Accessing Bernoulli-Numbers by Matrix-Operations

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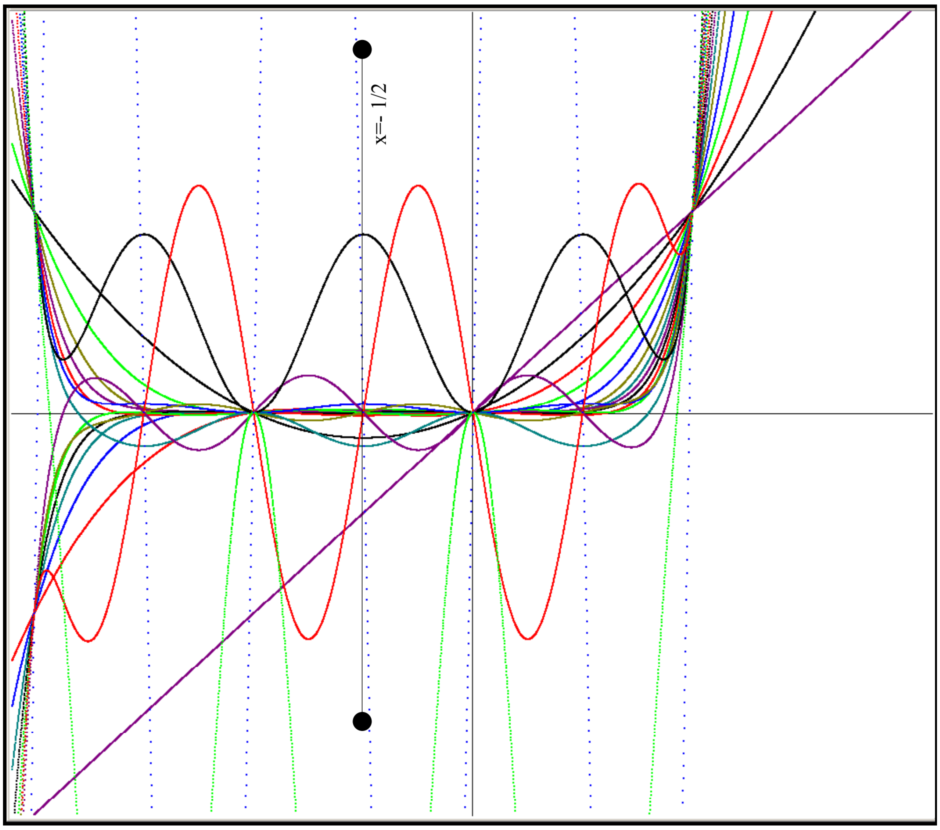
Version 2.3.2 (small corrections)

$$\begin{bmatrix}1 & . & . & . \\ 1 & 1 & . & . \\ 1 & 2 & 1 & . \\ 1 & 3 & 3 & 1\end{bmatrix} = exp\left(\begin{bmatrix}. & . & . & . \\ 1 & . & . & . \\ . & 2 & . & . \\ . & . & 3 & .\end{bmatrix}\right)$$

Binomial = exp ("counting")

Function $\boldsymbol{G_{p,1..18}(x)}$ = $\quad(x=-2..+2,\ y=-2..+2)$



Accessing Bernoulli-Numbers by Matrixoperations

1. Introduction

There is a lot of articles available on bernoulli-numbers - so: why another article?

It was my private fun and interest, to learn about the problems and applications of bernoulli-numbers; also I like to access such problems with the tools of matrix-algebra, which is suitable for many problems, for instance puzzles in the field of "recreational mathematics". The results, which I'm to present here, are not really new in most cases; but I find some expressions, which are not really prominent in most articles, to say the least; so some seem new and even surprising in their simplyness.

The most interesting results, that I achieved were:

- *The binomialmatrix* **P** *is expressible by the EXP()-Function applied to a vector of natural numbers*
- *Polynomials, which are constructed of the coefficicients from the binomialmatrix* **P** *, have roots at complex x of the form x=1/2 + i*tan(t), where t represent equidistant points on the circle*
- *The bernoulli-numbers can be found in the first eigenvector of the signed binomialmatrix* $\mathbf{P}_c$ *(=***P*J***) bzw* $\mathbf{P}_r$*; (=***J*P***); the matrix of eigenvalues is just the unit-matrix* ***J*** *with alternating signs*
- *The frequently discussed definition of the bernoulli-number* $ß_1$ *as + or - 1/2 (here as bernoullivectors* $\mathbf{B}_p$ *resp.* $\mathbf{B}_m$*) can be resolved as specific solution of the same systematic concept (like transposed solutions in matrixalgebra following from the non-commutativity) and formulae, based on that definitions, can be translated from one to another (as far as the matrix-concept reaches)*
- *The set of eigenvectors (in the following named as* ***G****-matrix) contains the coefficients of the integrals of the bernoulli-functions; this integrals provide directly the Bernoullian sums-of-powers for natural arguments, but also common recursive definitions and remarkable graphs for real-valued arguments.*

Most results were primarily heuristical; the relations to the known formulae are usually not easy to recognize; it was useful, that in at least some handbooks some tables with actual printed coefficients were provided, so that besides the explication of the matrix-formulae also the results could be compared.

For an experienced number-theorist that all may possibly be known (and much is, as I already learned) - in the case, the reader of this article belongs to this category, he/she might feel at least a philatelistic pleasure of this small collection of views...

Perspectives:

> *The matrix-tool is immanently restricted by the natural-numbered indexing of matrix-rows. Given, that the matrix row n provides the coefficient of the power-sum woth exponent n, then the same is not obvious with rational n or even real or complex-valued exponents x. This problem, as well as the now possible parametrizition of the eigenvalue-matrix* ***J*** *(which leads to possibly interesting variants of the binomial matrix* ***P****) is subject to further study.*

One of the most triggering impulses was ironically a tiny error, which appears in many internet-ressources; a small, but significant one.

2. A common equation of recursion (containing a significant error)

In the webpage "bernoullinumbers" in *[mathworld]* the known recursion for the computation of bernoullinumbers is given as

$$(B+1)^{[n]} = B^{[n]} \qquad \textit{(Eq 31)}$$

To keep the following text consistent I reformulate the symbols of this equation and the order of the terms as:

(2.1) $$(1+ß)^{[n]} = ß^{[n]}$$

where, as mentioned in that webpage, the *i*'th powers $i=0..n$ of $ß$ are to be replaced by the *i*'th bernoulli-numbers: $ß[i] \to ß_i$

If one resolves this recursion and puts the rows for running *n* together, then the following scheme evolves:

```
ß0 = 1*ß0
ß1 = 1*ß0 +      1*ß1
ß2 = 1*ß0 +      2*ß1         + 1*ß2
ß3 = 1*ß0 +      3*ß1         + 3*ß2  + 1*ß3
...
ßn = 1*ß0 + ch(n,1)*ß1  + ch(n,2)*ß2        .... + ch(n,n-1) ßn-1  + 1*ßn
```

where "*ch(n,k)*" indicates the binomial coefficients:

(2.2) $$ch(n,k) = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

This scheme can be seen as matrix-formula, where **B** is the columnvector of the bernoulli-numbers and **P** is the lower triangular matrix of coefficients, which represents the Pascal-triangle.

Thus the given recursion says:

(2.3) $$\mathbf{B} = \mathbf{P} * \mathbf{B} \qquad \begin{bmatrix} ß_0 \\ ß_1 \\ ß_2 \\ ß_3 \end{bmatrix} = \begin{bmatrix} 1 & . & . & . \\ 1 & 1 & . & . \\ 1 & 2 & 1 & . \\ 1 & 3 & 3 & 1 \end{bmatrix} * \begin{bmatrix} ß_0 \\ ß_1 \\ ß_2 \\ ß_3 \end{bmatrix}$$

- an interesting equation, which has the form of an eigenvector-problem which can also be expressed as ("***O***" indicates a zero-column-vector):

(2.4) $$\begin{aligned} \mathbf{O} &= \mathbf{P} * \mathbf{B} - \mathbf{B} \\ \mathbf{O} &= (\mathbf{P} - \mathbf{I}) * \mathbf{B} \end{aligned} \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & . & . & . \\ 1 & 0 & . & . \\ 1 & 2 & 0 & . \\ 1 & 3 & 3 & 0 \end{bmatrix} * \begin{bmatrix} ß_0 \\ ß_1 \\ ß_2 \\ ß_3 \end{bmatrix}$$

From the 2'nd row in (2.4) however, $ß_0$ needs to be zero, and from that follows, that in the 3'rd row also $ß_1=0$ etc.; in short: the vector of bernoulli-numbers **B** needs to equal the zero-vector; but which is not true.

3. Two versions B_m and B_p of bernoulli-numbers?

The mistake is in the recursion-equation (2.1), and obviously with $n=1$. From (2.1) it is:

(3.1) $$\beta_1 = 1*\beta_0 + 1*\beta_1$$

The first bernoulli-number β_0 is defined as 1, the 2'nd β_1 as $-½$. Plugged into the equation we have a contradiction:

(3.2) $$-½ = 1*1 + 1*(-½) = ½ \quad \text{// contradiction!}$$

For row $n = 1$ the equations needs to be:

(3.3a) $$(-1)*(1+\beta)^{[1]} = \beta^{[1]}$$

or

(3.3b) $$(1+\beta)^{[1]} - 1 = \beta^{[1]}$$

Since the following bernoulli-numbers with odd index are all zero, it is not obvious, whether this mistake should be corrected for all following odd-n bernoulli-numbers. But it seems from the following, that this is a wise assumption, since this leads to a useful and consistent representation.

Let **I** be the unitmatrix with *1* in the diagonal, always of the appropriate size, fitting the current matrixequation:

(3.4) $$\mathbf{I} := diag(1,1,1,1,...)$$

and let **J** be the resp. diagonalmatrix with alternating signs:

(3.5) $$\mathbf{J} := diag(1,-1,1,-1,...) \qquad \text{from which} \qquad \mathbf{J}*\mathbf{J} = \mathbf{I} \text{ and } \mathbf{J} = inv(\mathbf{J})$$

From this is (since only the sign of β_1 in $\mathbf{B}_p$ and $\mathbf{B}_m$ are affected and is insignificant for all $\beta_3 = \beta_5 = \beta_7 = ... = 0$),

(3.6) $$\mathbf{B}_p = \mathbf{J} * \mathbf{B}_m$$
and $$\mathbf{B}_m = \mathbf{J} * \mathbf{B}_p$$

The mistake can now simply be corrected by rewriting:

(3.7) $$\mathbf{J} * \mathbf{P} * \mathbf{B}_m = \mathbf{B}_m \qquad \begin{bmatrix} 1 & . & . & . \\ -1 & -1 & . & . \\ 1 & 2 & 1 & . \\ -1 & -3 & -3 & -1 \end{bmatrix} * \begin{bmatrix} 1 \\ -1/2 \\ 1/6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/2 \\ 1/6 \\ 0 \end{bmatrix}$$

or, after reformulation of $\mathbf{B}_m$ into $\mathbf{B}_p$

(3.8) $$\mathbf{J}*\mathbf{P} *(\mathbf{J}*\mathbf{B}_p) = (\mathbf{J}*\mathbf{B}_p)$$

(3.9) $$(\mathbf{P}*\mathbf{J}) * \mathbf{B}_p = \mathbf{B}_p \qquad \begin{bmatrix} 1 & . & . & . \\ 1 & -1 & . & . \\ 1 & -2 & 1 & . \\ 1 & -3 & 3 & -1 \end{bmatrix} * \begin{bmatrix} 1 \\ 1/2 \\ 1/6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/6 \\ 0 \end{bmatrix}$$

Since in **J*P** only the row-signs alternate and in **P*J** the signs in the colums, I introduce the shorthands for this variants:

(3.10) $\mathbf{P}_r := \mathbf{J} * \mathbf{P}$ *//signs of rows alternate*

$\mathbf{P}_c := \mathbf{P} * \mathbf{J}$ *//signs of columns alternate*

The following three notations are then correct and useful:

(3.11.1) $\mathbf{P}_r * \mathbf{B}_m = \mathbf{B}_m$

(3.11.2) $\mathbf{P}_c * \mathbf{B}_p = \mathbf{B}_p$

and since

$$\mathbf{P}_c * \mathbf{B}_p = (\mathbf{P}*\mathbf{J}) * \mathbf{B}_p = \mathbf{P} * (\mathbf{J}*\mathbf{B}_p) = \mathbf{P} * \mathbf{B}_m$$

one can also use the unsigned version of the binomialmatrix:

(3.11.3) $\mathbf{P} * \mathbf{B}_m = \mathbf{B}_p$

The two recursion equations read hence correctly:

(3.12)
$$(-1)^n \quad (1+\beta)^{[n]} = \beta^{[n]} \quad // \text{to compute } B_m$$
$$(1-\beta)^{[n]} = \beta^{[n]} \quad // \text{to compute } B_p$$

Table 3.1: $\mathbf{B}_\mathrm{m}$ und $\mathbf{B}_\mathrm{p}$: vectors of the bernoulli-numbers:

n	$\mathbf{B}_m$	$\mathbf{B}_p$	$\mathbf{I}_1$
0	1	1	0
1	-1/2	+1/2	1
2	1/6	1/6	0
3	0	0	0
4	-1/30	-1/30	0
5	0	0	0
6	1/42	1/42	0
7	0	0	0
8	-1/30	-1/30	0

Table 3.2: recursions $\mathbf{P}_\mathrm{c} * \mathbf{B}_\mathrm{p} = \mathbf{B}_\mathrm{p}$ and $\mathbf{P}_\mathrm{r} * \mathbf{B}_\mathrm{m} = \mathbf{B}_\mathrm{m}$

Rek.1: $(1 - \beta_p)^{[n]} = \beta_p{}^{[n]}$

Rek.2: $(-1)^n*(1 + \beta_m)^{[n]} = \beta_m{}^{[n]}$

$$\mathbf{P}_c * \mathbf{B}_p = \begin{bmatrix} 1 & . & . & . \\ 1 & -1 & . & . \\ 1 & -2 & 1 & . \\ 1 & -3 & 3 & -1 \end{bmatrix} * \begin{bmatrix} 1 \\ 1/2 \\ 1/6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/6 \\ 0 \end{bmatrix}$$

$$\mathbf{P}_r * \mathbf{B}_m = \begin{bmatrix} 1 & . & . & . \\ -1 & -1 & . & . \\ 1 & 2 & 1 & . \\ -1 & -3 & -3 & -1 \end{bmatrix} * \begin{bmatrix} 1 \\ -1/2 \\ 1/6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/2 \\ 1/6 \\ 0 \end{bmatrix}$$

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4. Computation of bernoulli-numbers by matrixinversion of (P-I)

Using this corrected recursion-formula the bernoulli-vector can now be computed. the following definition helps for notation:

(4.1) $\mathbf{I}_1 := \mathbf{B}_p - \mathbf{B}_m = \{0,1,0,0,....0\}$

from (3.11.3) $\mathbf{P} * \mathbf{B}_m = \mathbf{B}_p$
with (4.1) $\mathbf{P} * \mathbf{B}_m = \mathbf{B}_m + \mathbf{I}_1$
$(\mathbf{P} - \mathbf{I})\ \mathbf{B}_m = \mathbf{I}_1$
follows
(4.2) $\mathbf{B}_m = inv(\mathbf{P} - \mathbf{I}) * \mathbf{I}_1$

and the bernoulli-vector $\mathbf{B}_m$ can be found in the 2'nd column ($i=1$) of the inverted (**P-I**) - matrix. Analoguously the bernoulli-vector $\mathbf{B}_p$ can be determined:

from (3.11.3) $\mathbf{P} * \mathbf{B}_m = \mathbf{B}_p$
with (4.1) $\mathbf{P} * (\mathbf{B}_p - \mathbf{I}_1) = \mathbf{B}_p$
$\mathbf{B}_p - \mathbf{I}_1 = inv(\mathbf{P}) * \mathbf{B}_p$
$- \mathbf{I}_1 = (inv(\mathbf{P}) - \mathbf{I}) * \mathbf{B}_p$

follows:

(4.3) $\mathbf{B}_p = - inv(inv(\mathbf{P}) - \mathbf{I}) * \mathbf{I}_1$

and the bernoullivektor $\mathbf{B}_p$ can be found in the 2'nd column (*i=1)* of the negative of the inverted *(inv(***P***) -* **I***)* matrix.

This computation however cannot immediately be done. By the subtraction of **I** the matrices **P**-**I** as well as *(inv(***P***) -* **I***)* are singular and cannot be inverted.

(4.3.1)
$$\begin{matrix} inv(P)-I & * & B_p & = & -I_1 \\ \begin{bmatrix} 0 & & & \\ -1 & 0 & & \\ 1 & -2 & 0 & \\ -1 & 3 & -3 & 0 \end{bmatrix} & * & \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} & = & -\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \end{matrix}$$

But as it is easy to see, one can determine solutions for all bernoulli-numbers up to β_n, if the left-hand matrix has rows up to $n+1$; row *1* determines β_0, from that row *2* determines β_1 and so on. This means practically, that one has only to use the appropriate submatrix of a **P**-matrix with the highest row-number of $n+1$.

The both matrices computed this way are denoted as $\mathbf{G}_p$ (resp. $\mathbf{G}_m$) in the following.

In the following formula $\mathbf{P}^{[n+1]}$ indicates an extension of 1 row/column of the **P**-matrix, and the $_{[r,c]}$-indexes show the then selected ranges of rows/columns:

(4.4) $\mathbf{G}_m := inv((\mathbf{P}^{[n+1]} - \mathbf{I})_{[1..n+1,\ 0..n]})$
(4.5) $\mathbf{G}_p := - inv((inv(\mathbf{P})^{[n+1]} - \mathbf{I})_{[1..n+1,\ 0..n]})$

If we remove the same first row from the column-vector $\mathbf{I}_1^{[n+1]}$ as well, we get:

(4.6) $\mathbf{I}_0 := \{1,0,0...0\}'$

and we are able to determine the bernoulli-vectors from the inverses $\mathbf{G}_p$ resp. $\mathbf{G}_m$

(4.7) $\mathbf{B}_p = \mathbf{G}_p * \mathbf{I}_0 = \mathbf{G}_{p\ [0..n,\ 0]}$
(4.8) $\mathbf{B}_m = \mathbf{G}_m * \mathbf{I}_0 = \mathbf{G}_{m\ [0..n,\ 0]}$

just by extraction of the first column from **G**.

5. *J contains the eigenvalues, and G_m resp. G_p contain the eigenvectors of P_r resp. P_c*

The interesting news are hence, that $\mathbf{B}_p$ (resp. $\mathbf{B}_m$), which are computed from the expanded matrix $\mathbf{P}^{[n+1]}$, are just the first eigenvectors of the *n*-rowed matrix $\mathbf{P}_c$ (resp $\mathbf{P}_r$). Analoguously, and much more interesting, this is true for the matrices $\mathbf{G}_p$ (resp. $\mathbf{G}_m$), which contain the full set of eigenvectors each.

Thus we have the following identities and properties:

Let

$\mathbf{P}$ *the lower triangular matrix of binomial-coefficients*
$\mathbf{J}$ = *diag(1,-1,1,-1...) the identitymatrix with alternating signs*

then

(5.1) $\mathbf{J} = inv(\mathbf{J})$
(5.2) $\mathbf{J} * \mathbf{P} * inv(\mathbf{J}) = \mathbf{J} * \mathbf{P} * \mathbf{J} = inv(\mathbf{P})$

(5.3) $\mathbf{P} * \mathbf{J} = \mathbf{P}_c$ *the binomialmatrix having alternating signs in columns*
(5.4) $\mathbf{J} * \mathbf{P} = \mathbf{P}_r$ *the binomialmatrix having alternating signs in rows*

(5.5) $\mathbf{P}_c * \mathbf{G}_p = \mathbf{G}_p * \mathbf{J}$ *or* $\mathbf{P}_c = \mathbf{G}_p * \mathbf{J} * inv(\mathbf{G}_p)$
(5.6) $\mathbf{P}_r * \mathbf{G}_m = \mathbf{G}_m * \mathbf{J}$ *or* $\mathbf{P}_r = \mathbf{G}_m * \mathbf{J} * inv(\mathbf{G}_m)$

(5.7) $\mathbf{J}$ *the diagonalmatrix of eigenvalues of* $\mathbf{P}_c$ *and* $\mathbf{P}_r$
(5.8) $\mathbf{G}_p$ *the matrix of eigenvectors of* $\mathbf{P}_c$
(5.9) $\mathbf{G}_m$ *the matrix of eigenvectors of* $\mathbf{P}_r$

(5.10) $\mathbf{J} * \mathbf{G}_p \ \mathbf{J} = \mathbf{G}_m$
(5.11) $\mathbf{J} * \mathbf{G}_m \ \mathbf{J} = \mathbf{G}_p$

See a proof for this identity in the internet-conversation, copied to

ProofOfGpBeingEigensystem.htm

Table 5.1: $\mathbf{G}_m$ and $\mathbf{G}_p$

Zl	$\mathbf{G}_m$, *first 5 columns (0..4)*					$\mathbf{G}_p$, *first 5 columns (0..4)*				
0	1	.	.	.	.	1	.	.	.	.
1	-1/2	1/2	.	.	.	1/2	1/2	.	.	.
2	1/6	-1/2	1/3	.	.	1/6	1/2	1/3	.	.
3	0	1/4	-1/2	1/4	.	0	1/4	1/2	1/4	.
4	-1/30	0	1/3	-1/2	1/5	-1/30	0	1/3	1/2	1/5
5	0	-1/12	0	5/12	-1/2	0	-1/12	0	5/12	1/2
6	1/42	0	-1/6	0	1/2	1/42	0	-1/6	0	1/2

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6. The Binomial-Matrix and the Matrixexponential

For me an amazing property of the binomial-matrix is, that it can be expressed as an matrixexponential of an most elementary parameter: namely of a matrix consisting only of the sequence natural numbers 1..(n-1) in the first principal subdiagonal, which may be denoted here as **T**

(6.1) $\boldsymbol{P} = \exp(\boldsymbol{T})$

Example: $$\begin{bmatrix} 1 & . & . & . \\ 1 & 1 & . & . \\ 1 & 2 & 1 & . \\ 1 & 3 & 3 & 1 \end{bmatrix} = \exp\left(\begin{bmatrix} . & . & . & . \\ 1 & . & . & . \\ . & 2 & . & . \\ . & . & 3 & . \end{bmatrix}\right)$$

A proof is in *[Helms1]* or more in detail in *[Edelman]*.

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7. Bernoulli-vectors and the Matrixexponential

Using the matrixexpoential familiar formulae are popping up, if one replaces

$$inv(\mathbf{P}) = exp(\mathbf{T})^{-1} = exp(-\mathbf{T})$$

and denotes the submatrix *[1..n,0..n-1]* of a matrix $P^{[n+1]}$, reduced by one row/column, with the symbol $^{\vdots}\mathbf{P}$:

(7.1) $\mathbf{B}_m = {}^{\vdots}(exp(\mathbf{T}) - \mathbf{I})^{-1} * I_0$

(7.2) $\mathbf{B}_p = -{}^{\vdots}(exp(-\mathbf{T}) - \mathbf{I})^{-1} * I_0$

which gives for $\mathbf{B}_m$ nicely visible the scalar term of the generatingfunction for the bernoulli-numbers:

$$\frac{1}{exp(t)-1}$$

and for $\mathbf{B}_p$

$$\frac{-1}{exp(-t)-1} = \frac{exp(t)}{exp(t)-1}$$

(see the formulae, for instance in [mathworld] or [A&S], where in the numerator also the parameter t occurs).

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8. The structure of the remaining coefficients in the matrices G_m - and G_p

Heuristicaly for small n the following -much plausible, but not yet analytically verified- representation for the coefficients can be found. Here the row-/column-indices are understood as starting at zero (I use "r" for rows and "c" for columns):

(8.1) $$\mathbf{G}_m[r,c] = \binom{r}{c} * \frac{1}{s+1} * \mathbf{B}_m[r-c]$$

(8.2) $$\mathbf{G}_p[r,c] = \binom{r}{c} * \frac{1}{s+1} * \mathbf{B}_p[r-c]$$

Let the symbol "*#" denote an elementwise multiplication ("*Hadamard multiplication*") of two matrices of same dimension, then:

$$\mathbf{G} = \begin{bmatrix} 1 & . & . & . \\ 1 & 1 & . & . \\ 1 & 2 & 1 & . \\ 1 & 3 & 3 & 1 \end{bmatrix} *\# \begin{bmatrix} \beta_0 & . & . & . \\ \beta_1 & \beta_0 & . & . \\ \beta_2 & \beta_1 & \beta_0 & . \\ \beta_3 & \beta_2 & \beta_1 & \beta_0 \end{bmatrix} * \begin{bmatrix} \frac{1}{1} & . & . & . \\ . & \frac{1}{2} & . & . \\ . & . & \frac{1}{3} & . \\ . & . & . & \frac{1}{4} \end{bmatrix}$$

where the numerators (in left matrix) contain just the binomialmatrix **P**.

Table 8.1: recall matrices $\mathbf{G}_m$ and $\mathbf{G}_p$:

Zl	$\mathbf{G}_m$ *, first 5 columns (0..4)*					$\mathbf{G}_p$ *, first 5 columns (0..4)*				
0	1	.	.	.	.	1	.	.	.	.
1	-1/2	1/2	.	.	.	1/2	1/2	.	.	.
2	1/6	-1/2	1/3	.	.	1/6	1/2	1/3	.	.
3	0	1/4	-1/2	1/4	.	0	1/4	1/2	1/4	.
4	-1/30	0	1/3	-1/2	1/5	-1/30	0	1/3	1/2	1/5
5	0	-1/12	0	5/12	-1/2	0	-1/12	0	5/12	1/2
6	1/42	0	-1/6	0	1/2	1/42	0	-1/6	0	1/2

The rowsums in $\mathbf{G}_\mathrm{m}$ equal zero and in $\mathbf{G}_\mathrm{p}$ equal 1,

$\mathbf{G}_m * \mathbf{V}(1) = \mathbf{V}(0)$ $\qquad$ $\mathbf{G}_p * \mathbf{V}(1) = \mathbf{V}(1)$

and thus also

(8.3) $$\beta_{m\,z} = \mathbf{G}_m[z,0] = -\sum_{s=1}^{z} \mathbf{G}_m[z,s] \qquad \beta_{p\,z} = \mathbf{G}_p[z,0] = 1 - \sum_{s=1}^{z} \mathbf{G}_p[z,s]$$

Here again we find known recursion-formulae for the Bernoulli-numbers , which differ by just $2*\beta_1 = 1$, reflecting the both definitions for β_1:

(8.4.1) $$\beta_{m,n} = 0 - \sum_{k=1}^{n} \left(\binom{n}{k} \frac{1}{k+1} * \beta_{m,n-k} \right) \qquad \beta_{p,n} = 1 - \sum_{k=1}^{n} \left(\binom{n}{k} \frac{1}{k+1} * \beta_{p,n-k} \right)$$

9. The original problem of Jacob Bernoulli: "Powersums" - from G_p

The computation of the powersums $\mathbf{S}_{m,n} = 1^m + 2^m + 3^m + ... + n^m$ is then a simple matrixmultiplication using $\mathbf{G}_p$. We need the Vandermondevector

(9.1) $\mathbf{V} := \mathbf{V}_m(n) = \{ 1,n,n^2,n^3,...,n^m \}$

need only multiply to get the vector of all powersums **S** with the *m* and dimensions matching:

(9.2) $\mathbf{S} := \mathbf{G}_p * \mathbf{V}(n) * n$

and find in row **S***[m]* the sum of the *m'th* powers from *1* to *n*:

Table 9.1: Powersums

$$\mathbf{G}_p \quad * \mathbf{V}(3)*3 = \mathbf{S}(3)$$

$$\begin{bmatrix} 1 & . & . & . \\ 1/2 & 1/2 & . & . \\ 1/6 & 1/2 & 1/3 & . \\ . & 1/4 & 1/2 & 1/4 \end{bmatrix} * \begin{bmatrix} 3^1 \\ 3^2 \\ 3^3 \\ 3^4 \end{bmatrix} = \begin{bmatrix} 1^0+2^0+3^0 \\ 1^1+2^1+3^1 \\ 1^2+2^2+3^2 \\ 1^3+2^3+3^3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 14 \\ 36 \end{bmatrix}$$

Table 9.2 the original problem of Jakob Bernoulli (Quelle:[MICH])

Wahrscheinlichkeitsrechnung (Ars conjectandi). 99

Die Summe der Potenzen der natürlichen Zahlen.

$$S(n) = \tfrac{1}{2} n^2 + \tfrac{1}{2} n,$$
$$S(n^2) = \tfrac{1}{3} n^3 + \tfrac{1}{2} n^2 + \tfrac{1}{6} n,$$
$$S(n^3) = \tfrac{1}{4} n^4 + \tfrac{1}{2} n^3 + \tfrac{1}{4} n^2,$$
$$S(n^4) = \tfrac{1}{5} n^5 + \tfrac{1}{2} n^4 + \tfrac{1}{3} n^3 - \tfrac{1}{30} n,$$
$$S(n^5) = \tfrac{1}{6} n^6 + \tfrac{1}{2} n^5 + \tfrac{5}{12} n^4 - \tfrac{1}{12} n^2,$$
$$S(n^6) = \tfrac{1}{7} n^7 + \tfrac{1}{2} n^6 + \tfrac{1}{2} n^5 - \tfrac{1}{6} n^3 + \tfrac{1}{42} n,$$
$$S(n^7) = \tfrac{1}{8} n^8 + \tfrac{1}{2} n^7 + \tfrac{7}{12} n^6 - \tfrac{7}{24} n^4 + \tfrac{1}{12} n^2,$$
$$S(n^8) = \tfrac{1}{9} n^9 + \tfrac{1}{2} n^8 + \tfrac{2}{3} n^7 - \tfrac{7}{15} n^5 + \tfrac{2}{9} n^3 - \tfrac{1}{30} n,$$
$$S(n^9) = \tfrac{1}{10} n^{10} + \tfrac{1}{2} n^9 + \tfrac{3}{4} n^8 - \tfrac{7}{10} n^6 + \tfrac{1}{2} n^4 - \tfrac{1}{12} n^2,$$
$$S(n^{10}) = \tfrac{1}{11} n^{11} + \tfrac{1}{2} n^{10} + \tfrac{5}{6} n^9 - 1\, n^7 + 1\, n^5 - \tfrac{1}{2} n^3 + \tfrac{5}{66} n.$$

Wer aber diese Reihen in Bezug auf ihre Gesetzmässigkeit genauer betrachtet, kann auch ohne umständliche Rechnung die Tafel fortsetzen. Bezeichnet *c* den ganzzahligen Exponenten irgend einer Potenz, so ist

10. Polynomials of the coefficients of the Binomialmatrix P_c

If one uses the entries of the (signed) matrix $\mathbf{P}_c$ as coefficients for polynomials in x, then interesting functions pop up. Let the Vandermonde -vector $\{1,x,x^2,x^3,...x^z\}' = V_z(x)$ of the length $z+1$

(10.1) $$\mathbf{V}_z(x) = \{1,x,x^2,x^3,...x^z\}'$$

and define the polynomials in x using the coefficients of a fixed row from $\mathbf{P}_c$

(10.2) $$f_{m,z}(x) := (\mathbf{PJ} - \mathbf{I})_z * \mathbf{V}_z(x)$$

(10.3) $$f_{p,z}(x) := (\mathbf{PJ} + \mathbf{I})_z * \mathbf{V}_z(x)$$

the one finds as roots of $f_{m,z}(x)$ complex values for x, which are connected to the cyclotomic functions:

(10.4) $$f_{m,z}(x)=0 \quad \text{<-->} \quad x = \frac{1}{2} + \frac{1}{2} tan\left(\frac{\pi}{2} * \frac{m}{z}\right) * \hat{\imath} \quad \text{for } m=2k \text{ if } z \text{ is even}$$

(10.5) $$f_{p,z}(x)=0 \quad \text{<-->} \quad x = \frac{1}{2} + \frac{1}{2} tan\left(\frac{\pi}{2} * \frac{m}{z}\right) * \hat{\imath} \quad \text{for } m=2k+1 \text{ if } z \text{ is odd}$$

and conversely, if z is odd, so that the zeroes for both functions can be computed from the cyclotomical roots. Both cases together can then be represented by:

(10.6) $$f_{m,z}(x) * f_{p,z}(x)=0 \quad \text{<-->} \quad x = \frac{1}{2} + \frac{1}{2} tan\left(\frac{\pi}{2} * \frac{m}{z}\right) * \hat{\imath} \qquad \text{for } m=0..z-1$$

11. Values of the G_m- and G_p- polynomials for real x

The above equations produce the inverses $\mathbf{G}_m$ and $\mathbf{G}_p$ of the (by **I** reduced) (signed) binomial-matrices $^{\vdots}$(**J*****P** - **I**) resp $^{\vdots}$(**P*****J** - **I**), which interestingly -together with **J**- define an eigensystem of the original unreduced (signed) binomialmatrices.

If we use the entries of a fixed row of $\mathbf{G}_m$ or $\mathbf{G}_p$ with the rowindex *n* as coefficients for polynomials in general *x* (as in **(9.)**, but without the goal to find the powersums for a natural exponent *n*), then we have the more general functions $G_{m,n}(x)$ and $G_{p,n}(x)$, which come out to be just the integrals of the bernoulli-polynomials (depending on the sign of β_1) .

A multiplication of **G** with a vandermonde-(column-)vektor like

$V(x) := \{ x, x^2, x^3,.., x^{n+1}\}'$

written as a polynomial looks like this:

$$G_{m,n}(x) = \mathbf{G}_m[n,0]*x + \mathbf{G}_m[n,1]*x^2 + \mathbf{G}_m[n,2]*x^3 + ... + \mathbf{G}_m[n,n]*x^{n+1} \quad (11.1)$$

$$G_{p,n}(x) = \mathbf{G}_p[n,0]*x + \mathbf{G}_p[n,1]*x^2 + \mathbf{G}_p[n,2]*x^3 + ... + \mathbf{G}_p[n,n]*x^{n+1} \quad (11.2)$$

We get a family of functions with the interesting property, that in the range $2<x<2$ the local minima and maxima get better periodical with increasing *n* and seemingly approximate to a *sin*/*cos*-shape. The graphs of that functions are shown in the appendix.

This functions have the special values:

Table 11.1:

	$G_{m,n}(x)$	$G_{p,n}(x)$
	$G_{m,n}(-1) = +/- 1$ $G_{m,n}(0) = 0$ $G_{m,n}(1/2) = 0$ or local extremum $G_{m,n}(1) = 0 \ (n>1)$ $G_{m,n}(2) = 1 \ (n>1)$	$G_{p,n}(1) = 1$ $G_{p,n}(0) = 0$ $G_{p,n}(-1/2) = 0$ or local extremum $G_{p,n}(-1) = 0$ $G_{p,n}(-2) = +/- 1 \ (n>1)$

The derivatives of $G_{m,n}(x)$ resp $G_{p,n}(x)$ are

$$B_{m,n}(x) = d\, G_{m,n}(x) / dx \qquad B_{p,n}(x) = d\, G_{p,n}(x) / dx \quad (11.3)$$

meaning

$$B_{m,n}(x) = \mathbf{G}_m[n,0] + 2*\mathbf{G}_m[n,1]*x + ... + (n+1)\, \mathbf{G}_m[n,n]*x^n \quad (11.5)$$

$$B_{p,n}(x) = \mathbf{G}_p[n,0] + 2*\mathbf{G}_p[n,1]*x + ... + (n+1)\, \mathbf{G}_p[n,n]*x^n \quad (11.6)$$

and the $B_{m,n}(x)$ are just the well-known Bernoulli-Polynomials.

Table 11.2: Special values:

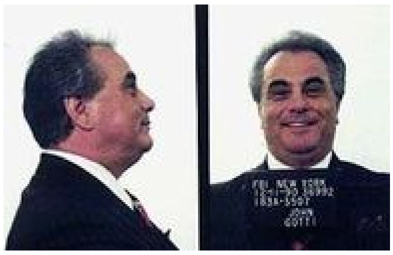
	$B_{m,n}(x)$	$B_{p,n}(x)$
	$B_{m,n}(-1) = +/- 1$ $B_{m,n}(0) = \beta_{m,n}$ $B_{m,n}(1) = 0$ $B_{m,n}(2) = 1/2$	$B_{p,n}(1) = 1$ $B_{p,n}(0) = \beta_{p,n}$ $B_{p,n}(-1) = 0$ $B_{p,n}(-2) = +/-.1/2 \ (n>1)$

All in all this *G*-functions seem to be much interesting; for instance one finds in OEIS the following entry: *[OEIS_A002425]*:

> A002425 Denominator of Pi^(2n)/(GAMMA(2n)(1-2^(-2n))*Zeta (2n)).*
> 1, 1, 1, 17, 31, 691, 5461, 929569, 3202291, 221930581,
> 4722116521, 968383680827, 14717667114151, 2093660879252671,
> 86125672563201181, 129848163681107301953, 868320396104950823611,
> 209390615747646519456961 (list)
>
> Consider the C(k)-summation process for divergent series: the series
> Sum((-1)^n(n+1)^k)==1-2^k+3^k-4^k+..., summable C(1) to the value*
> 1/2 for k=0, is for each k>=1 exactly summable C(k+1) to the sum s(k+
*> 1)=(2^(k+1)-1)*B(k+1)/(k+1) and so a(n)=Abs(numerator(s(2n))). -*
> Benoit Cloitre (abmt(AT)wanadoo.fr), Apr 27 2002
>
> Odd part of tangent numbers A000182 (even part is 2^A101921(n)). -
> Ralf Stephan, Dec 21 2004

These are just the odd factors of the denominators, which occur, if one computes the $\boldsymbol{G}_{p,n}(x)$-funktions for $x=-1/2$. Possibly there are other special values with complex x; but this is not yet obvious to me

12. G_p ("Gotti")-Matrix and Stirlingnumbers



For a continuous text it seems useful, to assign easy-to-remember names also to the **G-**matrices, as I was hinted by a reader in the german math-newsgroup; as a provisorial I just took my nickname "Gotti" (which equal also the name of a famous Mafiaboß from New York " ***John Gotti"***; who has his significance also from his subtractions (of the wealth of the wealthy) as well from the inversion (of the legal structure of New York). Picture taken from "wikipedia")

Here I use as the *"Gotti"*- matrix primarily the matrix $\mathbf{G}_p$; and its polynomials in *x*, constructed accordingly to chapter (10), may be calles as *"Gotti"*-polynomials.

Then we get:

(12.1) $$\mathbf{P}_c = \mathbf{G}_p * \mathbf{J} * \mathbf{G}_p^{-1} \qquad \mathbf{P}_r = \mathbf{G}_m * \mathbf{J} * \mathbf{G}_m^{-1}$$

The Gotti-matrix $\mathbf{G}_p$ own another interesting eigensystem. It seems to be:

(12.2) $$\mathbf{G}_p = \mathbf{S2} * \mathbf{R} * \mathbf{S2}^{-1} = \mathbf{S2} * \mathbf{R} * \mathbf{S1}$$

where

S2 : lower triangular matrix of the Stirlingnumbers 2. kind
S1 : lower triangular matrix of the Stirlingnumbers 1. kind
R : diagonalmatrix of the reciprocals of the natrual numbers

Of special interest here is the connection between **S1** and **S2**:

(12.3) $$\mathbf{S2} = \mathbf{S1}^{-1}$$

All entries of **S1** and **S2** are integers. (The entries of the analoguous eigensystem of $\mathbf{G}_m$ are rational).
For the basic binomialmatrix $\mathbf{P}_r$ this means the more detailed eigensystem:

(12.4) $$\mathbf{P}_r = (\mathbf{S2} * \mathbf{R} * \mathbf{S1}) \; * \; \mathbf{J} \; * \; (\mathbf{S2} * \mathbf{R}^{-1} * \mathbf{S1})$$

or, using **N** as diagonalmatrix of natural numbers **N** = *diag{1,2,3,...n}*,

(12.5.1) $$\mathbf{P}_r = (\mathbf{S1}^{-1} * \mathbf{N}^{-1} * \mathbf{S1}) \; * \; \mathbf{J} \; * \; (\mathbf{S1}^{-1} * \mathbf{N} * \mathbf{S1})$$

(12.5.2) $$= (\mathbf{S2} * \mathbf{N}^{-1} * \mathbf{S2}^{-1}) \; * \; \mathbf{J} \; * \; (\mathbf{S2} * \mathbf{N} * \mathbf{S2}^{-1})$$

which states an impressing hierarchy of known numbers of combinatorics.

Table 12.1

Zl	**S1** , *first 6 columns (0..5)*						**S2** , *first 6 columns (0..5)*					
0	1	0	0	0	0	0	1	0	0	0	0	0
1	-1	1	0	0	0	0	1	1	0	0	0	0
2	2	-3	1	0	0	0	1	3	1	0	0	0
3	-6	11	-6	1	0	0	1	7	6	1	0	0
4	24	-50	35	-10	1	0	1	15	25	10	1	0
5	-120	274	-225	85	-15	1	1	31	90	65	15	1

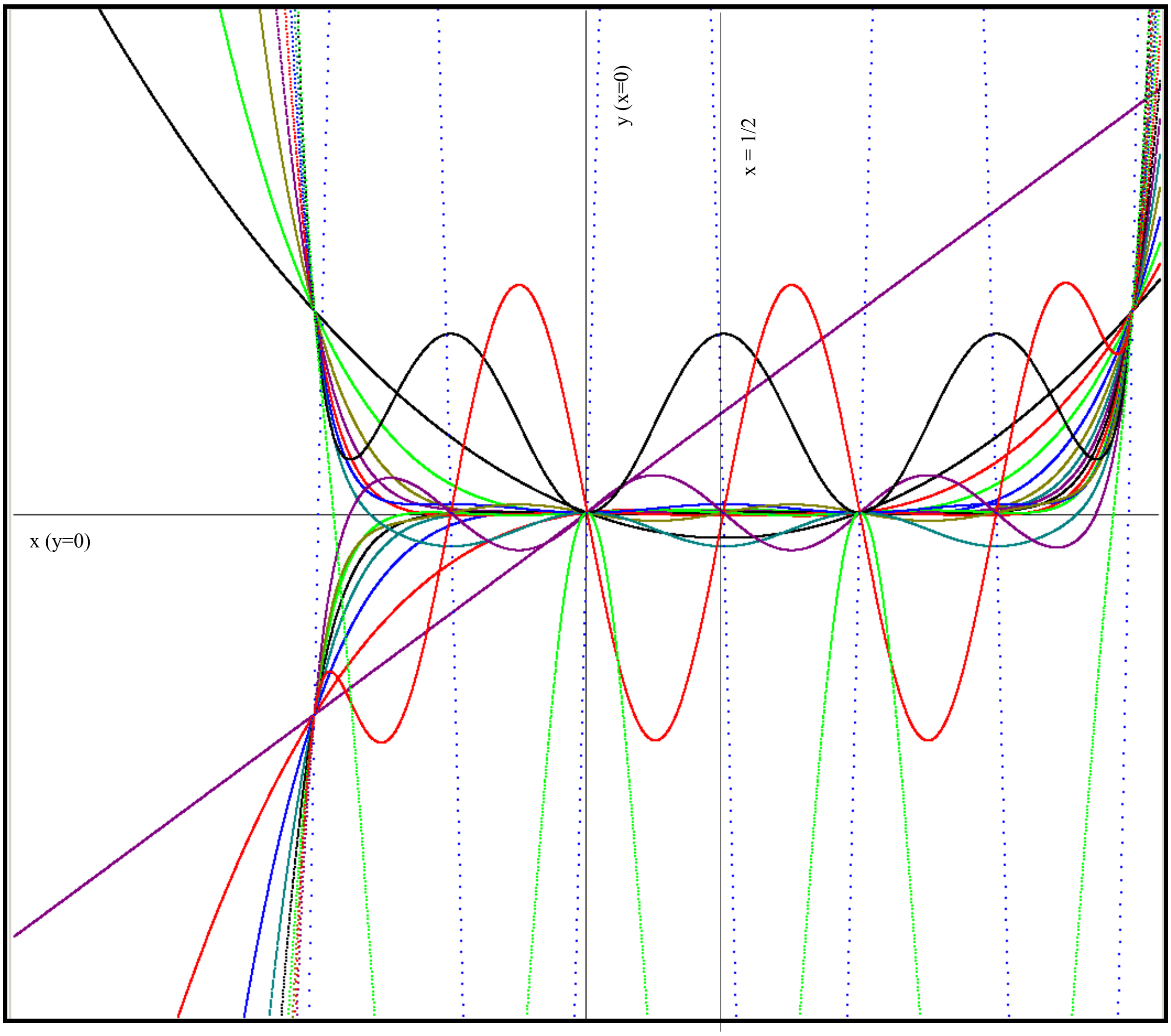
13. Sample Pari/Gp-Code

Here follows some example-coding(notation for the software Pari/GP):

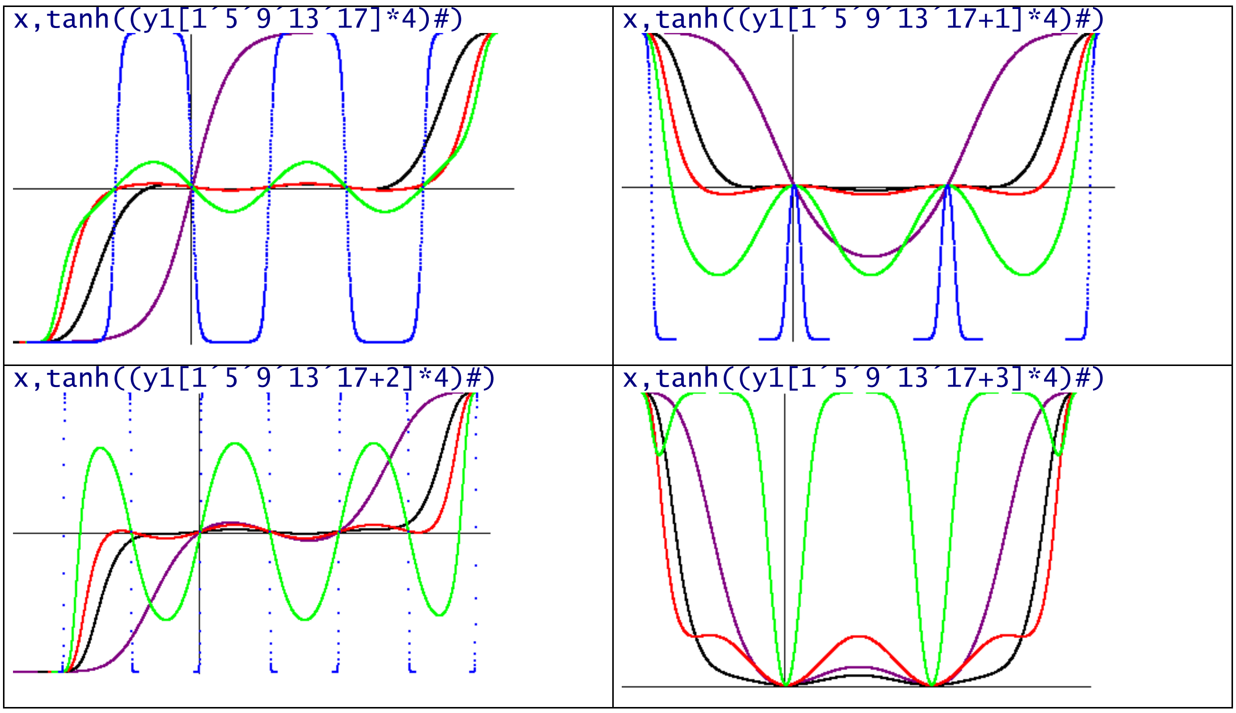
```
dim = 12
          \\ size for matrices
J = matdiagonal(vectorv(dim,r,(-1)^(r-1)))             \\ unit-matrix with altern. signs

\\ pascalmatrix and row/col-reduced pascalmatrix (P-I)* ------------------------

   P    = matpascal(dim-1)                                    \\ common pascalmatrix
        tmp = matpascal(dim) - matid(dim+1)
   P1_I = matrix(dim,dim,r,c,tmp[1+r,c])     \\ empty (first)row and (last)col: dimension is dim
\\----------------------------------------------------------------------------

\\ compute Gm, Gp
   Gm = P1_I^-1                                                  \\ compute Gm
   Gp = J * Gm * J                                  \\ compute Gp by formula (5.11)
\\----------------------------------------------------------------------------

\\ error-estimate: sum of absolute differences of entries of two matrices--------------------
              errest(M1,M2) = sum(r=1,#M[,1],  sum(c=1,#M[1,],  abs(M1[r,c]-M2[r,c]))))

\\ is Gp * J * Gp^-1 eigensystem of PJ ?
        print(errest(P*J , Gp * J * Gp^-1))

\\ is Gm * J * Gm^-1 eigensystem of JP ?
        print(errest(J*P , Gm * J * Gm^-1))

\\ is P * Gm = Gp ?
        print(errest(P *Gm , Gp))

\\-------------------------------------------------------------------------

\\ G - and H-funktions at x={-2, -1.5, -1,-0.5, 0, 0.5, 1, 1.5, 2} ----------
        VX = matrix(dim,9,r,c,(-2+(c-1)*0.5)^r)          \\ vandermondematrix for x-values

   fG(x,n) = Gp[n,] * x* V(x)                                    \\ function for G_n(x)
   for(n=1,12,ploth(x=-2,2,fG(x,n)))

   fH(x,n) = Gp[n,] * V(x)                                       \\ function for H_n(x)
   for(n=1,12,ploth(x=-2,2,fH(x,n)))
```

14. Pictures

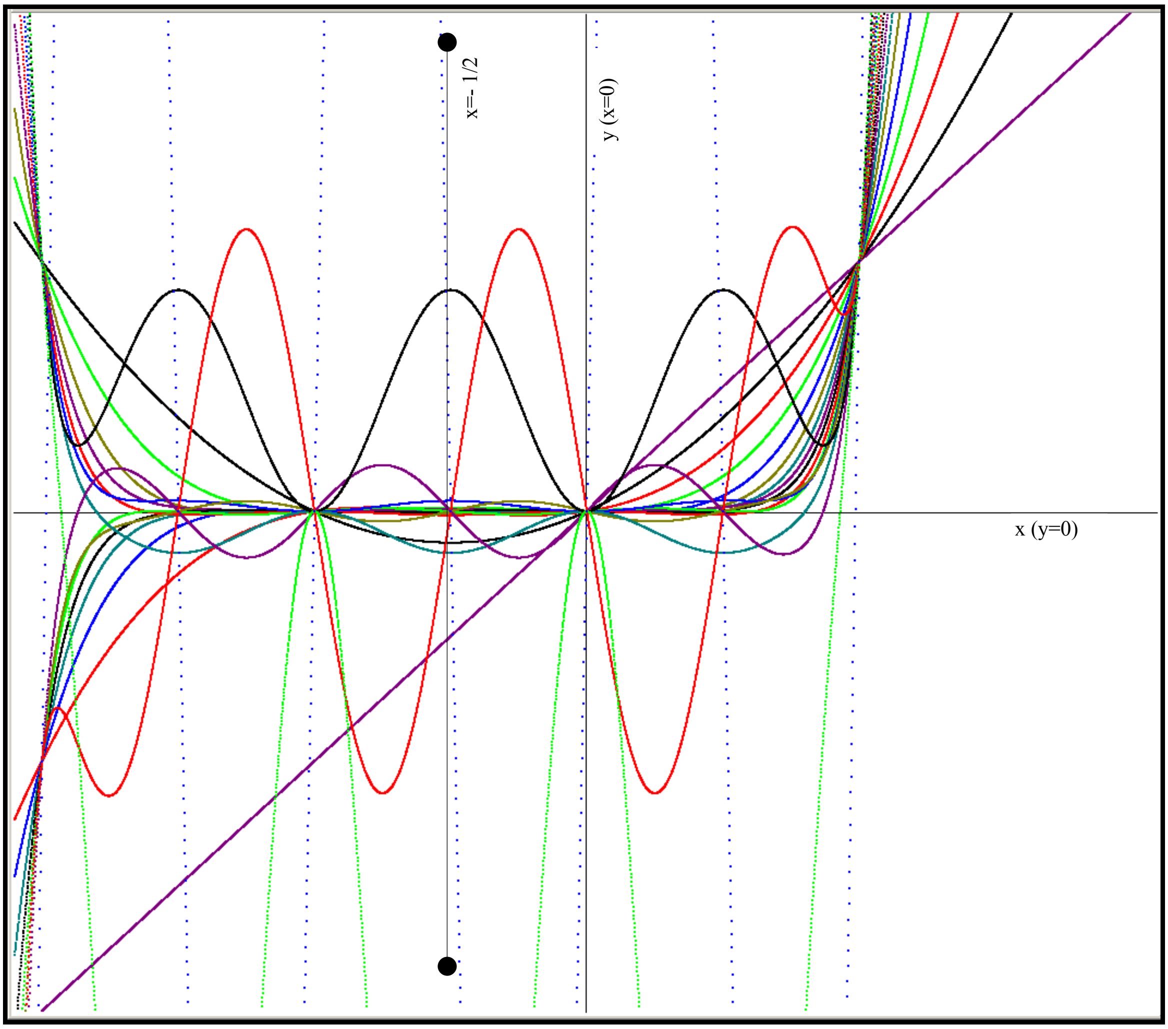
Function $G_{m,1..18}(\mathbf{x})$ (x=-2..+2, y=-2..+2)



and zoomed y'=*tanh*(y*4); in 4 groups of curves $G_{m,1'5'9'13}$ etc:

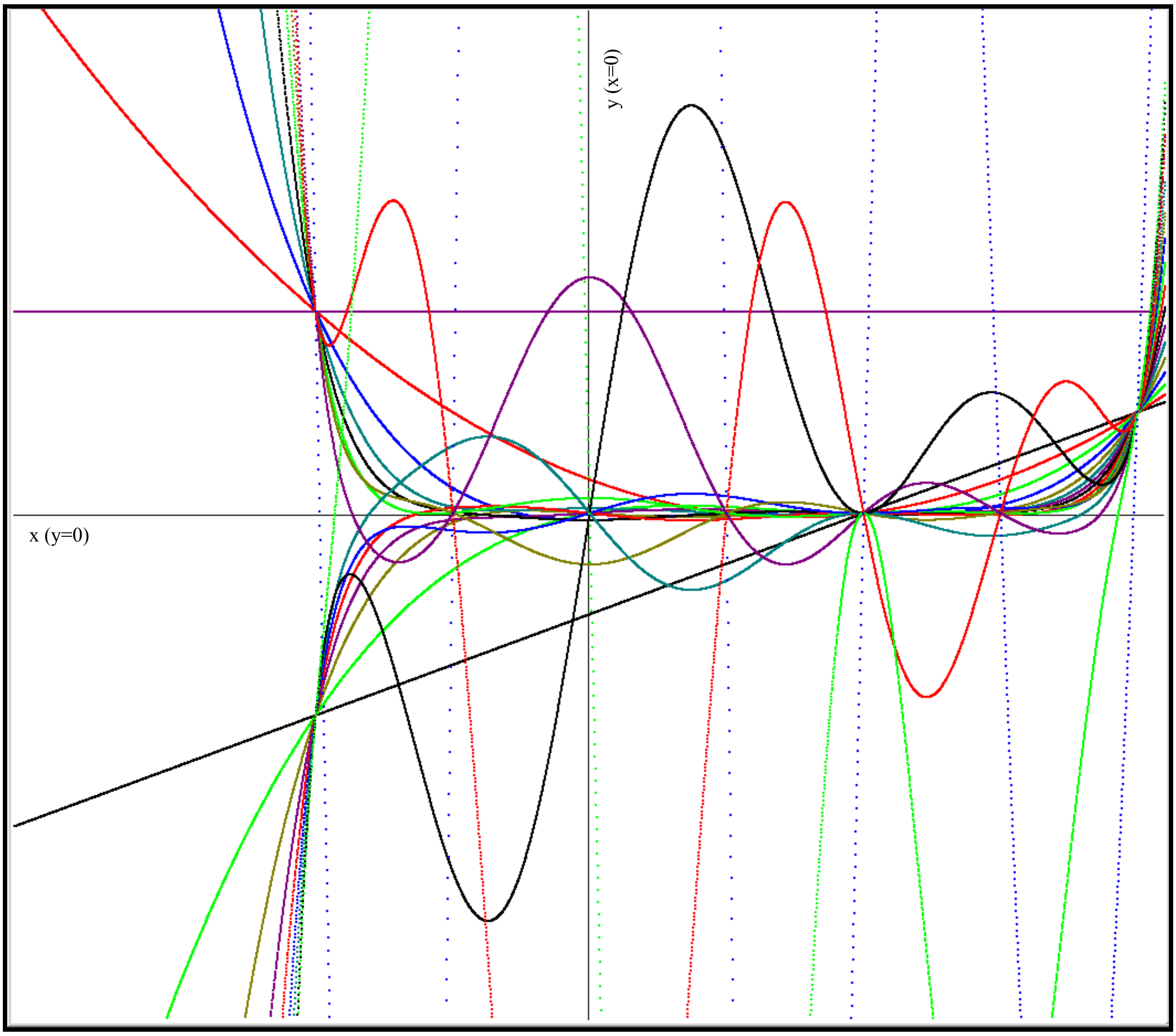


Function $\mathbf{G}_{p,1..18}(\mathbf{x})$ (x=-2..+2, y=-2..+2)

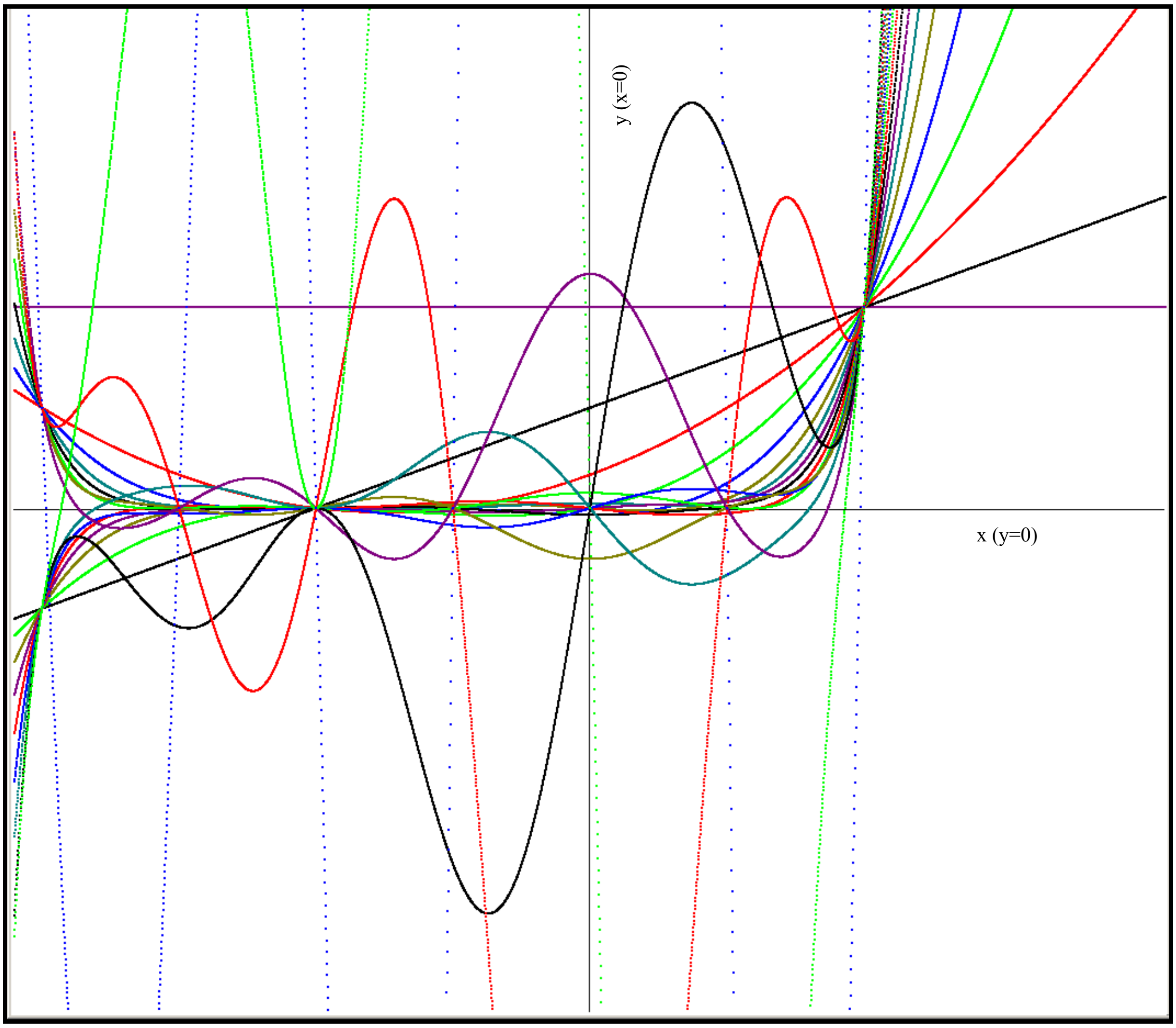


Note here (as before with $G_m(x)$) the similarity of the local extrema, which seems to improve with higher degree of n.

Funktion : $\boldsymbol{H_{m,1..18}(x)} = \boldsymbol{G_{m,1..18}(x)/x}$ *(x=-2..+2, y=-2..+2)*



Function : $\boldsymbol{H_{p,1..18}(x)} = \boldsymbol{G_{p,1..18}(x)/x}$ $(x=-2..+2, y=-2..+2)$



Literature & online-ressources

Online-Ressourcen:

The pascalmatrix as matrix-exponential:

[Helms1] Gottfried Helms (Kassel)
http://go.helms-net.de/math/binomial/PascalDreieckMatrixLog.pdf

[Edelman] Alan Edelman & Gilbert Strang, MIT
http://web.mit.edu/18.06/www/pascal-work.pdf

recursion-formula for bernoulli-numbers (eq. 30 and 31):

[mathworld] Eric Weissstein et al.
http://mathworld/wolfram.com/Bernoullinumbers

Scanned copy of the originaltext of Jakob Bernoulli:

[Mich] Bernoulli, Jakob: Wahrscheinlichkeitsrechnung (Ars conjectandi) von Jakob Bernoulli (1713)
translated and edited by R. Haussner.
(digitalized at University of Michigan, see:
http://www.hti.umich.edu/cgi/t/text/text-idx?c=umhistmath;idno=ABZ9501

A thorough discussion of the bernoulli-numbers, proposing the convention to use β_1=+*1/2* instead of –*1/2* (german):

> *"Dies ist vom systematischen Standpunkt aus einfacher und erweist sich als nützlicher als die konventionellen Definitionen über erzeugende Funktionen, denen eine gewisse Willkür anhaftet.*
>
> *So muss zum Beispiel die Frage, wie* B_1 *zu definieren ist, bei der Einführung mittels erzeugender Funktionen auf der Basis von Konventionen geregelt werden. Hingegen gibt die Bernoulli-Funktion eine konventionsfreie, eine analytische Antwort"* [Luschny], S.2.

[Luschny] : Peter Luschny (Straßburg)/ Hermann Kremer(Darmstadt)
http://www.dsmath.de/archiv/zahlen/BernoulliEuler.pdf

Webpage dedicated to the research on bernoulli-numbers, list of bernoulli-numbers of high index, free software for efficient computation of bernoulli-numbers:

[Kellner] Bernd Kellner (Göttingen)
http://www.bernoulli.org/

More on "Polynomials From Pascal's Triangle"

[Mathpages1] Mathforum@Drexel
http://mathpages.com/home/kmath304.htm

[OEIS_A002425] N.J.A.S. Sloane, "Online encyclopedia of integer sequences"
(*http://www.research.att.com/~njas/sequences/A002425*)

[Wikipedia] Picture of John Gotti (© public domain)
http://de.wikipedia.org/wiki/Bild:Johngotti1990.jpg

Version:

version 2.3.2	*(Aug 2009)*	*only textedits and translations of references, MatMate-code replaced by Pari/GP-code*
version 2.3	*(Mar 2006)*	*english version, better graphs*