mathbf{P}$(Pascal/Binomial), resp. r^2 in $\mathbf{P}_2$(Laguerre)),
- *binomial(r,2)* in the second subdiagonal ($\mathbf{GS}^{-1}$ (Hermitean)),

I assume (without verification) that ENDL's hierarchy is the obvious extension of this scheme.

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[2] "on a special property of the Laguerre and Hermite-polynomials" (see [*Endl*])

1.4. The matrix-logarithm of GS

GS^{-1} has a special simple and remarkable matrix-logarithm, very close to that of the binomialmatrix:

(1.4.1.) $\quad log(GS^{-1}) \ := lnGS_{c,c+2} = binomial(c+2,2)$

$$lnGS\hat{\ }{-1} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 6 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 10 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 15 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 21 & \cdot & \cdot \end{bmatrix}$$

where the second principal diagonal has the binomial-coefficients *binomial(c+2,2)* as entries in row/column (*r=c+2 , c)*.

The inverse result then of the simplem sign-change for the matrixlogarithm of ***GS***

(1.4.2.) $\quad GS \quad = exp(- log(GS^{-1}))$

Example

lnGS = log(GS)

$$lnGS = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -6 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -10 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -15 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -21 & \cdot & \cdot \end{bmatrix}$$

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2. Application to the system of derivatives of the Gauss-function N(z)

In matrix-notation the formula *(1.1.3)* gives a vector ***DN****(z)* of derivatives of *N(z)* depending on a powerseries-vector ***V****(z)* in the variable *z*.

This also means, each *r*'th row-entry of ***DN****(z)* contains the value of the *r*'th normalized Hermite-polynomial evaluated at -*z*, additionally scaled by the *cE(z)*-term in *(1.1.3)*:

(2.1.1.) $DN(z)= cE(z) * GS * V(z)$

or more explicitely

(2.1.2.) $$DN(z)=\frac{1}{\sqrt{2\pi}}exp\left(-\frac{z^2}{2}\right)*GS*V(z)$$

with the value of the *r*'th derivative at *z* in row *r* of the result.

ENDL mentions another benefit, if such polynomials (as given here) own the self-reciprocity:

Since ***jGS*** is its own reciprocal, a right-multiplication with a powerseries ***V****(z)*, which results in a vector *Y(z)*, and the inverse operation to calculate ***V****(z)* from the given ***Y****(z)* can be interchanged with the same coefficients-matrix (which applies then analoguously to P_J and P_{2J}):

(2.1.3.) let $jGS * V(z) = Y(z)$
(2.1.4.) then $V(z) = jGS * Y(z)$
with the same set of coefficients

3. Extension to negative derivatives: integrals

The simple form of the matrix-logarithm suggests to extend it to negative indexes.

For instance, a shift by *5* rows/*5* columns gives

(3.1.1.) $log(GS) := lnGS_{c,c+2}$
$= binomial(abs(c-5)+2,2)$

(3.1.2.) $$exp\left(\begin{bmatrix} . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ -6 & . & . & . & . & . & . & . \\ . & -3 & . & . & . & . & . & . \\ . & . & -1 & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & -1 & . & . \end{bmatrix}\right)=\begin{bmatrix} 1 & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . \\ -6 & . & 1 & . & . & . & . & . \\ . & -3 & . & 1 & . & . & . & . \\ 3 & . & -1 & . & 1 & . & . & . \\ . & . & . & . & . & 1 & . & . \\ . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & -1 & . & 1 \end{bmatrix}$$

To evaluate this for the integrals, this matrix must be assumed with infinite extension to the left and the ***V****(z)*-vector infinite extension to the top, representing the negative powers of *z*. The extension of ***GS*** to the left is simply a reflection of the original matrix ***GS*** at the 45-deg line, so the summation of a row of ***GS*** with the (infinitely extended) ***V****(z)*-vector implies the sum of products of factorials and negative powers of *z*, which is divergent for each *z<>0*.

If this heuristic is true, then the first integrals of *N(z)* could be written derived from expression for the first integral:

(3.1.3.)
$$\int N(z)dz \quad = N(z)*\left(\frac{1}{z}-\frac{1}{z^3}+\frac{3}{z^5}-\frac{15}{z^7}+\frac{105}{z^9}+...\pm\frac{(2k-1)!!}{z^{2k+1}}...\right)$$

Then the extension to higher integrals should be:

(3.1.4.)
$$\int^{(1)} N(z)dz \quad =\frac{exp(\frac{(z\hat{\imath})^2}{2})}{\sqrt{2\pi}}*\frac{1}{z}\left(1+\frac{1}{(z\hat{\imath})^2}+\frac{3}{(z\hat{\imath})^4}+\frac{15}{(z\hat{\imath})^6}+\frac{105}{(z\hat{\imath})^8}+...+\frac{(2k-1)!!}{(z\hat{\imath})^{2k}}...\right)$$
$$\int^{(2)} N(z)dz \quad =\frac{exp(\frac{(z\hat{\imath})^2}{2})}{\sqrt{2\pi}}*\frac{1}{z^2}\left(1+\frac{1*3}{(z\hat{\imath})^2}+\frac{3*5}{(z\hat{\imath})^4}+\frac{15*7}{(z\hat{\imath})^6}+\frac{105*11}{(z\hat{\imath})^8}+...+\frac{(2k-1)!!}{(z\hat{\imath})^{2k}}\binom{2k+1}{1}...\right)$$
$$\int^{(3)} N(z)dz \quad =\frac{exp(\frac{(z\hat{\imath})^2}{2})}{\sqrt{2\pi}}*\frac{1}{z^3}\left(1+\frac{1*6}{(z\hat{\imath})^2}+\frac{3*15}{(z\hat{\imath})^4}+\frac{15*28}{(z\hat{\imath})^6}+\frac{105*45}{(z\hat{\imath})^8}+...+\frac{(2k-1)!!}{(z\hat{\imath})^{2k}}\binom{2k+2}{2}...\right)$$
$$\int^{(4)} N(z)dz \quad =...$$

I don't know, whether a handy analytical expression for such a divergent summation depending on a variable parameter *z* can be given; but conversely, for instance a value for the summation of the first series at *z=1* could be given by the known value of the integral at the same *z.*

I found a related remark in *[Knopp]*, which reflects the series for the first integral:

304. § 66. B. Beispiele für das Summierungsproblem. **571**

Durch partielle Integration stellt man leicht fest, daß diese Funktion mit der in 1 angetroffenen identisch ist.

b) Wenn die asymptotische Reihe

$$1-\frac{1}{2\cdot x}+\frac{1\cdot 3}{2^2\cdot x^2}-+\cdots+(-1)^n\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2^n\cdot x^n}+\cdots$$

vorgelegt ist, hat man $\Phi\left(\frac{u}{x}\right)=\left(1+\frac{u}{x}\right)^{-\frac{1}{2}}$ und folglich

$$F(x)=\sqrt{x}\int_0^\infty\frac{e^{-u}}{\sqrt{u+x}}\,du=2\,e^x\sqrt{x}\int_{\sqrt{x}}^\infty e^{-t^2}\,dt.$$

Dies liefert noch die asymptotische Entwicklung

$$G(z)=\int_z^\infty e^{-t^2}\,dt=\frac{1}{2}\,e^{-z^2}\left(\frac{1}{z}-\frac{1}{2\,z^3}+\cdots\right)$$

für das sog. GAUSSsche Fehlerintegral, das in der Wahrscheinlichkeitsrechnung von besonderer Bedeutung ist.

Konrad Knopp: Unendliche Reihen, S 571, digicenter Univ Göttingen

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4. References

[Project-Index] http://go.helms-net.de/math/binomial/index

[Intro] http://go.helms-net.de/math/binomial/intro.pdf

[binomialmatrix] http://go.helms-net.de/math/binomial/01_1_binomialmatrix.pdf

[signed binomial] http://go.helms-net.de/math/binomial/01_2_signedbinomialmatrix.pdf

[Gaussmatrix] http://go.helms-net.de/math/binomial/04_1_gaussmatrix.pdf

[Stirlingmatrix] http://go.helms-net.de/math/binomial/05_1_stirlingmatrix.pdf

[Hasse] http://go.helms-net.de/math/binomial/01_x_recihasse.pdf

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[Gaussian-function] http://mathworld.wolfram.com/GaussianFunction.html

[Erf-function] http://mathworld.wolfram.com/Erf.html

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