Identities involving binomial-coefficients, Bernoulli- and Stirlingnumbers

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Mathematical Miniatures

Euler-MacLaurin using the ZETA-matrix

Abstract: The representation of the Euler-MacLaurin-summation formula in terms of the ZETA-matrix (modified Bernoulli-polynomials) is given. It is shown, that the Euler-MacLaurin-formula can be seen as another variant of summation using the ZETA-matrix, where simply the order of summation is changed.

In the text I use the term "matrixoperator". I called some of my matrices this way before I became aware, that this type of matrices was already known as "Carleman"-matrices, or, in a factorially similarity-scaled form, as "Bell"-matrix and are already well studied as tools for the expression of iteration of functions when considered in terms of their formal power series.

I've left my own naming convention here because of reasons of convenience for myself, perhaps I'll adapt this later.

(Vers 0.4)

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1. Sum of values when a function is evaluated at consecutive arguments

1.1. The Euler-MacLaurin-formula

We begin with the Euler-/MacLaurin-formula[1] where we also simply the notation from *f(n)* to f_n as given in [Knopp]'s book:

(1) $$f_0 + f_1 + f_2 + ... + f_n = \int_0^n f(t)dt + \frac{1}{2}(f_n + f_0) + \frac{b_2}{2!}(f_n' - f_0') + \frac{b_4}{4!}(f_n^{(3)} - f_0^{(3)}) + R_k$$

where the b_k are the bernoulli-numbers and R_k reflects the remaining error, if that series is truncated at the *k*'th term.

The formula can be made more smooth by cancelling of *f(0)*: this also adapts sign in the second summand. Also we interpret *+1/2* by $-b_1$ to connect this term with the sequence of bernoulli-numbers:

(1b) $$f_1 + f_2 + ... + f_n = \int_0^n f(t)dt - \frac{b_1}{1!}(f_n - f_0) + \frac{b_2}{2!}(f_n' - f_0') + \frac{b_4}{4!}(f_n^{(3)} - f_0^{(3)}) + R_k$$

Then we replace the Bernoulli-numbers by zeta-function representations:

$$b_1 = -1\zeta(1-1) \qquad b_2 = -2\cdot\zeta(1-2) \qquad b_3 = -3\cdot\zeta(1-3) \qquad ....$$

see for instance [Woon] 98)

write the translated version,

$$f_1 + f_2 + ... + f_n = \int_0^n f(t)dt - \zeta(0)\frac{(f_n - f_0)}{0!} - \zeta(-1)\frac{(f_n^{(1)} - f_0^{(1)})}{1!} - \zeta(-3)\frac{(f_n^{(3)} - f_0^{(3)})}{3!} + R_k$$

assume the limit where *r->inf* (and *R_r->0*) and then get the rewritten expression to which I'll refer in the following:

(2) $$S_n = \sum_{k=1}^n f(k) = \int_0^n f(t)dt - \sum_{k=0}^\infty \zeta(-k)\frac{(f^{(k)}(n) - f^{(k)}(0))}{k!}$$

[1] taken from K. Knopp, "infinite series"

1.2. The notation in terms of matrixoperators/Carlemanmatrices and dotproducts

Now, in the notation of matrixoperators/Carlemanmatrices this gets the following representation:

First we assume a columnvector *F* of infinite size which contains the coefficients of the formal powerseries for *f(x)*, say: $f(x) = K + ax + bx^2 + cx^3 + \dots$ such that *F = [K,a,b,c,d,...],*

Next we introduce the notation of a Vandermondevector ***V(x)*** having a formal indeterminate argument and has the form: ***V(x)***$=[1, x, x^2, x^3, \dots]$.

Then the function *f(x)* can formally be written as dot-product of ***V(x)*** and ***F***:

$$f(x) = V(x)\sim \cdot F$$

Since our goal is to represent the Euler-MacLaurin-formula in this framework, we restate analoguously

$$(3) \qquad S_n = f(1) + f(2) + f(3) + \dots + f(n)$$
$$= (V(1) + V(2) + V(3) + \dots + V(n))\sim \cdot F$$

1.3. The use of the ZETA-matrix/bernoulli-polynomials

Now we use a modified version of the bernoulli-polynomials to express the sum *V(1)+...+V(n)*. For this I introduced in "sums-of-like-powers" [H 2007] the ***ZETA***-matrix[2], which contains the coefficients of the ***integrals*** of the bernoulli-polynomials, and can immediately be used for the problem of sums-of-like-powers in the following way:

$$ZETA \cdot (V(a) - V(b)) = V(a+1) + V(a+2) + \dots + V(b)$$

The ***ZETA***-matrix is of infinite size, triangular with one upper subdiagonal filled and has the following aspect (top-left segment is shown):

$$ZETA = \begin{bmatrix} -1/2 & -1 & 0 & 0 & 0 & 0 \\ -1/12 & -1/2 & -1/2 & 0 & 0 & 0 \\ 0 & -1/6 & -1/2 & -1/3 & 0 & 0 \\ 1/120 & 0 & -1/4 & -1/2 & -1/4 & 0 \\ 0 & 1/30 & 0 & -1/3 & -1/2 & -1/5 \\ -1/252 & 0 & 1/12 & 0 & -5/12 & -1/2 \end{bmatrix}$$

The structure is simple; it contains just the values of the zeta at negative integers cofactored by binomials (like in the Pascalmatrix ***P***):

(4) *ZETA =*

ζ(0)·1	-1	.	.	...
ζ(-1)·1	ζ(0)·1	-1/2	.	...
ζ(-2)·1	ζ(-1)·2	ζ(0)·1	-1/3	...
ζ(-3)·1	ζ(-2)·3	ζ(-1)·3	ζ(0)·1	...
ζ(-4)·1	ζ(-3)·4	ζ(-2)·6	ζ(-1)·4	...
ζ(-5)·1	ζ(-4)·5	ζ(-3)·10	ζ(-2)·10	...
...	...	...	...	...

Note, that the negative reciprocals in the upper subdiagonal can consistently be understood as containing the limit s->1 of ζ(s)/Γ(s-1)

[2] which is an extension of the original matrix of Faulhaber by the leading column containing zeta-values (see the grey-shaded column

The expression, for instance from *a+1* to *b* is then indeed (using ***ZETA*** for the sum of like powers)

$$ZETA\cdot(V(a) - V(b)) = V(a+1)+V(a+2)+\dots+V(b)$$

or from *1 to n*

$$ZETA\cdot(V(0) - V(n)) = V(1)+V(2)+\dots+V(n)$$

					example V(a)-V(b)	example V(0)-V(n)
$ZETA\cdot(V(a) - V(b)) = V(a+1)+V(a+2)+\dots+V(b)$ $ZETA\cdot(V(0) - V(n)) = V(1)+V(2)+\dots+V(n)$					$1 - 1$ $a - b$ $a^2 - b^2$ $a^3 - b^3$ $a^4 - b^4$ $a^5 - b^5$ $a^6 - b^6$ ...	$1 - 1$ $0 - n$ $0 - n^2$ $0 - n^3$ $0 - n^4$ $0 - n^5$ $0 - n^6$
$\zeta(0)\cdot 1$	-1	.	.	...	$(a+1)^0+(a+2)^0+\dots+b^0$	n
$\zeta(-1)\cdot 1$	$\zeta(0)\cdot 1$	-1/2	.	...	$(a+1)+(a+2)+\dots+b$	$1+2+3+4+\dots+n$
0	$\zeta(-1)\cdot 2$	$\zeta(0)\cdot 1$	-1/3	...	$(a+1)^2+(a+2)^2+\dots+b^2$	$1^2+2^2+3^2+4^2+\dots+n^2$
$\zeta(-3)\cdot 1$	0	$\zeta(-1)\cdot 3$	$\zeta(0)\cdot 1$	...	$(a+1)^3+(a+2)^3+\dots+b^3$	$1^3+2^3+3^3+4^3+\dots+n^3$
0	$\zeta(-3)\cdot 4.$	0	$\zeta(-1)\cdot 4$	...	$(a+1)^4+(a+2)^4+\dots+b^4$	$1^4+2^4+3^4+4^4+\dots+n^4$
$\zeta(-5)\cdot 1$	0	$\zeta(-3)\cdot 10$	0	...	$(a+1)^5+(a+2)^5+\dots+b^5$	$1^5+2^5+3^5+4^5+\dots+n^5$
0	$\zeta(-5)\cdot 6$	0	$\zeta(-3)\cdot 20$	...	$(a+1)^6+(a+2)^6+\dots+b^6$	$1^6+2^6+3^6+4^6+\dots+n^6$
...	...	...	...	...	...	...

The same in terms of bernoulli numbers: $(b_1= -1/2)$

arbitrary†	$-b_0$	.	.	...	$(a+1)^0+(a+2)^0+\dots+b^0$	n
arbitrary	b_1 ‡	$-1/2\ b_0$	.	...	$(a+1)+(a+2)+\dots+b$	$1+2+3+4+\dots+n$
arbitrary	$-b_2$	$2/2\ b_1$	$-1/3\ b_0$	...	$(a+1)^2+(a+2)^2+\dots+b^2$	$1^2+2^2+3^2+4^2+\dots+n^2$
arbitrary	0	$-3/2\ b_2$	$3/3\ b_1$	...	$(a+1)^3+(a+2)^3+\dots+b^3$	$1^3+2^3+3^3+4^3+\dots+n^3$
arbitrary	$-b_4$	0	$-6/3\ b_2$	...	$(a+1)^4+(a+2)^4+\dots+b^4$	$1^4+2^4+3^4+4^4+\dots+n^4$
arbitrary	0	$-5/2\ b_4$	0	...	$(a+1)^5+(a+2)^5+\dots+b^5$	$1^5+2^5+3^5+4^5+\dots+n^5$
arbitrary	$-b_6$	0	$-15/3\ b_4$	...	$(a+1)^6+(a+2)^6+\dots+b^6$	$1^6+2^6+3^6+4^6+\dots+n^6$
arbitrary	...	...	...	...	...	...

†*Note: "arbitrary" means here that values are not covered by the common definition of/ansatz with the Bernoulli-polynomials resp. their integrals.*

‡ *Note: the Bernoulli-number b_1 is sometimes defined having the positive value instead*

1.4. Changing order of computation: the *ZETA*-matrix and the dot-product with *F*

For the following, to have the notation compatible with my other recent articles, we use the transposed of the above ***ZETA***-equation:

$$(V(0) - V(x))\sim \cdot ZETA\sim = V(1)\sim+V(2)\sim+...+V(n)\sim$$

K
a
b
c
d
e
f
...

Then that expression shall be multiplied by the coefficients-vector for the function $f(x) = K + ax + bx^2 + ...$:

$$(V(1)\sim+V(2)\sim+...+V(n)\sim) \cdot F = f(1) + f(2) + f(3) +...+ f(n)$$

$$= ((V(0) - V(n))\sim \cdot ZETA\sim) \cdot F$$

$(1-1)\cdot\zeta(0)$	$(1-1)\cdot\zeta(-1)\cdot 1$	0	$(1-1)\cdot\zeta(-3)\cdot 1$	0	$(1-1)\cdot\zeta(-5)\cdot 1$	0	...
$(0-n)\cdot 1$	$(0-n)\cdot\zeta(0)\cdot 1$	$(0-n)\cdot\zeta(-1)\cdot 2$	0	$-n\cdot\zeta(-3)\cdot 4$	0	$-n\cdot\zeta(-5)\cdot 6$	...
.	$(0-n^2)\cdot 1/2$	$(0-n^2)\cdot\zeta(0)\cdot 1$	$-n^2\cdot\zeta(-1)\cdot 3$	0	$-n^2\cdot\zeta(-3)\cdot 10$	0	...
.	.	$(0-n^3)\cdot 1/3$	$-n^3\cdot\zeta(0)\cdot 1$	$-n^3\cdot\zeta(-1)\cdot 4$	0	$-n^3\cdot\zeta(-3)\cdot 20$	...
.	.	.	...	...	...	...	...

The same in terms of bernoulli-numbers $(b_1=-1/2)$

$0\cdot b_1$	0	0	0	0	0	0	...
$-n\cdot 1$	$(-n)\cdot b_1$	$n\cdot b_2$	0	$n\cdot b_4$	0	$n\cdot b_6$	...
.	$-n^2\cdot 1/2$	$-n^2\cdot b_1$	$n^2\cdot b_2\cdot 3/2$	0	$n^2\cdot b_4\cdot 5/2$	0	...
.	.	$-n^3\cdot 1/3$	$-n^3\cdot b_1$	$n^3\cdot b_2\cdot 6/3$	0	$n^3\cdot b_4\cdot 15/3$	...
.	.	.	...	...	...	...	...

1.5. The Euler-MacLaurin sum S_n in terms of $V(n)$, *ZETA* and *F*

The expression for the Euler-MacLaurin formula occurs then by reordering of summation.

If we do not compute the dot-product of the rows of ***ZETA*** with ***F***, but evaluate using the subdiagonals (which contain the same zeta-value) separately:

K
a
b
c
d
e
f
...

(1-1)·ζ(0)	(1-1)·ζ(-1)·1	0	(1-1)·ζ(-3)·1	0	(1-1)·ζ(-5)·1	0	...
-n·1	-n·ζ(0)·1	-n·ζ(-1)·2	0	-n·ζ(-3)·4	0	-n·ζ(-5)·6	...
.	$-n^2$·1/2	$-n^2$·ζ(0)·1	$-n^2$·ζ(-1)·3	0	$-n^2$·ζ(-3)·10	0	...
.	.	$-n^3$·1/3	$-n^3$·ζ(0)·1	$-n^3$·ζ(-1)·4	0	$-n^3$·ζ(-3)·20	...
.	.	.	...	...	...	...	...

then we get the following, organized along the sums over the subdiagonals :

-n ·K/1	(1-1)·1 K·ζ(0)	.	.	.	.	.
$-n^2$·a/2	(0-n)·1 a·ζ(0)	(1-1)·1 a·ζ(-1)	.	.	.	.
$-n^3$·b/3	(0-n^2)·1 b·ζ(0)	-n ·2 b·ζ(-1)	0·b	.	.	.
$-n^4$·c/4	(0-n^3)·1 c·ζ(0)	$-n^2$·3 c·ζ(-1)	0·c	(1-1)· 1 c·ζ(-3)	.	.
$-n^5$·d/5	(0-n^4)·1 d·ζ(0)	$-n^3$·4 d·ζ(-1)	0·d	-n · 4 d·ζ(-3)	0·d	.
$-n^6$·e/6	(0-n^5)·1 e·ζ(0)	$-n^4$·5 e·ζ(-1)	0·e	$-n^2$·10 e·ζ(-3)	0·e	(1-1)·1 e·ζ(-5)
$-n^7$· f/7	(0-n^6)·1 f·ζ(0)	$-n^5$·6 f·ζ(-1)	0·f	$-n^3$·20 f·ζ(-3)	0·f	-n ·6 f·ζ(-5)
...	...	...	...	...	...	...

which, when summed up ***columnwise***, gives

$\int_0^n f(t)dt \frac{\zeta(1)}{-1!}$	$(f_0 - f_n)\zeta(0)$	$\frac{f_0^{(1)}-f_n^{(1)}}{1!}\zeta(-1)$	0	$\frac{f_0^{(3)}-f_n^{(3)}}{3!}\zeta(-3)$	0	$\frac{f_0^{(5)}-f_n^{(5)}}{5!}\zeta(-5)$

The sum of all that column-sums

$$\int_0^n f(t)dt + \sum_{k=0}^{\inf} \frac{f^{(k)}(0) - f^{(k)}(n)}{k!} \zeta(-k)$$

reflects precisely the Euler-MacLaurin sum as given in (2).

The same in terms of the bernoulli-numbers looks like:

$-n \cdot K/1$	$(0)\cdot 1\ K\cdot b_1$	.	.	.	.	.
$-n^2\cdot a/2$	$(-n)\cdot 1\ a\cdot b_1$	$(0)\cdot 1\ a\cdot b_2/2$	.	.	.	.
$-n^3\cdot b/3$	$(-n^2)\cdot 1\ b\cdot b_1$	$n \cdot 2\ b\cdot b_2/2$	$0\cdot b$	.	.	.
$-n^4\cdot c/4$	$(-n^3)\cdot 1\ c\cdot b_1$	$n^2\cdot 3\ c\cdot b_2/2$	$0\cdot c$	$(0)\cdot\ 1\ c\cdot b_4/4$	.	.
$-n^5\cdot d/5$	$(-n^4)\cdot 1\ d\cdot b_1$	$n^3\cdot 4\ d\cdot b_2/2$	$0\cdot d$	$n\ \cdot\ 4\ d\cdot b_4/4$	$0\cdot d$	.
$-n^6\cdot e/6$	$(-n^5)\cdot 1\ e\cdot b_1$	$n^4\cdot 5\ e\cdot b_2/2$	$0\cdot e$	$n^2\cdot 10\ e\cdot b_4/4$	$0\cdot e$	$(0)\cdot 1\ e\cdot b_6/6$
$-n^7\cdot f/7$	$(-n^6)\cdot 1\ f\cdot b_1$	$n^5\cdot 6\ f\cdot b_2/2$	$0\cdot f$	$n^3\cdot 20\ f\cdot b_4/4$	$0\cdot f$	$n\ \cdot 6\ f\cdot b_6/6$
...	...	...	...	...	...	...

When summed up columnwise this gives

$\int_0^n f(t)dt$	$(f_0 - f_n)b_1$	$\frac{f_n^{(1)}-f_0^{(1)}}{1}\frac{b_2}{2!}$	0	$\frac{f_n^{(3)}-f_0^{(3)}}{1}\frac{b_4}{4!}$	0	$\frac{f_n^{(5)}-f_0^{(5)}}{1}\frac{b_6}{6!}$

The sum of that column-sums

$$\int_0^n f(t)dt + \left(f^{(k)}(0) - f^{(k)}(n)\right)\frac{b_1}{1!} + \sum_{k=1}^{\inf}\left(\left(f^{(2k-1)}(n) - f^{(2k-1)}(0)\right)\frac{b_{2k}}{(2k)!}\right)$$

reflects the Euler-MacLaurin sum as given in *(1.b)* after signs are adapted.

2. References

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