Eulerian summation (part 2):

Aspects of the "Eulerian transformation" for some divergent and non-summable cases

The "Eulerian summation" for divergent series as described initially in my older article [Eulerian2007] is furtherly explorated for the cases, where the divergent series cannot be summed, that are for instance the geometric series with q>=1 and zeta-series with s<=1.

In 2014 a surprising connection with probability theory occured. We find a formula in the concept of "renewal-theory" which expresses coordinates in the composition of uniform distributions - and that formula matches perfectly the formula for the partial sums in the Eulerian-transform of the geometric series with quotient 1 resp of the zeta-series at the argument 0. I do not currently know whether this might point to some generalizations in the discussion of composition of distributions as well.

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version 5 (14.10.2017 - minor edits, some explications)
[version 4 (14.10.2017)/ version 3 (01.08.2016) /version 2 (10.07.2014)]

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1. Intro: Motivation and notation

1.1. Overview

A couple of years ago I've toyed around with a matrix-summation-method based on the matrix of the Eulerian numbers. It has some resemblance to the Borel-summation because it involves the transformation of the series-to-be-summed into another series, whose terms are scaled by reciprocal factorials, and - for instance - by the transformation of a geometric series produces an infinite sequence of exponential series. Exponential series are *entire* and thus each of them allows arguments from the whole complex plane. The Eulerian-summation, of course, means then a) to sum the evaluations of that sequence of exponential series and b) to see, whether that evaluations form a summable series.

While that "Eulerian" summation procedure extends the summability of the geometric series with quotient *q* to the complex half plane *Real(q)<1* and even allows to sum the alternating hypergeometric series (first time done by L. Euler by other means), ***non-***alternating divergent series cannot be summed[1]: while the transformed series has still well defined terms, that terms are increasing and would themselves form a series which is again divergent (with even more divergence).

However, even in that non-summable cases the initial "Eulerian" ***transformations*** alone have interesting properties and these properties of the "Eulerian" transformations of the geometric and the Dirichlet series for the divergent and nonsummable cases are the main focus of this article.

At the end I look at properties of some (improper) equations and (thus speculative) coefficients but for which I've not yet a meaningful interpretation in any other context.

1.2. Review of used mathematical tools and notational definitions

a) The description of the summation-method is based on the notion of matrices and vectors for my own convenience in notation and intuition.

Matrices and vectors are all of infinite size. By default, I assume a vector (say ***A*** for example) as a row-vector and the following notations to declare/use them

as column-vector: 'A
as diagonal-matrix: `A

The transposition-symbol for vectors or matrices

A^T *or A~* *(the latter taken from the convention in Pari/GP)*

The indices of a vector/matrix begin at zero: *n* rows are the rows *0* to rows *n-1*

$M_{r,c}$ *where the first index refers to the row and the second to the column*

and if I talk about the elements of a matrix or a vector I denote them with small letters preferably taken from the name of the matrix which in most cases are named by a bold capital letter:

$A = [a_0, a_1, a_2, \ldots]$ *(row) vector A and its scalar components*

Let's also use ***U*** as the (infinite) unit-vector ***U*** *= [1,1,1,1,...]* for ease of notation.

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[1] I have a remark in my ear (but forgot its source) which goes: "a valid/regular matrix summation method ***must*** fail for divergent non-alternating series" - in this case for instance for the geometric series *q>=1* , for Dirichlet series for *s<=1* or the nonalternating version of the hypergeometric series.

b) We introduce a notation with some argument x for a type of infinite-sized vector which I think is best called "Vandermonde-vector" so I give it the symbolical name "$V(x)$":

$$V(x) = [1, x, x^2, x^3, \dots]$$

The dot-product, initially taken as purely formal,

$$V(x) \cdot {}'A = a_0 + a_1 x + a_2 x^2 + \dots$$

is then a simple notation for a formal powerseries

$$f(x) = V(x) \cdot {}'A = \sum x^k a_k$$

Of course I use the same notation for the reference to the actual evaluation , for instance

$$f(2) = V(2) \cdot {}'A$$

and if, for instance, ***A*** contains the consecutive reciprocal factorials

$$A = [1/0! , 1/1!, 1/2!, 1/3!, \dots]$$

$V(x) \cdot {}'A = \exp(x)$ *in the sense of a formal power series but also*
$V(2) \cdot {}'A = e^2$ *in the sense of an actual evaluation*

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c) With the same logic I use a "Dirichlet" or "Zeta-vector" with one argument of the form

$$Z(m) = [1/1^m, 1/2^m, 1/3^m, 1/4^m, \dots]$$

such that we can write

$Z(m) \cdot {}'U = \zeta(m)$ *the formal notation for the series*
$Z(2) \cdot {}'U = \zeta(2) = \pi^2/6$ *the evaluation at some point*

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d) Finally I use a "factorial" vector of the form

$G = [0!, 1!, 2!, 3! , \dots]$ *(the letter G is taken to remind of the Gamma-function)*
$g = [1, 1, 1/2!, 1/3!, \dots]$ *(just use this as an additional notation for convenience*
$`g = `G^{-1}$

such that we can write

$V(x) \cdot {}'g = \exp(x)$ *the formal notation for the exponential series*
$V(2) \cdot {}'g = e^2$ *the evaluation at some point*

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e) Basis of the summation method is the matrix of Eulerian numbers (which I've explained in more detail in [Eulerian2007])

Matrix of Eulerian numbers :

1	.	.	.	.	.	...
1	.	.	.	.	.	...
1	1	.	.	.	.	...
1	4	1	.	.	.	...
1	11	11	1	.	.	...
1	26	66	26	1	.	...
...	...	...	...	...	...	...

(Infinite size is always assumed!)

It has the interesting and well known[2] property, that the sums along its rows are just the factorials.

Thus, if we rescale its rows by the reciprocal factorials, we get a matrix with all row-sums equal *1*. Let's call that matrix "***E***" in the following.

We can thus for instance write:

$E \cdot {}'U = {}'U$	$\cdot$	$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \ldots \end{bmatrix}$	
$\begin{bmatrix} 1 & . & . & . & . & . & \ldots \\ 1 & . & . & . & . & . & \ldots \\ 1/2 & 1/2 & . & . & . & . & \ldots \\ 1/6 & 2/3 & 1/6 & . & . & . & \ldots \\ 1/24 & 11/24 & 11/24 & 1/24 & . & . & \ldots \\ 1/120 & 13/60 & 11/20 & 13/60 & 1/120 & . & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}$	$=$	$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \ldots \end{bmatrix}$	

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f) So for instance, the formal dot-product

$$V(x) \cdot E = Y(x)$$

defines a row-vector ***Y*** (with the argument x), whose entries contain formal exponential series and their derivatives (according to the definitions of the columns in ***E***),

$$Y(x) = [\exp(x), \exp(2x)\text{-}2x \exp(x), \ldots]$$ *(for more entries see below)*

and which can be evaluated for all arguments x, because the exponential-series is entire:

$$Y(x) = [e^x , e^{2x} - 2xe^x , \ldots] \quad \text{and}$$
$$Y(1) = [e, e^2 - 2e , \ldots]$$

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g) For a general function (not discussed here) with a formal powerseries like $f(x)=a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$ rewritten as dot-product with some vector $A=[a_0, a_1, a_2, a_3,\ldots]$

$$f(x) = V(x) \cdot {}'A$$

and the basic function-transform[3] :

$$g(x) = V(x) \cdot {}^`g \cdot {}'A = \sum_{k=0}^{oo} \frac{a_k x^k}{k!}$$

the (here newly and basically introduced) "*Eulerian transform*" is

$$V(x) \cdot {}^`A \cdot E = Y(x)$$

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[2] See for instance: Handbook of mathematical functions online [NIST:Eulerian] or Wikipedia [WP:EulerianNumbers]

[3] the same basic transformation is also employed in the Borel-summation for divergent series

and the entries in the resulting vector $Y(x)$ are

$$y_0 = g(x)$$
$$y_1 = g(2x) - (g(x)+ x\,g'(x))$$
$$y_2 = g(3x) - (g(2x)+ (2x)\,g'(2x)) + (x\,g'(x)+ x^2\,g''(x)/2!)$$
$$...$$

The full evaluation of the "Eulerian summation" requires now to evaluate the series of the sequence of y_k if this is possible: because it is a) either convergent, b) summable by some other summation-procedures (like for instance Cesaro-, Euler- etc summation) or c) by analytic continuation.

This requires the general description of the partial sums; we find here that the partial sums up to some columnindex c are

$$s_0 = y_0 \qquad = g(1x)$$
$$s_1 = y_0 + y_1 \qquad = g(2x) - (1x)\,g'(1x)/1!$$
$$s_2 = \qquad = g(3x) - (2x)\,g'(2x)/1! + (1x)^2\,g''(1x)/2!$$
$$...$$
$$\lim_{c\to\infty} \quad s_c = f(x) \qquad = g((c+1)x) - (cx)\,g'(cx)/1! + ((c-1)x)^2\,g''((c-1)x)/2! - ... +$$

(For more detail see appendix)

1.3. A surprising relation to a problem in mixing distributions (theory of probability)

In a question of mine in the discussion-board "math.stackexchange" [MSE] a contributor pointed to an earlier question in "mathoverflow" [MO], which had asked for the proof of a formula which involves the topic of "*composition of uniform distributions*" and which employs the same observation about the composition of the $Y(x)$ vector at $x=1$ for $Y(1) = V(1) \cdot E$, in that it looks at the partial sums of $Y(1)$ (but specifically with the argument $x=1$ only).

This is of course a nice coincidence; but because my discussion here gives a much more general framework in which this small formula is only a detail, this coincidence suggests generalizations of the "*composition of distributions*" matter itself (and from this possibly in a similarly generalized "renewal-process" problem). Unfortunately I cannot yet recognize, which of the two frameworks - that of $V(x) \cdot E$ or that of $Z(m) \cdot E$ - might provide some such meaningful generalization - if there is some meaningful generalization at all.

See also for instance one entry in the OEIS for the constant s_2 as the 3rd partial sum of the columns of $Y(1)=V(1)\cdot E$ (written as $Y(0) = [y_0, y_1, y_2, y_3,...]$ and $s_2 = y_0 + y_1 + y_2 \sim 6.66656564...$)

A090143	Decimal expansion of e^3-2e^2+e/2.
COMMENTS	Expected number of picks from a uniform [0,1] needed to first exceed a sum of 3.
LINKS	*Table of n, a(n) for n=1..102.* Eric Weisstein's World of Mathematics, Uniform Sum Distribution
EXAMPLE	6.66656564...
CROSSREFS	Cf. A001113, A090142, A089139, A090611.

where the first s_0, s_1, s_2, s_3, s_4 exist as constants in the OEIS-database:

$$A001113 = s_0,\ A090142 = s_1,\ A089139 = s_3,\ A090611 = s_4$$

2. Geometric series: Eulerian transformations (for the non-summable cases)

2.1. Intro

The usual summation of a geometric series can be written as a formal dotproduct:

$$f(x) = V(x) \cdot {}'U = 1 + x + x^2 + \dots$$

and it has, even for the divergent cases $|x| \ge 1$ except for $x=1$, the standard evaluation to

$$f(x) = \frac{1}{1-x} \qquad \textit{for instance} \qquad f(2) = \frac{1}{1-2} = -1$$

by analytic continuation.

I define the "Eulerian transformation" to be:

$$(2.1) \qquad V(x) \cdot E = Y(x)$$

and the meaning of the statement, that the "Eulerian summation" can sum the geometric series for some range of x means/implies, that in the formulae

$$V(x) \cdot {}'U = f(x) \qquad \text{(by geometric series)}$$
$$Y(x) \cdot {}'U = f(x) \qquad \text{(by sum of Eulerian transformation)}$$

the two versions evaluate to the same $f(x)$, which might also be displayed as

$$f(x) = \sum_{k=0}^{oo} x^k = \frac{1}{1-x} = \sum_{k=0}^{oo} y_k \qquad -oo < x < 1$$

or in words: the vector $V(x)$ and the resulting vector $Y(x)$ as its "Eulerian transformation" have the same vector-sum for some argument x - of course so far ***only in the range of summability*** when $-oo<x<1$ (as derived in the first part of this treatize).

But the goal of this part is to study the properties of the Eulerian transform for the cases where $x \ge 1$ ***exceeds the summability range*** - to perhaps understand and formalize the occuring differences of the results.

2.2. Analytic description of the entries in $Y(x)$

The basic eulerian transform of a geometric series $f(x) = 1 + x + x^2 + x^3 + \dots$ is simply taken by the dot-product of $V(x)$ and A_0 :

$$g_0(x) = V(x) \cdot A_0$$
$$= 1 + x/1! + x^2/2! + x^3/3! + \dots$$
$$= \exp(x)$$

From 1.2 we know, that $\mathbf{Y(x)}$ in (2.1) is well defined ***for all*** x, because the exponential series $g(x)$ (and all its derivatives) is/are *entire functions* and all entries in $Y(x)$ are computable as finite polynomials of powers *of* g(x) and its derivatives, and thus finally even simply in terms of powers of e^x .

The question of summability reduces then to the question of divergence/convergence of the transformed series $Y(x) \cdot {}'U$, so the analytic properties of $Y(x)$ are now of basic importance.

By the description of compositions[4] of the columns of the Eulerian matrix ***E*** we find that the first few entries in *Y(x)* are:

(2.2.1)
$$\begin{aligned} g_0(x) &= e^{1x} \\ g_1(x) &= (e^{2x} - 1x\, e^{1x}) - (e^{1x}) \\ g_2(x) &= (e^{3x} - 2x\, e^{2x} + (1x)^2 e^{1x}/2!) - (e^{2x} - 1x\, e^{1x}) \\ g_3(x) &= (e^{4x} - 3x\, e^{3x} + (2x)^2 e^{2x}/2! - (1x)^3 e^{1x}/3!) - (e^{3x} - 2x\, e^{2x} + (1x)^2 e^{1x}/2!) \\ &\ldots \end{aligned}$$

This can better be memorized by introduction of a substitution $\mu = -x\, e^{-x}$

$$\begin{aligned} y_0 &=_{\mu=-xe^{-x}} e^{1x}[1] \\ y_1 &=_{\mu=-xe^{-x}} e^{2x}[1 + (1\mu)/1!] - e^{1x}[1] \\ y_2 &=_{\mu=-xe^{-x}} e^{3x}[1 + (2\mu)/1! + (1\mu)^2/2!] - e^{2x}[1 - (1\mu)] \\ y_3 &=_{\mu=-xe^{-x}} e^{4x}[1 + (3\mu)/1! + (2\mu)^2/2! + (1\mu)^3/3!] - e^{3x}[1 + (2\mu)/1! + (1\mu)^2/2!] \\ &\ldots \end{aligned}$$

The general expression for the entry in column c is the finite sum:

(2.2.2)
$$y_c(x) = e^{(c+1)x} + \sum_{\substack{k=1 \\ t=c+1-k}}^{c} (-1)^k \frac{k\cdot(tx)^{k-1} + (tx)^k}{k!} e^{tx}$$

or, introducing an intermediate function:

$$h_c(x) \underset{\mu=-xe^{-x}}{=} e^{(c+1)x} \sum_{k=0}^{c} \frac{((c+1-k)\mu)^k}{k!} \qquad \text{where } h_0(x) = e^x$$

then

$$y_c(x) = h_c(x) - h_{c-1}(x) \qquad \text{where } y_0(x) = h_0(x)$$

The partial sums $s_c = y_0 + y_1 + \ldots y_c$ are then the simpler expressions, which occur due to telescoping/cancelling of consecutive terms - to be just the function $h_c(x)$:

$$s_c(x) = e^{(c+1)x} + \sum_{\substack{k=1 \\ t=c+1-k}}^{c} (-1)^k \frac{(tx)^k e^{tx}}{k!}$$

or

(2.2.3)
$$s_c(x) = \sum_{\substack{k=0 \\ t=c+1-k}}^{c} \frac{(-tx)^k e^{tx}}{k!}$$

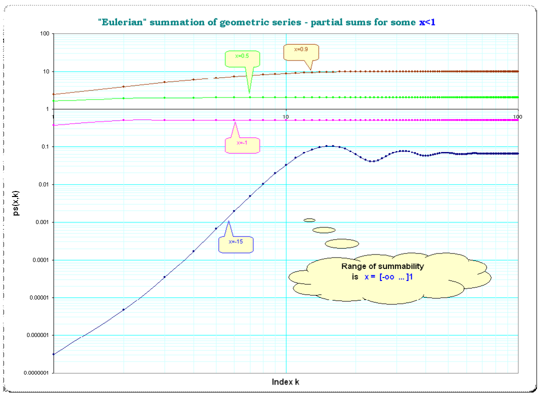
and for instance, the partial sum s_4 (up to column 4) is

$$s_4(x) = e^{5x} - (4x)e^{4x} + \frac{(3x)^2 e^{3x}}{2!} - \frac{(2x)^3 e^{2x}}{3!} + \frac{(1x)^4 e^x}{4!}$$

It is not obvious for which x the $\lim_{c\to\infty} s_c(x)$ is finite (actually this is in the range *-oo<x<1*), but each partial sum in the Eulerian transformation alone is for all x finite and thus well defined.

[4] see appendix for a more explicte description

In the earlier treatize [Eulerian2007] I've investigated the behaviour for the range of summability; here is a picture, how the partial sums behave for some selected *x*:



The picture shows the partial sums

$$s_{x,n} = y_{x,0} + y_{x,1} + y_{x,2} + \ldots + y_{x,n}$$

of the Eulerian transformation of the geometric series for some $x<1$*. We see, that for* $0<x<1$ *the sequence of partial sums increases monotonically but is bounded by its value* $1/(1-x)$ *(see brown and green curve for* $x=0.9$ *and* $x=0.1$*). Then* $x=-1$ *gives also a seemingly monotonically increasing curve and even for the divergent case* $t = -15$ *the curve of the partial sums behaves nicely and approaches its expected value with some diminuishing oscillation first time at the partial sum with* 12 *terms.*

2.3. Additional numerically heuristics

The initial eye-opener for this all was the sheer empirically/numerically observation of the remarkable values in the $Y(x)$ vector for some small x.

The table $V(x)\cdot E = Y(x)$ for a couple of x, where the convergent case $x=1/2$ is also shown for the larger picture:

x	$y(x)_0$	$y(x)_1$	$y(x)_2$	$y(x)_3$	$y(x)_4$	$y(x)_5$	$y(x)_6$
1/2	1.64872	0.245200	0.0755762	0.0218176	0.006213	0.00177	0.00050
1	**2.71828**	**1.95249**	**1.99579**	**2.00004**	**2.00006**	**2.00001**	**2.00000**
3/2	4.48169	8.88131	21.4394	51.4132	123.238	295.400	708.074
2	7.38906	32.4310	159.994	787.504	3875.75	19074.7	93877.1
5/2	12.1825	105.774	986.090	9185.54	85562.1	797000.	7.42395E6
3	20.0855	323.087	5429.72	91225.5	1.53268E6	2.57506E7	4.32636E8

For the value $x=1$ we get terms in $\mathbf{Y(1)}$, which seem to converge to 2 plus a (diminuishing) error - thus $Y(1)\cdot{}'U$ would again include the divergent series $1+1+1+\ldots$. And this surprising observation even extends to a relatively reliable pattern, when we look generally at the ***quotients*** between consecutive $y_k(x)$ in each row:

x	$y(x)_0$	$q(x)_1$	$q(x)_2$	$q(x)_3$	$q(x)_4$	$q(x)_5$	$q(x)_6$	$q(x)_7$
1/2	1.64872	0.148721	0.308223	0.288684	0.284792	0.284536	0.284639	0.284669
1	**2.71828**	**0.718282**	**1.02218**	**1.00213**	**1.00001**	**0.999974**	**0.999997**	**1.00000**
3/2	4.48169	1.98169	2.41399	2.39807	2.39701	2.39699	2.39700	2.39700
2	7.38906	4.38906	4.93338	4.92208	4.92156	4.92155	4.92155	4.92155
5/2	12.1825	8.68249	9.32257	9.31511	9.31487	9.31487	9.31487	9.31487
3	20.0855	16.0855	16.8058	16.8011	16.8010	16.8010	16.8010	16.8010

2.3.1. We get geometric series with another quotient q_x

This means: we have got terms approximately of another geometric series, or more precisely: we have to do with geometric series plus series of quickly vanishing residual/error terms.

This is a much interesting behave, completely unexpected in view of the complicated analytical expression of polynomials in e^x for the partial sums:

- for $0 < x < 1$ as well as for the non-summable cases $1 \le x < +oo$ the entries of the Eulerian-transform vector $Y(x)$ seem to ***form themselves geometric series*** with some other quotient, say "q_x" (which of course depends on x), scaled by some cofactor "a_x" (also depending on x) , plus ...
- ... we find a ***residual vector*** $R(x)$ whose entries diminuish rapidly and seem to ***have a finite sum*** *even in the non-summable cases;* so we may call it indeed a

"residual" (or: "error-term") and we'll look specifically at their values or their sum ρ_x ("*rho(x)*").

We'll write thus the following much plausible composition (based on numerically much accurate computations):

$$V(x) \cdot E = Y(x)$$

2.3.1 $$Y(x) = a_x V(q_x) + R(x) \qquad \text{// a,q depending on x}$$

For $1 <= x < oo$ (which define the divergent cases of the geometric series and which are the interesting cases here) we have with the observation, that the limit $\lim_{c\to oo} y_c(x) / y_{c-1}(x) = q_x$ approximates a constant dependend on x, the following expressions (suggested by numerical results):

2.3.2 $$q_x = \frac{-x}{-W(-x \cdot e^{-x})} \qquad \text{and its inverse relation} \qquad x_q = \frac{q_x \log(q_x)}{q_x - 1}$$

It should be mentioned as interesting feature, that for x in this range, the Lambert-W-function does not simply give back the value of $-x$ here, at least not when we ask for the principal branch $W_0()$ (we would get $-x$ if we used $W_{-1}()$ instead, according to Mathematica at www.Wolframalpha.com).

A meaningful scaling factor a_x can be found by the formula:

2.3.3 $$a_x = \frac{(q_x - 1)^2}{(q_x - 1) - \log(q_x)}$$ *for $x=q=1$ it is possible to do this using the limit*

Using this formula it seems, that the sequence of residual terms converges to zero and gives a (rapidly) converging series.

For $0<x<1$ the equation eq 2.3.2 using the principal branch of the Lambert-W-function (denoted as $W_0()$) gives only the constant value $q_x = 1$ - but we can use the second real branch, in Mathematica denoted as $W_{-1}()$. If we do not have access to $W_{-1}()$ we must do the approximation

2.3.4 $$q_x = \lim_{c\to oo} y_c(x) / y_{c-1}(x)$$ *(column index c ~ 20 suffices for, say, 12 digits precision)*

numerically to any desired precision.

The residual/error vector *R(x)*

Defining the residual-vector $R(x)$ using (2.3.2) or (2.3.4) and (2.3.3)

2.3.5 $$R(x) = Y(x) - a_x \cdot V(q_x)$$

we find first empirically the impressive diminuishing of residuals $r_{x,c}$:

x	$r_{x,0}$	$r_{x,1}$	$r_{x,2}$	$r_{x,3}$	$r_{x,4}$	$r_{x,5}$	$r_{x,6}$	$r_{x,7}$
1/2	0.703054	-0.0240013	-0.0010568	2.6727E-06	3.4646E-06	1.679E-07	-4.1005E-09	-8.3133E-10
1	0.718282	-0.0475076	-0.0042086	3.8851E-05	5.7579E-05	5.0728E-06	-3.5986E-07	-1.111E-07
3/2	0.748569	-0.0669708	-0.0096113	-4.8693E-05	0.00026763	4.1908E-05	-2.5978E-06	-1.7979E-06
2	0.782935	-0.0813966	-0.0171461	-0.000709	0.0007086	0.0001902	-1.1486E-06	-1.1477E-05
5/2	0.817358	-0.0903157	-0.0261955	-0.002534	0.0012894	0.0005815	4.7862E-05	-3.9362E-05
3	0.84977	-0.0937834	-0.0357236	-0.005993	0.0016615	0.0013203	0.00026287	-7.499E-05

and get by $\rho_x = \mathbf{R(x)} \cdot {}'\mathbf{U}$ well converging sums-of-residuals for all $x>0$ (this includes also the non-summable cases of the geometric series where $x>1, q>x>1$) .

Indeed we find empirically that that sum fits the expected geometric-series equation:

2.3.6 ρ_x $= R(x)\cdot{}'U$

$$\underset{empirically}{=}_{expected} \frac{1}{1-x} - \frac{a_x}{1-q_x} = \frac{1}{1-x} - \frac{1}{1-\log(q_x)/(1-q_x)}$$

such that finally, according to the hypothesis (based on the empirical observation), we can extend the range of summability due to the possible analytic continuation of the divergent series with quotient q_x to the rational expression $1/(1-q_x)$ (except when $x=q=1$)

2.3.7 $s(x) = \lim_{c\to oo} s_c(x) = 1/(1-x) = a_x/(1-q_x) + \rho_x$

to apparently arbitrary precision.

Here follows a table for the difficult cases in the near of $x=1$ where the geometric series as well as the series $Y(x) \cdot {}'U$ change from convergence to divergence and we see, that *(2.3.7)* holds empirically.

We define different values of $x=1 \pm eps$, where *eps* aproximates zero; the convergent cases are marked green and the divergent cases are marked orange. For the exact value of q_x we might select the appropriate branch of the *Lambert-W* in formula *(2.3.2)*) :

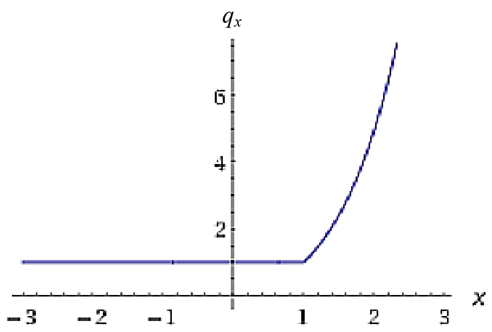
$x=1\pm 2^k$	q_x by ratio of columns	q_x by LambertW_0	q_x by LambertW_{-1}	a_x	ρ_x	$a_x/(1-q_x)+\rho_x = 1/(1-x)$
$1-2^{-1}=$	0.284668137041	1	0.284668137041	0.945666776835	0.678002720411	2
$1-2^{-2}=$	0.576834004142	1	0.576834004142	1.40961009463	0.668895638054	4
$1-2^{-3}=$	0.769993632100	1	0.769993632100	1.68659712899	0.667172024896	8
$1-2^{-4}=$	0.880101616339	1	0.880101616339	1.83842740033	0.666787456187	16
$1-2^{-5}=$	0.938788632410	1	0.938788632410	1.91795437558	0.666696218458	32
1=	**1.**	**1.**	**1.**	**2.**	**0.666666666667**	**->inf**
$1+2^{-5}=$	1.06381576032	1.06381576032	1	2.08464997945	0.666695012174	-32
$1+2^{-4}=$	1.13031866790	1.13031866790	1	2.17199227979	0.666777790260	-16
$1+2^{-3}=$	1.27173094812	1.27173094812	1	2.35511772204	0.667094191416	-8
$1+2^{-2}=$	1.59076137589	1.59076137589	1	2.75782540401	0.668256112480	-4
$1+2^{-1}=$	2.39699882630	2.39699882630	1	3.73312032168	0.672242990758	-2
$1+2^{0}=$	4.92155363457	4.92155363457	1	6.60612090601	0.684567271446	-1

Summary so far:

- ***divergent cases***: If $x\geq 1$ the resulting q_x are also positive and even $q_x \geq x$; q_x (and then a_x) can be evaluated by equation (2.3.2) ; if $x=1$, then $q=1$ and -interestingly- a_x and ρ_x can still be described by the exact formulae (2.3.3) and (2.3.4) to $a = 2$, $\rho = 2/3$.
- ***convergent cases:*** If $0<x<1$ then formula (2.3.2) is no more of use because the LambertW-expression evaluates then to -1 and the result for q_x is then useless. In this cases we must approximate q_x by the evaluation of the quotients of two consecutive entries in ***Y(x)***, which is well approximated by column-numbers smaller than, say , $c=20$.
- ***alternating geometric series:*** If x is negative, then the entries in ***Y(x)*** are no more approximate terms of a geometric progression but instead terms of a rapidly diminishing series with alternating or even chaotic sign.

Something more about the quotient q_x, the *LambertW*-function and another asymptotic series-expression for q_x

The inconsistency of the *LambertW*-expression, as for instance defined in the software Mathematica, is shown by the graph for $q_x = -x/W(-x \cdot e^{-x})$ (taken by the public version of WolframAlpha), where we see the knee and the slope changing to constant zero when $x<1$:



$q_x = -x/W(-x \cdot e^{-x})$
(graph by Mathematica at WolframAlpha
plot -x/Productlog(-x exp(-x)))

A closer look at the characteristic of the quotient q_x when taken as limit of quotient of consecutive terms $q(x) = \lim_{n\to oo} \ y_n/y_{n-1}$ in comparision to the description using the *LambertW* might give some more insight.

If we understand $y_n(x)$ at $x=1$ as polynomial in e as given in (2.2.2) then the polynomial long division of the two concecutive entries in $Y(1)$ gives a series in e, with its leading coefficients:

2.3.8 $$\lim_{n\to\infty} q_n(1) = 1e - 1 - 1/2!\, e^{-1} - 2^2/3!\, e^{-2} - 3^3/4!\, e^{-3} - 4^4/5!\, e^{-4} - 5^5/6!\, e^{-5} - \dots$$

This indicates a fairly obvious pattern in the leading coefficients; and when n increases then always the first n coefficients of the quotient's series in e follow exactly that pattern and the remaining coefficients deviate from that pattern by small residuals $r(1)$

$$q_n(1) = \frac{y_n(1)}{y_{n-1}(1)} = e - \sum_{k=0}^{n-1} \frac{(k/e)^k}{k!} \frac{1}{k+1} + \sum_{k=n}^{oo} r_{n,k}(1)$$

by polynomial long division

where the $r_{n,k}(1)$-terms are slightly different from that scheme and are also depending on the selection of n.

The obvious assumption for the limit is then the series without the residuals-expression

2.3.9 $$q(1) = e - \sum_{k=0}^{oo} \frac{(k/e)^k}{k!} \frac{1}{k+1}$$

To check convergence of this series we introduce Stirling's approximation-formula for the Gamma-function

2.3.9a $$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

and rearrange to get:

2.3.9b $$\left(\frac{n}{e}\right)^n \frac{1}{n!} \sim \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{n}}$$

This shows that the inner terms in the sum decrease sufficiently fast, such that the series converges. We get for the order of decrease of the terms approximately

2.3.10 $$\sum_{k=0}^{oo} \frac{(k/e)^k}{k!} \frac{1}{k+1} \sim \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{oo} \frac{1}{\sqrt{k}} \frac{1}{k+1}$$

and this converges because the resulting terms are all smaller than $1/k^{1.5}$ and we know that $\zeta(s)$ with $s = 1.5\ (> 1)$ converges.

We find even more: the formula can apparently be generalized for the indeterminate argument x:

2.3.11 $$q(x) = e^x - x \sum_{k=0}^{oo} \frac{\left(\frac{xk}{e^x}\right)^k}{k!} \frac{1}{k+1}$$

which, for $x>1$ is (by the above argument) even more rapidly convergent than the series for $x=1$.

2.4. Conclusion

The geometric series at positive arguments $x \ge 1$ cannot be summed by the Eulerian summation, because the resulting Eulerian transform of it involves again a geometric series with an even higher (positive) quotient $q_x \ge x$, making $Y(x) \cdot {}'U$ divergent.

However, we might say, that that resulting geometric series, in combination with its analytical continuation, and the residual-series together reproduce the correct/expected value of the geometric series, and only the case for $x=1$ remains with an unremovable singularity and an additional residue with the (so far unexplained) value of $2/3$.

3. Dirichlet-(Zeta) series: Eulerian transformation (for non-summable cases)

3.1. Intro

In this chapter we look at the Dirichlet-(zeta-)-type series[5], so we look at the Eulerian transform (and possibly -sum) of the series $\zeta(m)= 1/1^m + 1/2^m +1/3^m + \ldots$ for integer $m<=1$ (thus for the divergent cases again) but which we shall call now $\boldsymbol{\tau_m}$ ("*tau(m)*") to avoid confusion between the $\zeta(m)$-values and our experimental evaluations.

The basic formula is

3.1.1 $Z(m) \cdot E = Y(m)$

where we also note, that for the convergent cases $m>1$

$$Z(m) \cdot {}'U = \zeta(m)$$

and denote the Eulerian transformation by

3.1.2 $Z(m) \cdot E = Y(m)$ and

$$Y(m) \cdot {}'U = \tau_m$$

and expect -at least for the convergent cases- that

$$\zeta(m) = \tau_m$$

For the convergent cases $m>1$ this seems to hold by all done numerical checks, but for the divergent cases we get some systematic error and this current chapter is devoted to study that error and properties of τ_m.

A quick numerical check for some $m<=1$ (which are the divergent cases) give the following entries in the resulting $Y(m)$-vectors:

Y	[0]	[1]	[2]	[3]	[4]	[5]	[6]	...
Y(1)	1.7182818	0.47624622	0.33193046	0.25000971	0.20001152	0.16666751	0.14285709	...
Y(0)	2.7182818	1.9524924	1.9957914	2.0000389	2.0000576	2.0000051	1.9999996	...
Y(-1)	5.4365637	8.5757592	12.653940	16.666760	20.666950	24.666698	28.666664	...
Y(-2)	13.591409	40.505390	84.627779	144.66648	220.66803	312.66686	420.66665	...

Surely we get more than a vague impression of the kind of the resulting ***Y***-vectors - and by that of the deviations from the corresponding (and expected) $\tau(m)$-values.

[5] In recent discussions it appeared to me, that I should possibly replace the complete reference to Dirichlet-/Zeta series by that to Polylogs. But I've not yet a decisive answer for this.

3.2. Analytic description of the entries in the Eulerian transform *Y(x)*

Of course, there are as in chap. *2* not too difficult analytical expressions for the entries in ***Y(x)***.

a) For *Y(0)* we have the same case as in chap. *2* with the transforms of the geometric series with *x=1*. We get by the explicite decomposition[6] of the columns in the Eulerian matrix:

3.2.1: Composition of entries of Y(m) as polynomials in e

Y	[0]	[1]	[2]	[3]	[4]
Y(1)	-1 $+1\cdot e$	-1/2 $-1^0\cdot e+2^{-1}\cdot e^2$	-1/3 $+1^1\cdot e/2! - 2^0\cdot e^2+3^{-1}\cdot e^3$	-1/4 $-1^2\cdot e/3!+ 2^1\cdot e^2/2!- 3^0\cdot e^3+ 4^{-1}\cdot e^4$	...
Y(0)	$1\cdot e$	$-2\cdot e+ 1\cdot e^2$	$+3\cdot e/2!- 3\cdot e^2+ 1\cdot e^3$	$-4\cdot e/3!+ 8\cdot e^2/2!- 4\cdot e^3+ 1\cdot e^4$	...
Y(-1)	$2\cdot e$	$-5\cdot e+ 3\cdot e^2$	$+10\cdot e/2!- 11\cdot e^2+ 4\cdot e^3$	$-17\cdot e/3!+ 36\cdot e^2/2!- 19\cdot e^3+ 5\cdot e^4$	...
Y(-2)	$5\cdot e$	$-15\cdot e+ 11\cdot e^2$	$+37\cdot e/2!- 47\cdot e^2+ 19\cdot e^3$	$-77\cdot e/3!+ 180\cdot e^2/2!- 103\cdot e^3+ 29\cdot e^4$	...
Y(-3)	$15\cdot e$	$-52\cdot e+ 47\cdot e^2$	$+151\cdot e/2!- 227\cdot e^2+103\cdot e^3$	$-372\cdot e/3!+ 988\cdot e^2/2!- 622\cdot e^3+ 189\cdot e^4$	...
Y(-4)	$52\cdot e$	$-203\cdot e+227\cdot e^2$	$+674\cdot e/2!- 1215\cdot e^2+622\cdot e^3$	$-1915\cdot e/3!+ 5892\cdot e^2/2!- 4117\cdot e^3+1357\cdot e^4$	...
...	...	...	...	...	...

3.2.1a: The partial sums (horizontally, along a row) in that table are:

S	[0]	[1]	[2]	[3]	[4]
S(1)	-1 $+1\cdot e$	-3/2 $+1/2\cdot e^2- 0\cdot e$	-11/6 $+1/3\cdot e^3- 1/2\cdot e^2+ 1\cdot e/2!$	-50/24 $+1/4\cdot e^4- 2/3\cdot e^3 + 1\cdot e^2/2!+ 2\cdot e/3!$	
S(0)	$1\cdot e$	$1\cdot e^2- 1\cdot e$	$1\cdot e^3- 2\cdot e^2+ 1\cdot e/2!$	$1\cdot e^4- 3\cdot e^3 + 4\cdot e^2/2!- 1\cdot e/3!$	...
S(-1)	$2\cdot e$	$3\cdot e^2- 3\cdot e$	$4\cdot e^3- 8\cdot e^2+ 4\cdot e/2!$	$5\cdot e^4- 15\cdot e^3 + 20\cdot e^2/2!- 5\cdot e/3!$	...
S(-2)	$5\cdot e$	$11\cdot e^2-10\cdot e$	$19\cdot e^3- 36\cdot e^2+17\cdot e/2!$	$29\cdot e^4- 84\cdot e^3 + 108\cdot e^2/2!- 77\cdot e/3!$	...
S(-3)	$15\cdot e$	$47\cdot e^2-37\cdot e$	$103\cdot e^3-180\cdot e^2+77\cdot e/2!$	$189\cdot e^4- 622\cdot e^3 + 988\cdot e^2/2!-372\cdot e/3!$	...
...	...	...	...	...	...

b) Because in table 3.2.1 we can recognize the sequences of coefficients along the columns (beginning at row *0* downwards) as Bell-numbers $B_{r,c}$ (where the rowindex *r* is taken from the *Y(-m)* parameter *m*) and Stirling numbers 1st kind (marked red) we can thus prognose the coefficients of the following rows and columns:

3.2.2: Composition of Y(-m) expressed by Bell-numbers (using r=m, m >=0):

Y	[0]	[1]	[2]	[3]	[4]
Y(-m)	$1B_{r,0}\cdot e$	$1B_{r,1}\cdot e^2$ $-(1B_{r+1,0})/1!\cdot e^1$	$1B_{r,2}\cdot e^3$ $-(1B_{r+1,1})/1!\cdot e^2$ $+(-1B_{r+1,0}+1B_{r+2,0})/2!\cdot e^1$	$1B_{r,3}\cdot e^4$ $-(1B_{r+1,2})/1!\cdot e^3$ $+(-1B_{r+1,1}+1B_{r+2,1})/2!\cdot e^2$ $-(2B_{r+1,0}-3B_{r+2,0}+1B_{r+3,0})/3!\cdot e^1$	...

and where the generalization for the Bell-numbers to the row-index *-1* is taken as

3.2.2a $B_{-1,c} = 1/(1+c)$

where *c* is the column-number.

3.2.2b The "generalized" Bell-matrix B has its top-left segment as:

1	1/2	1/3	1/4	1/5	1/6
1	1	1	1	1	1
2	3	4	5	6	7
5	11	19	29	41	55
15	47	103	189	311	475
52	227	622	1357	2576	4447

(Infinite size is always assumed!)

[6] into their defining sequences of geometric-series type and their derivatives as shown in [Eulerian2007]

Because the Bell-numbers result from the row-sums of the matrix of Stirling numbers *2nd* kind this can even be coded by that two matrices of Stirling-coefficients alone.

c) Another decomposition involves a -perhaps- simpler matrix-expression.

We use the "Pascalmatrix" ***P*** :

3.2.3 *The "Pascal" matrix **P** has its top-left segment as:*	$$P = \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix}$$ *(Infinite size is always assumed!)*

Combined with the above described (generalized) Bell-matrix ***B*** we can find expressions for each complete column by polynomials in *e* with (matrix-) cofactors of powers of ***P*** and appropriate columns of ***B***:

3.2.4: Composition of Y(m) expressed by Pascal- and Bell-matrix

Y	[0]	[1]	[2]	[3]	[4]	[5]
Y= e·	$B_{,0}$	$e\cdot B_{,1} - P\cdot B_{,0}$	$e^2\cdot B_{,2} - P\cdot(e\cdot B_{,1} - P\cdot(B_{,0}))$	$e^3\cdot B_{,3} - P\cdot(e^2\cdot B_{,2} - P\cdot(e\cdot B_{,1} - P\cdot(B_{,0})))$		

More explicitely this is:

3.2.4a

$$Y_{,0} = + \quad B_{,0}\cdot e^1$$
$$Y_{,1} = -(1\cdot P)^1/1!\cdot B_{,0}\cdot e^1 \quad + \quad B_{,1}\cdot e^2$$
$$Y_{,2} = +(1\cdot P)^2/2!\cdot B_{,0}\cdot e^1 \quad -(2\cdot P)^1/1!\cdot B_{,1}\cdot e^2 \quad + \quad B_{,2}\cdot e^3$$
$$Y_{,3} = -(1\cdot P)^3/3!\cdot B_{,0}\cdot e^1 \quad +(2\cdot P)^2/2!\cdot B_{,1}\cdot e^2 \quad -(3\cdot P)/1!\cdot B_{,2}\cdot e^3 \quad +B_{,3}\cdot e^4$$
$$\dots = \dots$$

3.2.4b

$$Y_{,c} = e^{c+1}\sum_{k=0}^{c}\frac{(-\frac{c+1-k}{e})^k}{k!}\cdot P^k B_{,c-k}$$

For instance, the first terms of the consecutive partial sums $S_{,c}$ are then:

3.2.5

$$(\text{in } S_{,0}:) \quad (I \quad)\cdot B_{,0}\cdot e^1$$
$$(\text{in } S_{,1}:) \quad (I - (1\cdot P)^1/1! \quad)\cdot B_{,0}\cdot e^1$$
$$(\text{in } S_{,2}:) \quad (I - (1\cdot P)^1/1! + (1\cdot P)^2/2! \quad)\cdot B_{,0}\cdot e^1$$
$$(\text{in } S_{,3}:) \quad (I - (1\cdot P)^1/1! + (1\cdot P)^2/2! - (1\cdot P)^3/3! \quad)\cdot B_{,0}\cdot e^1$$

...

and in general:

3.2.5a

$$s(-m)_c = \sum_{k=0}^{c}\left(\frac{(-1)^k}{k!}\cdot P^k\right)\cdot eB_{,0} + \sum_{k=0}^{c-1}\left(\frac{(-2)^k}{k!}\cdot P^k\right)\cdot e^2 B_{,1} + \dots + \sum_{k=0}^{0}\left(\frac{(-(c+1))^k}{k!}\cdot P^k\right)\cdot e^{c+1}B_{,c}$$

```
*****(The set of Pari/GP-routines shall be displayed in a next edition of this article)*****
```

3.3. Additional descriptions by numerial observations

3.3.1. Basic observations

Here is again the beginning of the list of results for the ***Y***-vectors for $m=1,0,-1,-2,..-9$

Y	[0]	[1]	[2]	[3]	[4]	[5]	[6]	...
Y(1)	1.7182818	0.47624622	0.33193046	0.25000971	0.20001152	0.16666751	0.14285709	...
Y(0)	2.7182818	1.9524924	1.9957914	2.0000389	2.0000576	2.0000051	1.9999996	...
Y(-1)	5.4365637	8.5757592	12.653940	16.666760	20.666950	24.666698	28.666664	...
Y(-2)	13.591409	40.505390	84.627779	144.66648	220.66803	312.66686	420.66665	...
Y(-3)	40.774227	205.93498	596.72485	1307.5066	2434.1842	4072.8457	6319.5110	...
Y(-4)	141.35066	1125.5045	4431.5618	12298.108	27732.399	54510.600	97176.814	...
Y(-5)	551.81121	6593.7700	34613.011	120291.57	326186.63	749418.11	1529346.5	...
Y(-6)	2383.9332	41260.335	283855.47	1222538.6	3958934.9	10580273.	24627867.	...
Y(-7)	11253.687	274721.07	2439791.8	12897622.	49554696.	1.5333751E8	4.0571837E8	...
Y(-8)	57483.506	1939081.8	21938662.	1.4110443E8	6.3933029E8	2.2804055E9	6.8357591E9	...
Y(-9)	315252.74	14458661.	2.0600574E8	1.5992142E9	8.4962001E9	3.4786334E10	1.1775796E11	...

We observe in the first rows (when read column-by-column) that the entries approximate "obvious" values which depend in a simple way on the column-index c.

For instance in ***Y(1)*** they converge to $y_c \sim 1/(c+1)$, in ***Y(0)*** to $y_c \sim 2$, in ***Y(-1)*** to $y_c \sim 4\cdot(c+1)+2/3$ and so on. (The result for $m=0$ is the same as in the previous chapter for the transformation of the geometric series with $x=1$). Very obviously we can rewrite the first few rows (per column c) as

3.3.1
$$Y(1)_c = R(1)_c + 1\cdot(c+1)^{-1}$$
$$Y(0)_c = R(0)_c + 2\cdot(c+1)^0$$
$$Y(-1)_c = R(-1)_c + 2/3\cdot(c+1)^0 + 4\cdot(c+1)^1$$
$$\dots$$

That rule is suggestive enough to invest some effort to extend that pattern in the obvious way and we can actually find a table of coefficients ***C*** for lower indexes m as well and by which the residuals always diminish rapidly with increasing column-indexes c.

In general, we can indeed express the resulting vectors $Y(m)$ by this ansatz as compositions of vectors $Z(0) \dots Z(m)$ cofactored by coefficients of the newly created table ***C***:

Let as in 3.1.1 defined

$$Y(m) = Z(m) \cdot E$$

be the Eulerian transform of the "zeta"-vector $Z(m)$, then let $W(m)$ denote the part of the systematic composition in $Y(m)$:

3.3.2
$$W(m) = c_{m,0}\cdot Z(0) + c_{m,1}\cdot Z(-1) + c_{m,2}\cdot Z(-2) + \dots + c_{m,m}\cdot Z(m)$$

with coefficients $c_{m,c}$ according to the table ***C*** below and let us then express $Y(m)$ by $W(m)$ and a residual $R(m)$ such that

3.3.2a
$$Y(m) = W(m) + R(m)$$

Then the entries of the ***R(m)*** form rapidly diminishing sequences whose sums converges quickly to some (rational) residual value ρ_m . The empirically observed good convergence supports the hypothese about the general type of composition of $Y(m)$.

Example decomposition of the vector *Y(-3) = Z(-3) · E* into component vectors:

Z(-3) · E = Y(-3)
= R(-3) + W(-3)
= R(-3) + 38/45·Z(0) + 20/3·Z(-1) + 16·Z(-2) + 16·Z(-3) *(see this in the table below at C(-3)*

Remember we have already analytical expressions for the compositions in table *3.2.1*, such that for instance the residuals in *R(1)* are exactly:

$[e - 2,\ e^2/2 - e\ -2/2,\ e^3/3 - e^2 + e/2!\ -2/3, \ldots]$

where the terms without *e* as cofactor are *2·[1 , 1/2 , 1/3 , 1/4 , ...]*.

3.3.2. The coefficients of the matrix *C* - heuristically and "most likely" analytically

The coefficients of the heuristic matrix ***C*** can be guessed by approximations of the numerical solutions with some reasonable effort for *m* from *m=1* down to, say, *m=-7* . For even higher negative *m* this becomes too much heavy work and seems unreasonable.

3.3.2.1 Table C - compiled due to guesses based on "obvious" numerical approximations:

C	-1	0	1	2	3	4	5	6	7	...
C(1)	1	0	0	0	0	0	0	0	0	...
C(0)	0	2	0	0	0	0	0	0	0	...
C(-1)	0	2/3	4	0	0	0	0	0	0	...
C(-2)	0	2/3	4	8	0	0	0	0	0	...
C(-3)	0	38/45	20/3	16	16	0	0	0	0	...
C(-4)	0	34/27	116/9	40	160/3	32	0	0	0	...
C(-5)	0	130/63	28	976/9	560/3	160	64	0	0	...
C(-6)	0	458/135	596/9	952/3	672	2240/3	448	128	0	...
C(-7)	0	1846/405	22244/135	8912/9	22736/9	10304/3	2688	3584/3	256	...
...	...	...	...	...	...	...	...	...	...	...
	· Z(1)	· Z(0)	· Z(-1)	· Z(-2)	· Z(-3)	· Z(-4)	· Z(-5)	· Z(-6)	· Z(-7)	...

Note: the indexes of the matrix **C** *are here adapted to match the values* m *of the transformed* Z(m)*-vector.*

I did not succeed to find a direct or recursive pattern in the rows or columns of ***C*** to extend the found initial guesses to arbitrarily wide ranges of *m* and *c*.

But by accident[7] I found another way which now looks like the most likely analytical solution. This resulted initially from an analysis for the ***convergent*** cases *m>1* where similarly we can find a composition-table for the resulting *Y(m)* by a -now infinite- composition of the vectors *Z(m)* for *Z(m) .. Z(oo)*. Let's call this matrix of coefficients ***D***.

Its top left segment begins as follows:

*3.3.2.2 Table **D** - compiled due to guesses based on "obvious" numerical approximations:*

*Coefficients-matrix **D** for compositions of*

$$Y(m) = R(m) + \sum_{k=m}^{oo} D_{k-1,m-1} \cdot Z(k)$$

for m>1

$$D = \begin{bmatrix} 1 & . & . & . & . & . & . \\ 0 & 1/2 & . & . & . & . & . \\ 0 & 1/12 & 1/4 & . & . & . & . \\ 0 & 0 & 1/8 & 1/8 & . & . & . \\ 0 & -1/120 & 1/48 & 1/8 & 1/16 & . & . \\ 0 & 0 & -1/48 & 5/96 & 5/48 & 1/32 & . \\ 0 & 1/252 & -1/96 & -13/576 & 5/64 & 5/64 & 1/64 \end{bmatrix}$$

(Infinite size is always assumed!)

[7] The accident was a mail, which I posted in the seqfan-mailing list asking for help for an analytic description for the third column in the empirically guessed matrix ***D***. The longtime seqfan-correspondent Paul D. Hanna had the idea that if the second column if ***D*** has the exponential generating function *h(x)*, (which could then be found to be *h(x)=log((exp(x)-1)/x))* the third column has *h(x)²* as its generating function - and this gave immediately the generalization for all other columns in ***D***.

This means for example

$$Z(2)\cdot E = Y(2) = 1/2\cdot Z(2) \quad + 1/12\cdot Z(3) \qquad\qquad - 1/120\cdot Z(5)... \; + R(2)$$
$$Z(3)\cdot E = Y(3) = \qquad\qquad + 1/4 \cdot Z(3) \quad +1/8\cdot Z(4)+ 1/48 \cdot Z(5)... \; + R(3)$$

Interestingly, by the first few guessed coefficients in ***D***, the exponential generating function for the second column in ***D*** seems to be the function

3.3.2.3 $$h(x) = \log\left(\frac{\exp(x)-1}{x}\right)$$

and simply that of the following columns its powers $h(x)^0, h(x), h(x)^2, h(x)^3, \dots$, so that I assume with a very strong likelihood, that the Carlemanmatrix[8] ***H*** of the function $h(x)$ and the matrix ***D*** are related by the formula:

3.3.2.4 $$D = \grave{}G \cdot H \cdot \grave{}g$$

thus

3.3.2.5 ***D*** is a factorially similarity scaling of the Carlemanmatrix ***H*** for the function $h(x)$.

Now our empirically guessed matrix ***C*** seems to be just the inverse of ***D*** and so by the same strong likelihood

3.3.2.6 $$C = D^{-1} = \grave{}G \cdot H^{-1} \cdot \grave{}g$$

and the function $b(x)$, for which $\boldsymbol{H^{-1}}$ is the Carlemanmatrix, is the inverse of $h(x)$, so the function $b(x) = -(-h)^{[-1]}(x)$ seems to be the exponential generating function for the second column of ***C*** and for the other columns in ***C*** the consecutive powers

3.3.2.7 $$[\, b(x)^0 , b(x) , b(x)^2 , b(x)^3 , \dots] .$$

3.3.3. An exponential generating function for the columns of *C*

The inverse $b(x) = -(-h)^{[-1]}(x)$ can, according to the public available tool at WolframAlpha, be expressed as

3.3.3.1 $$b(x) = -\left(LambertW(-e^{-x-e^{-x}}) + e^{-x}\right)$$

but in the version with public access only it cannot give a power series for this.

Fortunately we can get the leading terms of the power series by formal *series inversion* using the software Pari/GP by the following:

```
h(x) = log(-x/(exp(-x)-1))
```

gives $= 1/2\, x - 1/24\, x^2 + 1/2880\, x^4 - 1/181440\, x^6 + O(x^8)$

which is $= -\zeta(0)/1!\, x + \zeta(-1)/2!\, x^2 + \zeta(-3)/4!\, x^4 + \zeta(-5)/6!\, x^6 + O(x^8)$

```
b(x) = -serreverse(-h(x))§      \\ = h[-1](x)
```

gives $= 2\, x + 2/3\, x^2/2! + 2/3\, x^3/3! + 38/45\, x^4/4! + 34/27\, x^5/5! + O(x^6)$

which matches perfectly the guesses from the heuristics in the relevant $Y(m)$ written into the second column of the matrix ***C***.

[8] see [WP:Carlemanmatrix] the page about Carlemanmatrices. Note that I use the Carleman-matrices in the transposed form here.

§ notation for a function-call in the software Pari/GP

3.3.4. The entries in the compositions *W(m)* and in the residuals *R(m)*

If we write the vectorial compositions *W(m)* as

3.3.4.1 $W(m) = Y(m) - R(m)$

by which ***W(m)*** denotes the systematic part in ***Y(m)*** then the individual columns in ***W(m)*** look like

*3.3.4.2 Table **W** as "systematic" part in **Y** (based on compositions by matrix **C**)*

W\k	0	[1]	[2]	[3]	[4]	[5]		k
W(1)	1	1/2	1/3	1/4	1/5	1/6		1/(1+k)
W(0)	2	2	2	2	2	2		2
W(-1)	4+2/3	8+2/3	12+2/3	16+2/3	20+2/3	24+2/3		2/3+4·(1+k)
W(-2)	8+4+2/3	32+8+2/3	72+12+2/3	128+16+2/3	200+20+2/3	288+24+2/3		$2/3+4\cdot(1+k)+8\cdot(1+k)^2$
W(-3)	1778/45	9278/45	26858/45	58838/45	109538/45	183278/45		
W(-4)	3766/27	30394/27	119662/27	332050/27	748774/27	1471786/27		
W(-5)	34598/63	415370/63	2180686/63	7578386/63	20549750/63	47213338/63		
W(-6)	321518/135	5569538/135	38320838/135	165042938/135	534456158/135	1428336818/135		
...	...	...	...	...	...	...		...

Finally, the residuals in the vectors ***R(m)*** (columnwise and at the right margin their sum) look numerically like

*3.3.4.3 Table **R** as "residual" part **Y-W**, and rowsums ρ_m:*

R	[0]	[1]	[2]	[3]	[4]	[5]	...	$\sum(R)=\rho_m$
R(1)	0.71828183	-0.023753779	-0.0014028770	0.0000097126192	0.000011515718	0.00000084545928		log(2)
R(0)	0.71828183	-0.047507558	-0.0042086309	0.000038850477	0.000057578590	0.0000050727557		2/3
R(-1)	0.76989699	-0.090907512	-0.012726920	0.000093225273	0.00028329490	0.000030911246		2/3
R(-2)	0.92474248	-0.16127701	-0.038887949	-0.00018668861	0.0013658960	0.00019163872		98/135
R(-3)	1.2631163	-0.24279621	-0.11959776	-0.0044815623	0.0064000543	0.0012076617		122/135
R(-4)	1.8691736	-0.19918042	-0.36414371	-0.039748735	0.028658228	0.0077033125		82/63
R(-5)	2.6366080	0.59539347	-1.0528713	-0.27565729	0.11821945	0.049355040		5858/2835
R(-6)	2.3183487	4.4978882	-2.5873734	-1.6754118	0.40691276	0.31398533		1318/405
...								

3.4. Conclusion and a speculation about some "magic" coefficients

By the above discussion we find that we cannot correctly Eulerian-sum the zeta-series at arguments $m<=1$.

However, the surprising observation, that, for instance in the first three transformations,

3.4.1a $Z(1) \cdot E = R(1) + 1 \; Z(1)$
3.4.1b $Z(0) \cdot E = R(0) + 2 \; Z(0)$
3.4.1c $Z(-1) \cdot E = R(-1) + 2/3 \, Z(0) + 4 \, Z(-1)$

the same vector $Z(m)$ occurs at the left as well as on the right hand of the transformation but with different multiplicities might introduce some speculation about possibly meaningful insertion of finite values for the occuring infinite expressions/sums.

If -for instance for $m=0$- we assume some meaningful finite replacement value $\tau(0)$ for the infinite expression $Z(0)\cdot {}'U$, being equally valid on both sides of the equation and demand equality in:

3.4.2 $Z(0)\cdot{}'U = R(0)\cdot{}'U + 2\cdot Z(0)\cdot{}'U$

then we see, that besides the obvious solution by an "infinity" in $\tau(0)=Z(0)\cdot{}'U$ there is also one possible solution in a finite value because $\rho(0)$ (for $R(0)\cdot{}'U$) has a well defined value $\rho(0)=2/3$.

Thus we might rearrange in the rhs and lhs the assumed sums

$$\tau(0) = \rho(0) + 2\,\tau(0)$$
$$\tau(0) = \rho(0)/(1-2) = -\rho(0)$$

to arrive at

3.4.2a $\tau(0)= -2/3$

Similarly, if we assume some meaningful finite value $\tau(-1)$ for $Z(-1)\cdot{}'U$ (while again $\rho(-1)$ for $R(-1)\cdot{}'U$ has a well defined value $\rho(-1)=2/3$) then we get, beginning from:

$$\tau(-1) = \rho(-1) + 2/3\,\tau(0) + 4\,\tau(-1)$$
$$-3\,\tau(-1) = (\rho(-1) + 2/3\,\tau(0))$$
$$= 2/3 - 4/9$$

by rearranging:

3.4.2b $\tau(-1) = -2/27$

The table for the first few possible finite $\tau()$-function - assignments comes out to be:

3.4.3c	$\tau(0) = -2/3$	=	$-2/3$		$/(2-1)$
	$\tau(-1) = -2/27$	=	$-2/3$	$/3$	$/(2-1)(4-1)$
	$\tau(-2) = 2/945$	=	$2/5$	$/3^2$	$/(2-1)(4-1)(8-1)$
	$\tau(-3) = 338/42525$	=	$338/5$	$/3^3$	$/(2-1)(4-1)(8-1)(16-1)$
	$\tau(-4) = -58/112995$	=	-406	$/3^4$	$/(2-1)(4-1)(8-1)(16-1)(32-1)$
	$\tau(-5) = -27982/7118685$	=	-587622	$/3^5$	$/(2-1)\dots(64-1)$
	$\tau(-6) = 224594/645766425$	=	$99045954/5$	$/3^5$	$/(2-1)\dots(128-1)$

However, for $\tau(1)$ we get by

3.4.3d $\tau(1) = \rho(1) + \tau(1)$ where $\rho(1) = \log(2)$

that there is ***no*** possible finite insertion and thus we observe a remaining singularity with some "residue"(?) of $\log(2)$.

However, I don't know yet whether these finite values might have any sensical/meaningful interpretation in any other context.

4. Appendix

Expansions in the entries in $Y(x) = [y_0, y_1, y_2, \ldots]$ of the "Eulerian transform" $V(x)\cdot E = Y(x)$ of the geometric series $V(x)\cdot {}'U = f(x) = 1 + x + x^2 + x^3 + \ldots$. The entries in the numerators of the sums are defined by the explicite representation of the entries in the Eulerian triangle (see: [NIST:Eulerian] or [Eulerian2007]):

$$\begin{aligned}
y_0(x) &= \sum_{k=0}^{\inf} \frac{(1x)^k}{k!} \\
&= e^{1x} \\
y_1(x) &= \sum_{k=0}^{\inf} \frac{(2x)^k - \binom{k+1}{1}(1x)^k}{k!} \\
&= e^{2x} - (1x+1)e^{1x} \\
&= \left(\frac{1}{0!}e^{2x} - \frac{(1x)^1}{1!}e^{1x}\right) + \left(0 - \frac{1}{0!}e^{1x}\right) \\
y_2(x) &= \sum_{k=0}^{\inf} \frac{(3x)^k - \binom{k+1}{1}(2x)^k + \binom{k+1}{2}(1x)^k}{k!} \\
&= e^{3x} - (2x+1)e^{2x} + 1x(1x+2)\frac{e^{1x}}{2!} \\
&= \left(\frac{1}{0!}e^{3x} - \frac{(2x)^1}{1!}e^{2x} + \frac{(1x)^2}{2!}e^{1x}\right) + \left(0 - \frac{1}{0!}e^{2x} + \frac{(1x)}{1!}e^{1x}\right) \\
y_3(x) &= \sum_{k=0}^{\inf} \frac{(4x)^k - \binom{k+1}{1}(3x)^k + \binom{k+1}{2}(2x)^k - \binom{k+1}{3}(1x)^k}{k!} \\
&= e^{4x} - (3x+1)e^{3x} + 2x(2x+2)\frac{e^{2x}}{2!} - (1x)^2(1x+3)\frac{e^{1x}}{3!} \\
&= \left(\frac{1}{0!}e^{4x} - \frac{(3x)^1}{1!}e^{3x} + \frac{(2x)^2}{2!}e^{2x} - \frac{(1x)^3}{3!}e^{1x}\right) + \left(0 - \frac{1}{0!}e^{3x} + \frac{(2x)}{1!}e^{2x} - \frac{(1x)^2}{2!}e^{1x}\right)
\end{aligned}$$

...

The partial sums of the above

$$\begin{aligned}
s_0(x) &= e^{1x}(1) \\
s_1(x) &\underset{y=-xe^{-x}}{=} e^{2x}\left(1 + \frac{(1y)^1}{1!}\right) \\
s_2(x) &\underset{y=-xe^{-x}}{=} e^{3x}\left(1 + \frac{(2y)^1}{1!} + \frac{(1y)^2}{2!}\right) \\
s_3(x) &\underset{y=-xe^{-x}}{=} e^{4x}\left(1 + \frac{(3y)^1}{1!} + \frac{(2y)^2}{2!} + \frac{(1y)^3}{3!}\right) \\
s_4(x) &\underset{y=-xe^{-x}}{=} e^{5x}\left(1 + \frac{(4y)^1}{1!} + \frac{(3y)^2}{2!} + \frac{(2y)^3}{3!} + \frac{(1y)^4}{4!}\right)
\end{aligned}$$

...

Computation of the column-values for the ZETA-summation

$$gb[r,c] = \sum_{k=0}^{c} s2_{r,k} \cdot (c+1)^{k-1} \qquad \text{- matrix-indexes beginning at zero}$$

```
\\ Computation of the column-values for the EU(lerian) summation of ZETA-vectors
   n=32 \\ standard size of matrices and vectors
   \\ install set of extended Bell-numbers as constants in a matrix "GenBell"
   GenBell = S2 * matrix(n,n,r,c,c^(r-2)) ;\\ GenBell = S2*ZV~ * dZ(1) ;
GB(r,c)=GenBell[1+r,1+c]  \\ install a functional call into Bell numbers

\\ analytic computation of Z(m)*E = Y(m)
{EU_zeta_Y(m,dim=8,e='e) = local(res,tmp,Pm,Gm,m1);
 m=1-m;m1=m+1;
 res=vector(dim);
 Pm = vector(1+m,c,binomial(m,c-1));       \\ Pm=PPow(1,m+1)[1+m,];
 Gm=VE(GenBell,1+m,dim);

 \\ initialize result "res"                \\
 for(c=1,dim, res[c]=GB(m,c-1)*e^c);       \\ res = Gm[1+m,] *e *dV(e,dim)
 for(p=1,dim,
      tmp= vector(m1,c,p^(m1-c)*Pm[c]) ;
   tmp = tmp * Gm;
      tmp= vector(dim,c,tmp[c]*c^p);       \\ tmp= PPow(p,m1) *Gm *dZ(-p,dim);
   for(k=p+1,dim,res[k] += (-1)^p*tmp[k-p]/p!*e^(k-p));
   );
 return(res);}
\\ ============================================================================

\\ test for m=-3

EU_zeta_Y(-3,4)
\\  [15*e,
     47*e^2 - 52*e,
     103*e^3 - 227*e^2 + 151/2*e,
     189*e^4 - 622*e^3 + 494*e^2 - 62*e
    ]

\\ ==================
```

5. References

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