Tetration

Mathematical Miniatures

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Continuous iteration of functions having a powerseries

A guide towards a general powerseries for continuous tetration

Abstract: In this article I discuss the formal process to determine integer, fractional and general iterates of a function which has, or: is given in a powerseries representation.

I intended to make this article is a very basic and self-containing introduction - just because I missed such a paper myself when I searched in my first weeks of considering iterating of functions. I give some examples of powerseries which are nicely configured for our purpose here to give the idea of such iterations and arrive at the powerseries representation of tetration (b^x iterated). Here "tetration" is understood as iterated exponentiation beginning at a start-parameter x. Also I show this for the "decremented iterated exponentiation" ($dxp_t(x)$ = t^x – 1) which I call U-tetration here.

Fractional iteration is introduced as interpolation of the coefficients of the (formal) power series of the integer iterations. Here my examples employ as a first and naive step a polynomial interpolation - so the coefficients of a fractional iterate of a power series are the interpolation between that of iterates at integer heights. A more general concept is then that of the logarithms or the diagonalization of the involved matrix-operators.

Thus the mathematical "engine" for this employs formal powers of the powerseries of the function f under discussion, and, although this can all be expressed by the functional notation, I find it much more convenient to represent all coefficients of the formal power series in ***one matrix-style*** *and operate with these matrices. This will prove extremely useful for the discussion of fractional iteration, since this can then be expressed by fractional powers of these matrices, for which a reliable and well-defined instrumentarium is already existent (and was in fact used by many authors, for instance Eri Jabotinsky, L. Comtet and more recently P.Walker, P. Gralewicz & K. Kowalski, R. Aldrovandi & L. P. Freitas and S. C. Woon and others).*

However – the polynomial interpolation of coefficients which is in different ways exemplified here is not the only path towards fractional iteration; there are various approaches to the (best) interpolation-method. One may recall the discussion of various approaches to define an interpolation for the factorial function, where one method has finally been singled out to have the "best"/most consistent properties for the use in numbertheory - namely the gamma-function as defined by L. Euler.

So even if the naive polynomial approach may not lead to "the best" solutions – the discussion here may still be advanteous for the general introduction into the whole concept (and of course of its possible shortcomings).

Contents

1. Iteration of functions, serial and matrix approach

1.1. Formal powerseries and their iterations

In the whole text we consider functions $f(x)$ which have a representation by a powerseries, basically

(1.1.1.) $f(x) = K + a x + b x^2 + c x^3 + d x^4 + \dots$

Iterations of f, where the function-value is reintroduced as its argument, like $f(x), f(f(x)),f(f(f(x)))$ may be indicated by an iterator-index h and the iteration may be denoted by a superscript-circle:

(1.1.2.) $f^{\circ h}(x) = f(f(f(\dots f(x))$ // with h-fold occurence of f

The following discussion of iterations is done by consideration of f as formal powerseries. In that context of formal powerseries the coefficients $(K,a,b,c,\dots)$ are discussed without respect to actual values of x, and thus considerations of convergence-radius wrt x are omitted as long as we look at algebraic properties and properties of composition.

To compute iterates of f, we insert $f(x)$ instead of x at each term in (1.1.1).

(1.1.3.) $f^{\circ 2}(x) = K + a f(x) + b f(x)^2 + c f(x)^3 + \dots$

Thus we need the formal expansions of each power of $f(x)$ in terms of its coefficients $(K,a,b,c,\dots)$ first, since now:

(1.1.4.)
$$\begin{aligned} f^{\circ 2}(x) &= K \\ &+ a (K + a x + b x^2 + c x^3 + d x^4 + \dots) \\ &+ b (K + a x + b x^2 + c x^3 + d x^4 + \dots)^2 \\ &+ c (K + a x + b x^2 + c x^3 + d x^4 + \dots)^3 \\ &+ \dots \end{aligned}$$

and the resulting powerseries for $f^{\circ 2}(x)$ can be given when the powers of the parentheses are expanded and equal powers of x are collected.

For instance the powerseries of $f(x)^2$, which is the third term only (in the above formula), begins with

(1.1.5.)
$$\begin{aligned} f(x)^2 = &\ K^2 \\ &+(\quad 2(Ka))\, x \\ &+(a^2+2(Kb))\, x^2 \\ &+(\quad 2(Kc+ab))\, x^3 \\ &+(b^2+2(Kd+ac))\, x^4 \\ &+(\quad 2(Ke+ad+bc))\, x^5 \\ &+(c^2+2(Kf+ae+bd))\, x^6 \\ &+ \dots \end{aligned}$$

and this may be thought continued analoguously to higher powers of $f(x)$.

For the second iterate we get by such expansions of the powers of $f(x)$:

(1.1.6.)
$$\begin{aligned} f^{\circ 2}(x) &= K \\ &+ a (K + a x + b x^2 + c x^3 + d x^4 + \dots) \\ &+ b (K^2+2Ka x + (a^2+2Kb) x^2 + (2Kc+2ba) x^3 + (2ac+b^2+2Kd) x^4 + \ ) \\ &+ c (K^3+3K^2a x + (3Ka^2+3K^2b) x^2+(3K^2c+(a^3+6Kba)) x^3+\dots) \\ &+ \dots \end{aligned}$$

and if the coefficients are collected according to their powers of x we get the following power series :

(1.1.7.)
$$\begin{aligned} f^{\circ 2}(x) &= K(1 + a + Kb + K^2c + \dots) \\ &+(a(\quad a + 2bK + 3cK^2 + \quad \dots)) \quad x \\ &+(\ b(\quad a + 2bK + 3cK^2 + 4dK^3 + \dots) \\ &\quad + a^2(\quad 1b + 3cK + 6dK^2 + \dots)) \quad x^2 \end{aligned}$$

$+(\quad c(\quad a + 2bK + 3cK^2 + 4dK^3 + \ldots)$
$\quad +2ab(\quad 1b + 3cK + 6dK^2 + \ldots)$
$\quad + a^3(\quad 1c + 4dK + \ldots) \quad x^3$
$\ldots$

Its coefficients must be determined by evaluation of the parentheses. Here we have the need of evaluation of series - and this series must of course be evaluated before $f^{\circ 2}(x)$ itself can be evaluated. The parentheses – btw – show the formal expansions of $f(x)$ and its derivatives at $x=K$, so it is also convenient, to express these parentheses as $f(K), f'(K), f''(K)/2!, \ldots$ and have a much shorter form at hand

(1.1.8.) $f^{\circ 2}(x) = f(K) + a f'(K) x + (a^2 f''(K)/2! + b f'(K))x^2 +$

but I'm not going to discuss this here in more detail. (see 3.2)

The sheer massiveness of such a formula as (1.1.7) explains, why we discuss simpler powerseries first. Most discussions about fractional iteration focus powerseries, whose K-term is zero. We get then the remarkable reduction:

(1.1.9.) $f(x) = a x + b x^2 + c x^3 + \ldots$

$f^{\circ 2}(x) = (\quad a(\quad 1a)) \quad x$

$+(\quad b(\quad 1a)$
$\quad + a^2(\quad 1b)) \quad x^2$

$+(\quad c(\quad 1a .)$
$\quad +2ab(\quad 1b)$
$\quad + a^3(\quad 1c) \quad x^3$
$\ldots$
$= a^2 x + (ba + ba^2)x^2 + (ca + 2ab^2 + ca^3) x^3 + \ldots$

An important further reduction occurs for functions, where[1] $a=1$:

(1.1.10.) $f(x) = x + b x^2 + c x^3 + \ldots$

$f^{\circ 2}(x) = x + 2b x^2 + (2c + 2b^2) x^3 + (b^3 + 5cb + 2d)x^4 + \ldots$
(see further development at 3.1)

In the following couple of examples we discuss only such functions, because we focus ourselves to understand the ***general idea*** of the interpolation to fractional iterates. Only in the description of the powerseries and iteration of (U-) tetration we'll reintroduce functions with a general coefficient a.

[1] Sometimes called "Schlicht"-functions

1.2. Matrices composed by coefficients of consecutive powers of formal powerseries

As we have so far introduced, the concept of formal powerseries ***and their powers*** is needed for the study of functional iteration.

However we have already seen, that writing the coefficients in their powers and compositions is a massive task. If instead we collect the coefficients of our powerseries ***and of its powers*** into a ***matrix***, such that we have

$$A=\begin{bmatrix} 1 & . & . & . & . & \dots \\ 0 & a & . & . & . & \dots \\ 0 & b & a^2 & . & . & \dots \\ 0 & c & 2ab & a^3 & . & \dots \\ 0 & d & 2ac+b^2 & 3a^2b & a^4 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

where in column *1* (counting begins at zero) occur the original coefficients of *f(x)*, in column *2* the coefficients of $(f(x))^2$ and so on, then this – together with the notation of matrix-algebra – allows us to make analysis of iteration much more readable: an explicite handling of the coefficents of higher iterates is very soon impossible otherwise.

So here is a short introduction into the matrix-notation of the coefficients of the formal powerseries.

First write the coefficients of the function *f* in a columnvector and the powers of *x* as a rowvector to denote the series as vector-multiplication:

To do this I introduce a vector *V(x)* of consecutive powers of a variable *x* (let's call it "Vandermondevector"):

(1.2.1.) $V(x) = \text{column}(1, x, x^2, x^3, \dots)$

(where "~" denotes the transpose) and with a column-vector A_1 of the original coefficients (we refer to the most general powerseries first)

(1.2.2.) $A_1 = \text{column}(K, a, b, c, d, \dots)$

we can then use the dot-product for the definition of *f(x)* :

(1.2.3.) $f(x) = V(x)\sim * A_1$

Since for iteration we need the vectors $A_0, A_1, A_2, \dots$ of coefficients for the consecutive powers of *f(x)* as well, we may arrange them in a matrix ***A*** to write

(1.2.4.) $A = \text{concatenate}(A_0, A_1, A_2, \dots)$

and then have the matrix-equation

(1.2.5.) $[1, x, x^2, x^3, \dots] * A = [1, f(x), f(x)^2, f(x)^3, \dots]$

The most interesting aspect of this is that in the result vector we have the same structure of consecutive powers of a certain value (here of *f(x)*), so the output-vector is again of the Vandermonde-type and thus can be reused as new input vector:

$V(x)\sim * A = V(f(x))\sim$
$V(f(x))\sim * A = V(f(f(x)))\sim$
and so on.

The top-left of ***A*** for a general powerseries *f* looks like

(1.2.6.)
$$A = \begin{bmatrix} 1 & \kappa & \kappa^2 & \kappa^3 & \kappa^4 & \dots \\ 0 & a & 2\kappa a & 3\kappa^2 a & 4\kappa^3\, a & \dots \\ 0 & b & a^2+2\kappa b & 3\kappa a^2+3\kappa^2 b & 6\kappa^2a^2+4\kappa^3\, b & \dots \\ 0 & c & 2ab+2\kappa c & a^3+6\kappa ab+3\kappa^2 c & 4\kappa a^3+6\kappa^2(2ab)+4\kappa^3\, c & \dots \\ 0 & d & b^2+2ac+2\kappa d & 3a^2b+3\kappa(b^2+2ac)+3\kappa^2 d & a^4+12\kappa a^2b+6\kappa^2(b^2+2ac)+4\kappa^3\, d & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

and in the columns we find the coefficients for the formal powerseries for $f(x)^0, f(x)^1, f(x)^2, \dots$.

In my first articles on this subject I called matrices like this, which can be used to transform a formal powerseries into another formal powerseries, a "***matrix-operator***" or simply "***operator***" - not knowing that canonical names "***Carlemanmatrix***" and "***Bellmatrix***" are already in use[2]. Here I leave it (in this third edition of this article) with the name "operator" because this name focuses more the functional aspect of such matrices.

So ***A*** is an *operator*, which transforms a powerseries in x into one in $f(x)$. This is expressed in the basic matrix-formula

(1.2.7.) $V(x)\sim * A = V(f(x))\sim$ // *if this is the case then A is called an "operator"*

For a better illustration of the composition of terms in a dot-product I like to write examples for matrix-multiplications in a grafic-like scheme, so we get an example for (1.2.7) and the most general case of some $f(x)$:

Example:

(1.2.8.)

$V(x)\sim * A = V(f(x))\sim$

	A_0	A_1	A_2	A_3	A_4	...
	1	K	K^2	K^3	K^4	...
	0	a	2Ka	$3K^2a$	$4K^3a$	...
	0	b	a^2+2Kb	$3Ka^2+3K^2b$	$6K^2a^2+4K^3b$	...
*	0	c	$2Kc+2ba$	$3K^2c+a^3+6Kba$	$4K^3c+4Ka^3+12K^2ba$	...
	...	...	...	...	...	
$[1, x, x^2, x^3, ...]$ =	[1,	f(x),	$f(x)^2$,	$f(x)^3$,	$f(x)^4$,	...]

1.3. K=0, triangular matrix-operators

If in the power series for $f(x)$ the coefficient $K=0$ (also[3] written as $f(0)=0$) then this picture of a *matrix-operator / Carlemanmatrix* simplifies considerably. We get triangular matrices ***A*** for such powerseries. Here is the matrix-multiplication-scheme with such a triangular matrix:

Example:

(1.3.1.)
$V(x)\sim * A = V(f(x))\sim$

	A_0	A_1	A_2	A_3	A_4
	1	.	.	.	.
	0	a	.	.	.
*	0	b	a^2	.	.
	0	c	2ab	a^3	.
	...	...	...	...	...
$[1, x, x^2, x^3, ...]$ =	[1,	f(x),	$f(x)^2$,	$f(x)^3$,	$f(x)^4$,...]

Since we have finitely many terms in each row now[4], powers of ***A*** are not affected by problems of non-convergence in the dot-products between multiple ***A***'s and the behave of the iteration of such functions is much easier to study. Infinite matrices of this type are called "rowfinite", and the "rowfiniteness" allows to compute the terms of powers of this matrix exactly up to the (finitely truncated) size of the matrix.

A typical property of these matrices is, that in the diagonal we have the consecutive powers of the second coefficient of the powerseries, a. The eigenvalues of a triangular matrix in the case of finite size are equal to its diagonal entries, and we want to assume here the same property being present for our matrices of infinite size, so we might say in generalization, that the eigenvalues of our triangular matrix-operator equals the set of consecutive powers of a. (This shall be discussed in context with fractional and general continuous iteration in later chapters).

[2] ... and extensively studied in the context of functional composition, see Eri Jabotinsky, P. Walker, R.Aldrovandi and more

[3] see for instance [AF97]

[4] see "rowfiniteness" of infinite matrices in [???]

1.4. Further reduced subcase a=1 (equivalent to "f(0)=0 and f '(0)=1 ")

An important subcase of this *K=0*-type functions are then functions, where *a=1*: then also the whole diagonal has the value *1*, and powers of the matrix (and thus iterations of the function) are again more simple to discuss. Here is a picture of such a matrix in the context of a matrix-product:

Example:

(1.4.1.)
$V(x)\sim * A = V(f(x))\sim$

	A_0	A_1	A_2	A_3	A_4
	1	.	.	.	.
	0	1	.	.	.
*	0	b	1	.	.
	0	c	2b	1	.
	...	...	...	...	...
$[1, x, x^2, x^3, ...]$ =	$[1,$	$f(x),$	$f(x)^2,$	$f(x)^3,$	$f(x)^4, ...]$

1.5. Iteration via matrix-operator

Iterations of *f(x)* can be computed, if in eq *(1.2.7), (1.3.1), (1.4.1)* in the vector *V(x)* on the lhs *f(x)* is inserted for *x* and the matrix-multiplication is repeated.

(1.5.1.)
$$V(x)\sim * A = V(f(x))\sim$$
$$V(f(x))\sim * A = V(f(f(x)))\sim = (V(x)\sim * A)*A$$
$$V(f^{\circ h}(x))\sim * A = V(f^{\circ h+1}(x))\sim = (((V(x)\sim * A)*A*...*A)*A$$

The product of multiple ***A*** can simply be written as matrix-power $\boldsymbol{A}^h$ by exploiting associativity in matrix-products:

(1.5.2.)
$$V(x)\sim * A = V(f(x))\sim$$
$$V(x)\sim * A^2 = V(f(f(x)))\sim$$
$$V(x)\sim * A^h = V(f^{\circ h}(x))\sim$$

Of course, because ***A*** is of infinite size, it must be made sure, that the dot-products are based on converging, or at least on summable, series expressions. So although basically we state, that we work on formal power series of some functions *f(x)* and thus have no concern about convergence of that series, we see now, that the question of convergence and/or summability occurs on a second level. In many cases similar to the examples here convergence shall be present -especially in the case of triangular matrix-operators where this is automatically the case, so we'll postpone the consideration of this problem for later.

2. Examples for functions without constant term (K=0)

2.1. Geometric series, a=1

The most simple example is the geometric series:

(1.1.1.1.) $f(x) = 1\,x + 1\,x^2 + 1\,x^3 + \ldots = x/(1-x)$

The radius of convergence is $-1 < x < 1$, and with the tools of divergent summation (Euler-summation) we may extend its domain to $-oo < x < 1$. Also we know, that we can extend its domain to all $x\neq 1$ due to analytic continuation. But this shall be of concern only as a sidenote since primarily we want to study the expansion of iterations into formal powerseries and the conversion into a matrix-problem.

2.1.2. The usual functional approach

Iteration means to substitute x by $f(x)$ and this means application of the binomial-theorem.

a) First we get

(1.2.1.1.) $f^{\circ 2}(x) = 1\,f(x) + 1\,f(x)^2 + 1\,f(x)^3 + \ldots$
$= 1\,(1\,x + 1\,x^2 + 1\,x^3 + \ldots)$
$+ 1\,(1\,x + 1\,x^2 + 1\,x^3 + \ldots)^2$
$+ 1\,(1\,x + 1\,x^2 + 1\,x^3 + \ldots)^3$
$+ ..$

Powers of $f(x)$ expanded

(1.2.1.2.) $f^{\circ 2}(x) = 1\,(1\,x + 1\,x^2 + 1\,x^3 + 1\,x^4 + 1\,x^5 + \ldots)$
$+ 1\,(1\,x^2 + 2\,x^3 + 3\,x^4 + 4\,x^5 + \ldots)$
$+ 1\,(1\,x^3 + 3\,x^4 + 6\,x^5 + \ldots)$
$+ 1\,(1\,x^4 + 4\,x^5 + \ldots)$
$+ \ldots.$

Equal powers of x collected

(1.2.1.3.) $f^{\circ 2}(x) = 1\,x + 2\,x^2 + 4\,x^3 + \ldots + 2^k\,x^{k+1} \ldots$

For $x=1/4$ we get

(1.2.1.4.) $f(1/4) = 1/4(1+1/4+1/4^2+\ldots) = 1/4 * 1/(1-1/4) = 1/3$
$f^{\circ 2}(1/4) = f(1/3) = 1/3\,(1+1/3+1/3^2+\ldots) = 1/3*1/(1-1/3) = 1/2$

other way:

(1.2.1.5.) $f^{\circ 2}(1/4) = 1/4(1 + 2/4 + 2^2/4^2 + 2^3/4^3 + \ldots)$
$= 1/4(1 + 1/2 + 1/2^2 + 1/2^3 + \ldots) = 1/4 * 2 = 1/2$

For the next iteration we need also the powers of $f^{\circ 2}(x)$ which becomes tedious to write down explicitely and we leave it aside.

b) In a view of the closed form of the function $f(x) = x/(1-x)$ we can also derive:

(1.2.1.6.) $$f^{\circ 2}(x) = \frac{f(x)}{1-f(x)} = \frac{\frac{x}{1-x}}{1-\frac{x}{1-x}} = \frac{\frac{x}{1-x}}{\frac{1-2x}{1-x}} = \frac{x}{1-x} * \frac{1-x}{1-2x} = \frac{x}{1-2x}$$

c***)*** Both derivations mean that the powerseries for the iterated $f(x)$ is

(1.2.1.7.) $f^{\circ 2}(x) = 1*x + 2*x^2 + 4*x^3 + 8*x^4 + \ldots = x * ((2x)^0 + 2x + (2x)^2 + (2x)^3 + \ldots)$

and by induction this can then be generalized to any positive integer power:

(1.2.1.8.) $$f^{\circ h}(x) = \frac{x}{1-hx} = x(1+hx+(hx)^2+\ldots) = f(hx)/h$$

Since h is here a simple parameter we may also conclude a version of ***fractional iteration*** from here!

2.1.3. The matrix-operator approach

If we use the matrix-notation for the coefficients of all formal powerseries $f(x)^0$, $f(x)$, $f(x)^2$,... as columns, we get the (shifted) pascal matrix ***P***[5] and then by matrix-multiplication the powers of $f(x)$:

(1.3.1.1.) $V(x)\sim * P = V(f(x))\sim$

$$P = \begin{bmatrix} 1 & . & . & . & . \\ 0 & 1 & . & . & . \\ 0 & 1 & 1 & . & . \\ 0 & 1 & 2 & 1 & . \\ 0 & 1 & 3 & 3 & 1 \end{bmatrix}$$

$$[\ 1 \quad x \quad x\verb|^|2 \quad x\verb|^|3 \quad x\verb|^|4\ ] \qquad [\ 1 \quad f(x) \quad f(x)\verb|^|2 \quad f(x)\verb|^|3 \quad f(x)\verb|^|4\ ]$$

and for $x=1/4$ we get -due to the entries in the second column of ***P*** - the geometric series with $q=1/4$

(1.3.1.2.) $f(1/4) = 1/4(1 + 1*1/4 + 1*1/4^2 + ...) = 1/4*(1/(1 - 1/4)) = 1/3$

The next iteration is perfomed as repeated matrix-multiplication using associativity of matrix operations

(1.3.1.3.) $V(x) * P = V(f(x))\sim$
$(V(x) * P)*P = V(x) * (P*P) = V(x)\sim*P^2 = V(f(f(x)))\sim$
...
$V(x) * P^h = V(f^{\circ h}(x))\sim$

The coefficients of the formal powerseries $f^{\circ 2}(x)$ are then in the second column of $\boldsymbol{P^2}$

(1.3.1.4.) $V(x)\sim * P^2 = V(f^{\circ 2}(x))\sim$

$$P^2 = \begin{bmatrix} 1 & . & . & . & . \\ 0 & 1 & . & . & . \\ 0 & 2 & 1 & . & . \\ 0 & 4 & 4 & 1 & . \\ 0 & 8 & 12 & 6 & 1 \end{bmatrix}$$

$$[\ 1 \quad x \quad x\verb|^|2 \quad x\verb|^|3 \quad x\verb|^|4\ ] \qquad [\ 1 \quad f^{\circ 2}(x) \quad f^{\circ 2}(x)\verb|^|2 \quad f^{\circ 2}(x)\verb|^|3 \quad f^{\circ 2}(x)\verb|^|4\ ]$$

and this agrees with the formula in the previous paragraphs.

2.1.4. Iteration to/with the inverse

The simpliness of the function $f(x)$ and of the associated matrix allows to introduce inverse iteration in a few lines of text.

a) By the functional approach we can ask:

(1.4.1.1.) $f(f^{\circ -1}(x)) = x$

Setting y for $f^{\circ -1}(x)$

(1.4.1.2.)
$y/(1-y) = x$
$1/(1/y-1) = x$
$1/y - 1 = 1/x$
$1/y = 1+1/x = (x+1)/x$
$y = x/(1+x)$

and this is then

(1.4.1.3.) $f^{\circ -1}(x) = x(1 - 1*x + 1*x^2 - 1*x^3 + ...-...)$

b) By the matrix-approach, the inverse, or better (since we have the case of infinite size) a "*matrix-reciprocal*", is defined, if by

(1.4.1.4.) $P * P^{-1} = I$

the matrix $\boldsymbol{P^{-1}}$ can be found.

[5] In my other texts I refer to the unshifted Pascalmatrix as ***P***. I do it here only in this example for simplicitiness. I hope, this does not introduce too much confusion

Since ***P*** is triangular, a principal reciprocal can also be triangular

(1.4.1.5.)

$$\begin{array}{cc} & * \begin{bmatrix} ? & . & . & . & . \\ ? & ? & . & . & . \\ ? & ? & ? & . & . \\ ? & ? & ? & ? & . \\ ? & ? & ? & ? & ? \end{bmatrix}_{P^{-1}} \\ \begin{bmatrix} 1 & . & . & . & . \\ 0 & 1 & . & . & . \\ 0 & 1 & 1 & . & . \\ 0 & 1 & 2 & 1 & . \\ 0 & 1 & 3 & 3 & 1 \end{bmatrix}_{P} & = \begin{bmatrix} 1 & . & . & . & . \\ . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 1 \end{bmatrix}_{I} \end{array}$$

This system of equations can iteratively be solved (evaluating rowwise) to get

(1.4.1.6.)

$P * P^{-1} = I$

$$\begin{array}{cc} & * \begin{bmatrix} 1 & . & . & . & . \\ 0 & 1 & . & . & . \\ 0 & -1 & 1 & . & . \\ 0 & 1 & -2 & 1 & . \\ 0 & -1 & 3 & -3 & 1 \end{bmatrix}_{P^{-1}} \\ \begin{bmatrix} 1 & . & . & . & . \\ 0 & 1 & . & . & . \\ 0 & 1 & 1 & . & . \\ 0 & 1 & 2 & 1 & . \\ 0 & 1 & 3 & 3 & 1 \end{bmatrix}_{P} & = \begin{bmatrix} 1 & . & . & . & . \\ . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 1 \end{bmatrix}_{I} \end{array}$$

and in the second column of ***P***$^{-1}$ we find the same coefficients as we got it by the functional approach above. The inverse function is then $f^{\circ-1}(x)$

(1.4.1.7.) $V(x)\sim * P^{-1} = V(f^{\circ-1}(x))\sim$

which occurs obviously, if the multiplication-scheme with ***P***$^{-1}$ is displayed (I abbreviated $f^{\circ-1}(x)$ by $F(x)$ here because of limitations of the bitmap):

(1.4.1.8.)

$V(x)\sim * P^{-1} = V(f^{\circ-1}(x))\sim = V(F(x))\sim$

$$\begin{array}{cc} & * \begin{bmatrix} 1 & . & . & . & . \\ 0 & 1 & . & . & . \\ 0 & -1 & 1 & . & . \\ 0 & 1 & -2 & 1 & . \\ 0 & -1 & 3 & -3 & 1 \end{bmatrix}_{P^{-1}} \\ \begin{bmatrix} 1 & x & x^2 & x^3 & x^4 \end{bmatrix} & = \begin{bmatrix} 1 & F(x) & F(x)^2 & F(x)^3 & F(x)^4 \end{bmatrix} \end{array}$$

Example evaluation: for $x=1/4$ we get

(1.4.1.9.) $f^{\circ-1}(1/4) = 1/4 \,/\, (1 + 1/4) = 1/(4+1) = 1/5$

Since this is assumed as the result of the inverse operation, we should get $1/4$ by applying the iteration on the result:

(1.4.1.10.) $f(1/5) = 1/5*(1 + 1/5 + (1/5^2 + \dots) = 1/5* 1/(1-1/5) = 1/5 * 5/4 = 1/4$

which is the expected result.

2.1.5. Fractional and general continuous iteration using the matrix-approach

The computation of the reciprocal in the previous is just a special case, of first negative power. This shall now be generalized to arbitrary powers / iteration heights.

Fractional powers of ***P*** are not obviously constructable. For finite matrices we have three options:

> a) (meaningful) ***interpolation of the list*** of consecutive integer powers of ***P***
> b) use of ***matrix-logarithm***
> c) ***eigensystem-decomposition***

For the matrix ***P*** we even have a fourth option

> d) ***similarity scaling by diagonal-matrices*** (comes out the be equivalent to matrix-logarithm-method)

Since ***P*** has a degenerated eigensystem, option c) is not applicable here.

a) Fractional powers by interpolation of a list of matrix-powers

Since we know, that the coefficients of the powerseries of $f^{\circ h}(x)$ occur in the 2'nd column of the h'th power of ***P***, we may collect all these columns from the consecutive powers of ***P*** and try, whether we can find a meaningful interpolation based on the progression of coefficients in equal rows. A list of these second columns for iteration heights $h=0,1,2,3,4,5,\ldots$ is

(1.5.1.1.)

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 4 & 9 & 16 & 25 \\ 0 & 1 & 8 & 27 & 64 & 125 \\ 0 & 1 & 16 & 81 & 256 & 625 \end{bmatrix}$$

Now we can apply a technique to find continuous polynomials in h, which interpolate each row, beginning with index $h=0$. What we get by any polynomial interpolation-procedure is the following matrix of coefficients for polynomials in h, where the first column is associated with h^0, the second column with h^1 and so on

(1.5.1.2.)

$$Poly = \begin{bmatrix} 0 & . & . & . & . & . \\ 1 & 0 & . & . & . & . \\ 0 & 1 & 0 & . & . & . \\ 0 & 0 & 1 & 0 & . & . \\ 0 & 0 & 0 & 1 & 0 & . \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

This is a very simple solution; it means, that for the interpolation of the rows in ***L*** for fractional h we have for the h'th entry:

> $row_1[h] = 1$
> $row_2[h] = 0 + 1*h$
> $row_3[h] = 0 + 0 \quad + 1*h^2$
> ...

and since the entries in the r'th row in ***L*** are the coefficients of x^r in the powerseries of $f^{\circ h}(x)$, we can construct this powerseries by

(1.5.1.3.) $f^{\circ h}(x) = 1x + hx^2 + h^2 x^3 + \ldots = x(1 + hx + (hx)^2 + (hx)^3 + \ldots)$

as we had by expansion and collection with the method of recursive series-substitution for the integer case.

Since the coefficients in ***Poly*** represent interpolation-polynomials, we may feel enabled to declare this as one meaningful interpolation-technique for fractional or even continuous and complex h, which means then the same type of iteration as well as an interpolation-technique for the coefficients in 2'nd column of an arbitrary power of ***P***.

So, $f^{\circ 1/2}(x)$ and its associated powerseries may now simply be determined by inserting $h=1/2$ in the above formula *(2.5.1.3)*.

b) Use of matrix-logarithm

The logarithm of a scalar *log(1+x)* is defined by a powerseries *("Mercator series")*

(1.5.1.4.) $\log(1+x) = x/1 - x^2/2 + x^3/3 - x^4/4 \ldots$
$\log(x) = (x-1)/1 - (x-1)^2/2 + (x-1)^3/3 - \ldots$

and a fractional *h*'th power is then defined as

(1.5.1.5.) $x^h = \exp(h * \log(x))$

The formula of the power series for the logarithm as well for the exponential can sometimes be formally extended to have matrices as their argument. With this simple triangular matrix this is possible and we can attempt to find

(1.5.1.6.) $\log(P) = (P - I)/1 - (P - I)^2/2 + (P - I)^3/3 - \ldots$

Since the diagonal of $(\mathbf{P} - \mathbf{I})$ is zero, $(\mathbf{P} - \mathbf{I})$ is nilpotent to the order of its size and we may approximate/extrapolate the case of infinite size by finite matrices of increasing size. We will always have only finitely many terms in the logarithm-series for consecutive sizes, and increasing the size does not affect the earlier computed results:

(1.5.1.7.)

$\log(P_{2x2}) = (P_{2x2} - I)/1 - (P_{2x2} - I)^2/2 + \mathbf{0} - \mathbf{0} + \ldots - \ldots$ $\begin{bmatrix} 0 & . \\ . & 0 \end{bmatrix}$

$\log(P_{3x3}) = (P_{3x3} - I)/1 - (P_{3x3} - I)^2/2 + (P_{3x3} - I)^3/3$ $\begin{bmatrix} 0 & . & . \\ 0 & 0 & . \\ 0 & 1 & 0 \end{bmatrix}$

$\log(P_{4x4}) = \Sigma_{k=1..4} (-1)^{k-1} * (P_{4x4} - I)^k/k$ $\begin{bmatrix} 0 & . & . & . \\ 0 & 0 & . & . \\ 0 & 1 & 0 & . \\ 0 & 0 & 2 & 0 \end{bmatrix}$

$\log(P_{5x5}) = \Sigma_{k=1..5} (-1)^{k-1} * (P_{5x5} - I)^k/k$ $\begin{bmatrix} 0 & . & . & . & . \\ 0 & 0 & . & . & . \\ 0 & 1 & 0 & . & . \\ 0 & 0 & 2 & 0 & . \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix}$

We may multiply *log(P)* by an arbitray constant *h* and compute the exponential similarly by applying the exponential series to the matrix-argument *(this can even be done symbolically keeping the height parameter h indeterminate)* and we get

(1.5.1.8.)
$P^h = \exp(h*\log(P)) =$ $\begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & h & 1 & . & . & . \\ 0 & h^2 & 2*h & 1 & . & . \\ 0 & h^3 & 3*h^2 & 3*h & 1 & . \\ 0 & h^4 & 4*h^3 & 6*h^2 & 4*h & 1 \end{bmatrix}$ P^h

from which we find, using

(1.5.1.9.) $V(x)\sim * P^h = V(f^{\circ h}(x))\sim$

and the vectorial product of the rowvector $V(x)\sim$ with the columns of $\mathbf{P}^h$ that the iterates $f^{\circ h}(x)$ and its powers are

(1.5.1.10.) $f^{\circ h}(x)^0 = 1$
$f^{\circ h}(x)^1 = 0 + 1\,x + h\,x^2 + h^2\,x^3 + \ldots = x\,(1 + 1hx + 1(hx)^2 + 1(hx)^3 + \ldots)$
$f^{\circ h}(x)^2 = 0 + 0\,x + 1\,x^2 + 2h\,x^3 + \ldots = x^2(1 + 2hx + 3(hx)^2 + 4(hx)^3 + \ldots)$
$f^{\circ h}(x)^3 = 0 + 0\,x + 0\,x^2 + 1\,x^3 + \ldots = x^3(1 + 3hx + 6(hx)^2 + 10(hx)^3 + \ldots)$
...

where the expression for $f^{\circ h}(x)^1 = f^{\circ h}(x)$ is the same result as we got with the computation via iterated substitution of the powerseries and the polynomial matrix-interpolation.

d) similarity scaling

Using the binomial-theorem one can also show, that the *similarity-scaling* by a diagonal-vector of the Vandermonde-type $V(h)$ holds:

(1.5.1.11.) $P^h = {}^dV(h) * P * {}^dV(h)^{-1}$

Again fractional and even complex iterates can this way be defined. This is because the similarity scaling of ***P*** (to get powers of itself) needs only dot products with diagonal matrices. For diagonal-matrices any power is defined by the same power at the scalar entries of its diagonal. So we may describe any fractional (or even complex!) "height" h of iteration by the matrix-formula

(1.5.1.12.)
$$\begin{aligned} P^h &= {}^dV(h) * P * {}^dV(h)^{-1} \\ V(x)\sim * P^h &= V(x)\sim * {}^dV(h) * P * {}^dV(1/h) \\ &= V(hx)\sim * P * {}^dV(1/h) \end{aligned}$$

and since $f^{\circ h}(x)$ occurs in the second column of the result (column-index 1)

(1.5.1.13.) $f^{\circ h}(x) \quad = V(f^{\circ h}(x))\sim [1]$

it is also

(1.5.1.14.)
$$\begin{aligned} f^{\circ h}(x) &= V(hx)\sim *P [1] *1/h \\ &= V(hx)\sim * [0,1,1,....]\sim /h \end{aligned}$$

(1.5.1.15.)
$$\begin{aligned} &= 1*0 + hx/h + (hx)^2/h + (hx)^3/h + ... \\ &= \quad x(1 + (hx) + (hx)^2 + ...) \end{aligned}$$

for any fractional or continuous value of h.

Note, that the similarity-scaling is essentially equivalent to the matrix-logarithm-method, since

(1.5.1.16.) ${}^dV(h)*P*{}^dV(1/h) = {}^dV(h)*exp(log(P))*{}^dV(1/h) = exp({}^dV(h)*log(P)*{}^dV(1/h))$

and the inner part comes out to be equivalent to the scalar-multiplication of $log(\mathbf{P})$ by h, since $log(\mathbf{P})$ is just the first principal subdiagonal containing $[1,2,3,...]$ in an otherwise empty matrix .

2.1.6. Conclusion

All shown matrix-methods ***a)***, ***b)***, ***d)*** give the same result for the fractional iteration for the geometric series as the usual methods in the previous subchapter:

(1.6.1.1.) $f(x) = x + x^2 + x^3 + \ldots$
$f^{\circ h}(x) = x(1 + (hx) + (hx)^2 + (hx)^3 + \ldots$

such that, for instance for $x=1/2$ the first half-iterate is

(1.6.1.2.) $f^{\circ 1/2}(1/2) = 1/2\ (1 + 1/4 + 1/4^2 + \ldots) = 1/2/(1-1/4) = 1/2\ /(3/4) = 2/3$

and the next half-iterate is

(1.6.1.3.) $f^{\circ 1/2}(2/3) = 2/3\ (1 + 1/3 + 1/3^2 + \ldots) = 2/3/(1-1/3) = 2/3\ /(2/3) = 1$

From the nice matrix-representation involving simply matrix-powers a first general law for the iterator-index can be derived ("***Additivity of iterations***"):

(1.6.1.4.) $f^{\circ a}(f^{\circ b}(x)) = f^{\circ a+b}(x)$

which is compatible with the matrix-power-approach:

(1.6.1.5.)
$$\begin{aligned} V(x)\sim\ * P^a &= V(f^{\circ a}(x))\sim \\ (V(x)\sim * P^a) * P^b &= V(x)\sim * (P^a * P^b) \\ &= V(x)\sim * P^{a+b} \\ &= V(f^{\circ a+b}(x))\sim \end{aligned}$$

according to the general rules of matrix-algebra and is fundamental for this approach to fractional iteration.

Note, that the various methods, interpolation of powers, matrix-logarithm and similarity-scaling allow (and provide) only interpolation based on fractional **powers** of the matrix, giving polynomials and powerseries of a parameter h, and is only "exact" as far as the coefficient a in the basic powerseries is $a=1$. For other conditions (and thus matrices) they are possibly uncomfortable and require considerations of convergence in the sequence of matrix-powers itself. The eigensystem-approach, if it is available, gives more flexibility and the convergence-criteria are much more obvious.

2.2. The sine-function[6], K=0, a=1

The formal powerseries for the sine-function is

(2.2.1.1.) $\sin(x) = x/1! - x^3/3! + x^5/5! - x^7/7! + ... - ...$

its coefficients can be written as infinite vector

(2.2.1.2.) $[0, 1, 0, -1/3!, 0, 1/5!, 0, -1/7!, ...]$

So we use another example of the type $K=0$, $a=1$ (or $f(0)=0$,$f'(0)=1$). The matrix-operator, computed by the method described above has the top-left-edge

(2.2.1.3.) $$SIN = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 0 & 1 & . & . & . \\ 0 & -1/6 & 0 & 1 & . & . \\ 0 & 0 & -1/3 & 0 & 1 & . \\ 0 & 1/120 & 0 & -1/2 & 0 & 1 \end{bmatrix}$$

Its application as coefficients for powerseries in x gives the powers of $\sin(x)$:

(2.2.1.4.) $V(x)\sim * SIN = V(\sin(x))\sim = [1, \sin(x), \sin(x)^2 , \sin(x)^3, ...]$

which is iterable, since the form of the result is of the same form as the left-multiplicator: a Vandermonde vector consisting of consecutive powers of a parameter.

So we extend the notation for the iterable sin-function to

(2.2.1.5.) $\sin^{\circ h}(x) = \sin(\sin(...\sin(x)...))$ //h iterations

For instance, the second power of ***SIN*** gives $\sin^{\circ 2}(x) = \sin(\sin(x))$, with an actual value for x, $x=1$ this should give

(2.2.1.6.) $\sin^{\circ 2}(1) = 0.745624141666...$

The second power of ***SIN*** has the top-left edge:

(2.2.1.7.) $$SIN^2 = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 0 & 1 & . & . & . \\ 0 & -1/3 & 0 & 1 & . & . \\ 0 & 0 & -2/3 & 0 & 1 & . \\ 0 & 1/10 & 0 & -1 & 0 & 1 \end{bmatrix}$$

and used as matrix-operator on the vandermonde-vector with parameter 1 this gives

(2.2.1.8.)

$V(1)\sim *SIN^2 =$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 0 & 1 & . & . & . \\ 0 & -1/3 & 0 & 1 & . & . \\ 0 & 0 & -2/3 & 0 & 1 & . \\ 0 & 1/10 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1.000 & 1.000 & 1.000 & 1.000 & 1.000 \end{bmatrix} \quad \begin{bmatrix} 1 & 0.7456 & 0.5560 & 0.4145 & 0.3091 & 0.2304 \end{bmatrix}$$

where the second column of the result is the above value $\sin^{\circ 2}(1) = 0.7456$ and the remaining entries its consecutive powers.

The inverse of $\sin(x)$ (= $\sin^{-1}(x)$, $\arcsin(x)$ or "negative height" $-h$)

The inverse of ***SIN*** should give the coefficients for the powerseries of $\sin^{\circ -1}(x)$ (or $\arcsin(x)$) . It can simply be computed by inversion of the SIN - matrix and also by the matrix-logarithm and using $h=-1$ for the matrix-exponential as shown below:

(2.2.1.9.) $SIN^{-1} = \exp(-\log(SIN))$

[6] See Hans Töpfer (1940) [Töpfer] for a rigorous discussion of iteration of the sine and the cosine (german)

The top left of ***SIN***$^{-1}$ looks like:

(2.2.1.10.) $$SIN^{-1} = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 0 & 1 & . & . & . \\ 0 & 1/6 & 0 & 1 & . & . \\ 0 & 0 & 1/3 & 0 & 1 & . \\ 0 & 3/40 & 0 & 1/2 & 0 & 1 \end{bmatrix}$$

where the entries of the second column are that of the Taylor-series for $sin^{-1}(x)$ according to the "Handbook of mathematical functions" *([A&S] pg 81)*

4.4.40

$$\arcsin z = z + \frac{z^3}{2\cdot 3} + \frac{1\cdot 3 z^5}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5 z^7}{2\cdot 4\cdot 6\cdot 7} + \ldots \quad (|z|<1)$$

(2.2.1.11.) $sin^{-1}(x) = x + 1/(2*3)\ x^3 + (1*3)/(2*4*5)\ x^5 + (1*3*5)/(2*4*6*7)\ x^7 + \ldots$

The "continuous" iteration to fractional or even complex "height"

a) Using interpolation of list and the POLY-matrix

The polynomial-interpolation-approach gives the matrix of coefficients ***POLY*** for the interpolation-polynomials (read by rows, first column associated to h^0) as already shown in the example with the geometric series:

(2.2.1.12.) $$POLY = \begin{bmatrix} 0 & . & . & . & . & . \\ 1 & 0 & . & . & . & . \\ 0 & 0 & 0 & . & . & . \\ 0 & -1/6 & 0 & 0 & . & . \\ 0 & 0 & 0 & 0 & 0 & . \\ 0 & -1/30 & 1/24 & 0 & 0 & 0 \end{bmatrix}$$

This means, assuming coefficients $a_{h,k}$ as entries of a vector A_h which define the powerseries

(2.2.1.13.) $sin^{\circ h}(x) = a_{h,1}\ x + a_{h,3}\ x^3 + a_{h,5}\ x^5 + \ldots = V(x)\sim * A_h$

to get the following polynomial expressions in the height parameter h:

(2.2.1.14.) $a_{h,1} = 1 \qquad a_{h,3} = -1/6\ h \qquad a_{h,5} = -1/30\ h + 1/24\ h^2$

and so on. The powerseries for non-integer iteration heights with h left indeterminate is then

(2.2.1.15.) $$\begin{aligned} sin^{\circ h}(x) = & \ 1\ * x \\ & + (0 - 1/6\ h) * x^3 \\ & + (0 - 1/30\ h + 1/24\ h^2) * x^5 \\ & + (0 - 41/378\ h + 1/45\ h^2 - 5/432\ h^3) * x^7 \\ & + (0 - 4/945\ h + 67/5670\ h^2 - 71/6480\ h^3 + 35/10368\ h^4) * x^9 \\ & \ldots \end{aligned}$$

The half-iterate uses $h=1/2$ and we get the powerseries

(2.2.1.16.) $sin^{\circ 1/2}(x) = x - 1/12\ x^3 - 1/160\ x^5 - 53/40320\ x^7 - 23/71680\ x^9 - 92713/1277337600\ x^{11} + O(x^{13})$

b) Interpolation using the matrix-logarithm

The matrix-logarithm of ***SIN*** can easily be determined, since its diagonal contains the unit only and so the matrix-terms for the logarithm-series are nilpotent to the order of matrix-size. The top-left edge of the matrix-logarithm is then:

(2.2.1.17.) $$log(SIN) = \begin{bmatrix} 0 & . & . & . & . & . \\ 0 & 0 & . & . & . & . \\ 0 & 0 & 0 & . & . & . \\ 0 & -1/6 & 0 & 0 & . & . \\ 0 & 0 & -1/3 & 0 & 0 & . \\ 0 & -1/30 & 0 & -1/2 & 0 & 0 \end{bmatrix}$$

and the general *h*'th power of ***SIN*** according to $SIN^h = exp(h*log(SIN))$ is

(2.2.1.18.) $$exp(h*log(SIN)) = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 0 & 1 & . & . & . \\ 0 & -1/6*h & 0 & 1 & . & . \\ 0 & 0 & -1/3*h & 0 & 1 & . \\ 0 & 1/24*h^2-1/30*h & 0 & -1/2*h & 0 & 1 \end{bmatrix}$$

Here the entries of the second column provide the (polynomial) coefficients for the $sin^{\circ h}()$-powerseries in x and we find, that this agrees with the solution by polynomial interpolation.

Then the half-power $SIN^{1/2}$ is

(2.2.1.19.) $$SIN^{1/2} = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 0 & 1 & . & . & . \\ 0 & -1/12 & 0 & 1 & . & . \\ 0 & 0 & -1/6 & 0 & 1 & . \\ 0 & -1/160 & 0 & -1/4 & 0 & 1 \end{bmatrix}$$

The powerseries for $sin^{\circ 1/2}(x)$ according to this is the same as in the previous subsection

(2.2.1.20.) $$sin^{\circ 1/2}(x) = x - 1/12\ x^3 - 1/160\ x^5 - 53/40320\ x^7 - 23/71680\ x^9 - 92713/1277337600\ x^{11} + O(x^{13})$$

We get from the matrix-formula in numbers for two subsequent half-iterates the expected integer-iterated value *(but see remarks on non-convergene in c) below!)*

(2.2.1.21.)

$$V(1)\sim * SIN^{1/2} = V(y) \sim$$
$$V(y)\sim * SIN^{1/2} = V(z)\sim$$
$$y = sin^{\circ 1/2}(1) \qquad z = sin^{\circ 2/2}(1)$$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 0 & 1 & . & . & . \\ 0 & -1/12 & 0 & 1 & . & . \\ 0 & 0 & -1/6 & 0 & 1 & . \\ 0 & -1/160 & 0 & -1/4 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1.000 & 1.000 & 1.000 & 1.000 & 1.000 \end{bmatrix} \qquad \begin{bmatrix} 1.000 & 0.9087 & 0.8258 & 0.7504 & 0.6819 & 0.6196 \end{bmatrix}$$

Next half-iteration arrives at the first integer iterate:

$$\begin{bmatrix} 1.000 & 0.9087 & 0.8258 & 0.7504 & 0.6819 & 0.6196 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0.8415 & 0.7081 & 0.5958 & 0.5014 & 0.4219 \end{bmatrix}$$

which is what we of course expect.

c) Divergent series for fractional iterates

Unfortunately, in every power series for fractional heights the coefficients increase above any bound and we cannot simply evaluate that power series to some exact limit. But interestingly that series can be interpreted as "asymptotic series" for which there is some standard handling for approximations[7]: we'll find some approximate value after evaluation to a certain number of terms - meaning we use only a conveniently truncated version of that power series.

Another option is a summation-method for divergent summation[8] - but we leave it here for the approximation by the truncated series. For $x=1$ we get this way in two steps of half-iteration with $h=1/2$

(2.2.1.22.)
$$\begin{aligned} sin^{\circ 1/2}(1) &= 0.908708... \\ sin^{\circ 1/2}(0.908708...) &= 0.841471.. \\ &= sin^{\circ 2/2}(1) = sin^{\circ 1}(1) = sin(1). \end{aligned}$$

This agrees with the direct computation $sin(1) = 0.841471...$

[7] See for instance the monography of K. Knopp, Chap XIV the subsection on asymptotic series

[8] See for a basic introduction wikipedia or see K. Knopp, Chap XIII on divergent series

3. Symbolic fractional iteration for schlicht-functions

Well, after we have seen two simple and practical examples we should look at the problem of fractional iteration in more generality. So we discuss the iteration with unknown/with general coefficients in a formal powerseries or we can say, the symbolic notation of integer and fractional iteration of functions based on their representation as formal powerseries.

3.1. Formula for interpolation for a general powerseries $f(x)= 1\,x + b\,x^2 + c\,x^3 + \ldots$

We restrict ourselves to the case $K=0$, $a=1$ here (or: $f(0)=0$, $f'(0)=1$)

(3.1.1.1.) $f(x) = 1\,x + b\,x^2 + c\,x^3 + d\,x^4 + e\,x^5 + \ldots$

First we build a table of differences of the second columns of powers of the associated matrix-operator, see table below.

In the first row we get the original coefficients of the powerseries of

(3.1.1.2.) $f^{\circ 0}(x) = 1\,x$
$f^{\circ 1}(x) = 1\,x + 1b\,x^2 + 1c\,x^3 + \ldots$
$f^{\circ 2}(x) = 1\,x + 2b\,x^2 + \ldots$
...

where the table-columns are associated with the according powers of the iteration-height h.

In the subsequent rows the forward-differences of the coefficients, where the forward-differences are of the order of rowindex r.

We find mixtures of differences of different order of progression for each symbolic coefficient (and their composites), where I highlighted different orders of progression by different colors.

(3.1.1.3.) Table of differences of symbolic coefficients along iterates of $f^{\circ h}(x)$, using $a=1$

diff. index	$f^{\circ 0}(x)$	$f^{\circ 1}(x)$	$f^{\circ 2}(x)$	$f^{\circ 3}(x)$
Δ^0 the formal powerseries itself	$1\,x$ 0 0 0 0	$1\ {*}x$ $1b * x^2$ $1c * x^3$ $1d * x^4$ $1e * x^5$ ...	$1\ {*}x$ $(2b) * x^2$ $(2c+2b^2) * x^3$ $(2d+5bc+b^3) * x^4$ $(2e+3(2bd+c^2)+5b^2c) * x^5$ ...	$1\,x$ $(3b) * x^2$ $(3c + 6b^2)* x^3$ $(3d + 15bc + 9b^3) * x^4$ $(3e + 9(2bd + c^2) + 41b^2c + 10b^4) * x^5$ . ...
Δ^1		$0\,x$ $1b\,x^2$ $1c\,x^3$ $1d\,x^4$ $1e\,x^5$	$0\,x$ $(1b)\,x^2$ $(1c + 2b^2)*x^3$ $(1d + 5bc + b^3)*x^4$ $(1e + 3(2bd + c^2)+5b^2c) * x^5$	$0\,x$ $(1b)\,x^2$ $(1c + 4b^2)*x^3$ $(1d + 10bc + 8b^3) * x^4$ $(1e + 6(2bd + c^2) + 36b^2c + 10b^4) * x^5$
Δ^2			$0\,x$ $0\,x^2$ $(2b^2)\,x^3$ $(5bc + b^3)*x^4$ $(3(2bd+c^2)+5b^2c) * x^5$	$0\,x$ $0\,x^2$ $(2bb)*x^3$ $(5bc + 7b^3) * x^4$ $(3(2bd + c^2) + 31b^2c + 10b^4) * x^5$
Δ^3				$0\,x$ $0\,x^2$ $0\ {*}x^3$ $(6b^3) * x^4$ $(26b^2c + 10b^4) * x^5$

Legend: linear progression quadratic progression cubic progression biquadratic progression of coefficients

If we write the numeric coefficients at each (combination) of the symbolic coefficients as function of the iteration-index h (which indexes also the columns in the table), then this gives – as denoted in the last column, where we order for like (compositions of) symbolic coefficients – the following:

diff. index	$f^{\circ 0}(x)$	$f^{\circ 1}(x)$	$f^{\circ 2}(x)$	$f^{\circ h}(x)$
Δ^0				$1\ x$ $(hb)*x^2$ $(hc + h(h-1)b^2)*x^3$ $(hd + 5/2h(h-1)bc + h(h-1)(2h-3)b^3)*x^4$ $(he + 3/2h(h-1)(2bd + c^2) + h(h-1)(26h-37)b^2c + h(h-1)(h-2)(3h-4)b^4)*x^5$ . ...

The same, collected for like powers of h instead:

(3.1.1.4.) Y = *(matrix of polynomials for coefficients at powers of x dependend on h):*

	$*h^0$	$*h$	$*h^2$	$*h^3$	$*h^4$	
x^0	.	.	.	.	.	
x *	1					
x^2 *	0	b				/0!
x^3 *	0	$-\ b^2$ $+\ c$	b^2			/1!
x^4 *	0	$+\ 3b^3$ $-\ 5bc$ $+\ 2d$	$-\ 5b^3$ $+\ 5bc$	$2b^3$		/2!
x^5 *	0	$-\ 16b^4$ $+\ 37b^2c$ $-\ 18bd$ $-\ 9c^2$ $+\ 6e$	$+\ 36b^4$ $-\ 63b^2c$ $+\ 18bd$ $+\ 9c^2$	$-26b^4$ $+26b^2c$	$6b^4$	/3!

The latter table shows how the bivariate powerseries of $f^{\circ h}(x)$ (depending on x and h) develops, given the general form $f(x) = 1\ x + b\ x^2 + c\ x^3 + \dots$

(3.1.1.5.)

$$\begin{aligned} f^{\circ h}(x) = x\ &* 1 \\ &+ x^2 * (\ h*(b)\ ) \\ &+ x^3 * (\ h*(-b^2 + c) \\ &\qquad + h^2*(\ b^2\ )\ )/1! \\ &+ x^4 * (\ h*(\ 3b^3 - 5bc + 2d) \\ &\qquad + h^2*(-5b^3+5bc) \\ &\qquad + h^3*(\ 2b^3\ )\ )/2! \\ &+ \dots \end{aligned}$$

Here we can describe coefficients at powers of x as polynomials $A_{ix,ih}$ in h . We have

$$f^{\circ h}(x) = 1 * x + (A_{2,1}\ h)/0!\ x^2 + (A_{3,1}\ h + A_{2,2}\ h^2)/1!\ x^3 + (A_{4,1}\ h + A_{4,2}\ h^2 + A_{4,3}\ h^3)/2!\ x^4 +$$

Then

$A_{2,1} = b$

$A_{3,2} = b^2$ $\quad A_{3,1} = c - b^2$

$A_{4,3} = 2b^3$ $\quad A_{4,2} = 5b\ A_{3,1}$ $\quad A_{4,1} = 2d - 2bc - 3b\ A_{3,1}$

$A_{5,4} = 6b^4$ $\quad A_{5,3} = 26b^2\ A_{3,1}$ $\quad A_{5,2} = 9b\ A_{4,1} + 9\ A_{3,1}{}^2$ $\quad A_{5,1} = 6e - 9c^2 - 9b\ A_{4,1}\ -8b^2c + 29b^4$

....

The highest exponent of h at x^{k+1} is k, but may be lower, if some of the coefficients $b,c,d,...$in the basic formal powerseries are zero. Because of this we might expect, that the convergence-radius of the powerseries of $f^{\circ h}(x)$ decreases with increasing h roughly proportionally, so if the convergence-radius μ for $f^{\circ h}(x)$ is $\mu_h = c/h$ then it is $\mu_{h+1} \sim c/(h+1)$. (This is not perfectly true, since we observe also increasing coefficients with the powers of h for higher powers of x, but I didn't investigate these progressions deeper).

A more direct approach is the log/exp-function for the matrix-argument

Let's define $\mathbf{F}$ as the matrix-operator for the function $f(x) = x + bx^2 + cx^3 + dx^4 + \ldots$. Truncated to size $6x6$ this looks like:

(3.1.1.6.)
$$\mathbf{F} = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & b & 1 & . & . & . \\ 0 & c & 2*b & 1 & . & . \\ 0 & d & 2*c+b^2 & 3*b & 1 & . \\ 0 & e & 2*b*c+2*d & 3*c+3*b^2 & 4*b & 1 \end{bmatrix}$$

Then the log/exp-solution for the h'th iterate, letting h being indeterminate/symbolical gives

(3.1.1.7.) $FH(h) = Exp\ (\ h * Log\ \ (F))$

and the matrix looks like

$\mathbf{FH(h)}=$

1	.	.	...
.	1	.	...
.	+(1 h) b	1	...
.	+(1 h² - h) b² (h) c	+(2 h)b	...
.	+(h³ - 5/2 h²+3/2 h) b³ +(5/2 h²-5/2 h) bc +(h) d	+(3 h² - 2 h) b² +(2 h) c	...
.	+(h⁴ - 13/3 h³ + 6 h² - 8/3 h) b⁴ +(13/3 h³ - 21/2 h² + 37/6 h) b²c +(3/2 h² - 3/2 h) c² +(3 h² - 3 h) bd +(h) e	+(4 h³ - 7 h² + 3 h) b³ +(7 h² - 5 h) bc +(2 h) d	...
..	...	...	...

with the polynomials in h at the coefficients. *Note that for our solution we need only the entries in column* 1 *of the matrix, thus only the first few columns (and rows) are shown here and that column* 1 *is displayed unshaded.*

The expression for the power series depending on x and h comes then from the expansion of the dot-product

(3.1.1.8.) $V(x) * FH(h) = V\ (\ f^{\circ h}\ (\ x\ )\ )$

and of course agrees with that formula *(3.1.1.5)* above.

Also remember, that this polynomial-interpolation-approach (as well as the matrix-logarithm-approach) is only exact for powerseries with $K=0$.

For $0 \neq a \neq 1$ the symbolic description is also much more complicated, and so the above symbolic form covers only a subset of interesting functions. For instance, for (**T**-) tetration this means restriction to base $e^{1/e}$ and for **U**-tetration to base e.

Additional remarks: I didn't find simpler rules for extrapolation of the polynomials so far, I can construct them only by the according matrix-operations. Their complicated explicite structure needs enormous amounts of memory if only matrices for size 64 x 64 should be stored. A file for the symbolic representation of coefficients in textfomat was about 200 Mb diskspace, and since matrices of this size allow only powerseries of 64 terms (which gives good approximations only for a small range of its parameters), it seems better to compute the coefficients numerically for a current function f and possibly also either for a current fixed iteration-height h (keeping only x as variable) or for a fixed x (keeping only h as variable)

4. Fractional and continuous iteration for Tetration

4.1. U-Tetration : $t^x - 1$

4.1.1. Definition

I call **U**-tetration, for what Andrew Robbins[9] proposed the canonical name "*decremented iterated exponential*", for shortness here. It is, for a base t, defined by the function:

(4.1.1.1.) $f_t(x) = t^x - 1$ *(tetration-forum:= $dxp_t(x)$)*

and I use the letter ***U*** here for better reading

(4.1.1.2.) $U_t(x) = t^x - 1$

If the base-parameter $t=e = exp(1)$ I abbreviate this to $U(x)$ simply. Also, for the logarithm of the base-parameter t I write usually the small letter u, so

(4.1.1.3.) $u=log(t)$

(Note: there exists a statement of Erdös/Jabotinsky contradictory to the possibility of real iterates for fractional heights, see footnote[10])

4.1.2. Function and matrix-operator for U_e-tetration

For $\mathbf{U_e}$-tetration the function $U(x)$ resp its iteration $U^{\circ h}(x)$ is defined as follows:

(4.1.2.1.) $U(x) = U_e(x) = exp(x)-1$
$U^{\circ h}(x) = U^{\circ h-1}(exp(x)-1)$
$U^{\circ 0}(x) = x$

and the powerseries for $U(x)$ is just the exponential-series in x, where the constant is removed:

(4.1.2.2.) $U(x) = 1/1!\, x + 1/2!\, x^2 + 1/3!\, x^3 + \dots$

so the symbolic coefficients $a,b,c,d,\dots$ as in ***(3.1.1.1)*** are

(4.1.2.3.) $a=1, \quad b=1/2!, \quad c=1/3!, \dots$

The matrix ***U*** (for the primary function $U(x)=U^{\circ 1}(x)$) is of infinite size and has the top-left edge:

(4.1.2.4.)
$$\mathbf{U} = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 1/2 & 1 & . & . & . \\ 0 & 1/6 & 1 & 1 & . & . \\ 0 & 1/24 & 7/12 & 3/2 & 1 & . \\ 0 & 1/120 & 1/4 & 5/4 & 2 & 1 \end{bmatrix}$$

which –in my usual notation- equals just the matrix ***fS2F*** (a factorially similarity-scaling of the matrix of Stirling-numbers 2'nd kind with offset as given for instance in the wikipedia-definition). The third column gives the coefficients for the powerseries in x for $U(x)^2$, the fourth column those for $U(x)^3$ and so on. So we have, in matrix-notation

(4.1.2.5.) $\mathbf{V}(x)\sim * \mathbf{U} = \mathbf{V}(U(x))\sim$

or more explicitely

$$[1, x, x^2, x^3, \dots] * \mathbf{U} = [1, U(x), U(x)^2, U(x)^3, \dots]$$

---

[9] see [RO08]

[10] Erdös/Jabotinski state in [EJ61], there are "***no real non-integer iterates***" for $f(x)=e^x-1$ (meaning: no real value for fractional height for U_e-tetration), attributing this to I.N.Baker in [BA58]. However, Baker states only, that "*the radius of convergence*" of the powerseries in x for noninteger heights and $base=e$ "*is zero*". Here the issue is *not nonexistence*, but *convergence*. Moreover, using the well developed concept of divergent summation we may extend the domain for h and x beyond the classical radius of convergence. A heuristical inspection of the coefficients suggest, that the absolute value of terms is asymptotically of order $exp(r^2)$, where r is the index of term (row-index of matrix). This would mean, that the resulting powerseries in x cannot be Euler-, but possibly be Borel-summed (see [KN]).

In Abramowitz/Stegun [A&S] we find exactly this (however without the notion of a matrix) in the formulae for stirlingnumbers 2^{nd} kind.

4.1.3. The polynomial-interpolation approach to U_e-tetration

The list of *2*'nd columns of the consecutive powers of $\mathbf{U}$ begins like

(4.1.3.1.) $$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1/2 & 1 & 3/2 & 2 \\ 0 & 1/6 & 5/6 & 2 & 11/3 \\ 0 & 1/24 & 5/8 & 5/2 & 77/12 \end{bmatrix}$$

The *2*'nd column (for $h=1$) gives the coefficients for the powerseries in x for $U(x)=U^{\circ 1}(x)$, the column for $h=2$ those for $U^{\circ 2}(x)$ and so on.

Now we build polynomials in h for the interpolation of the entries for each row across the columns. Some examples:

In row 1 we have only ones, so that is the constant function:

$L_{1,h}=1$

In row 2 we have the linear increase of ½ and the function is obviously:

$L_{2,h}= h/2$

In row 3 we can express the entries by quadratic polynomial:

$L_{3,h}= 1/4\, h^2 - 1/12\, h$

and so on.

The matrix ***POLY*** of coefficients of that interpolating polynomials in h, which interpolate the terms for the powerseries for iterated $U^{\circ h}(x)$ as given in $\mathbf{L}$ according to its "height" is

(4.1.3.2.) $$POLY = \begin{bmatrix} 0 & . & . & . & . \\ 1 & 0 & . & . & . \\ 0 & 1/2 & 0 & . & . \\ 0 & -1/12 & 1/4 & 0 & . \\ 0 & 1/48 & -5/48 & 1/8 & 0 \end{bmatrix}$$

To get the coefficients a_k for the powerseries in x for the general height h of $U^{\circ h}(x)$ we postmultiply ***POLY*** with the vandermonde-vector of h : $V(h) = [1, h, h^2, h^3, h^4, ...]$

(4.1.3.3.) $\mathbf{U_1} = \mathbf{POLY} * \mathbf{V}(h) = [a_0, a_1, a_2, a_3,...]\sim$

Inserted into the symbolic description for the iterable version with a given h as iteration (or "height") -parameter this means in matrix-notation:

(4.1.3.4.) $\mathbf{V}(x) \sim * \mathbf{U_1} = U^{\circ h}(x)$

In serial notation the previous is (the reintroduced index e at U_e shall remind, that this is base e here)

(4.1.3.5.) $U_e^{\circ h}(x) = 1\, x\;\; h/2! * x^2 +(h/3! + h(h-1)/2!^2)* x^3 + (h/4! + 5/2h(h-1)/2!/3! + h(h-1)(2h-3)/2!^3) * x^4 ...$

$$\begin{aligned} &= (1) * x \\ &+ (0 + 1h) * x^2/2! \\ &+ (0 - 1h + 3h^2)/2 * x^3/3! \\ &+ (0 + 1h - 5h^2 + 6h^3)/2! * x^4/4! \\ &+ (0 - 4h + 30h^2 - 65h^3 + 45h^4)/3! * x^5/5! \\ &+ (0 + 22h - 273h^2 + 890h^3 - 1155h^4 + 540h^5)/4!\, x^6\ /6! \\ &+ .... \end{aligned}$$

For $h=0$ this degenerates to

(4.1.3.6.) $U_e^{\circ 0}(x) = 1*x + 0 + 0 + ...$

for $h=1$ this gives

(4.1.3.7.) $U_e^{\circ 1}(x) = 1*x + 1/2!\, x^2 + 1/3!\, x^3 + ...$

the known exponential-series for $exp(x)-1$, and for $h=-1$ this gives

(4.1.3.8.) $U_e^{\circ -1}(x) = 1*x - 1/2\, x^2 + 1/3\, x^3 + ...$

the known powerseries for $log(1+x)$ which connects then the negative heights h with the inverse function to $exp(x)-1$ (iterated by $h=-1$).

4.1.4. The matrix-logarithm-approach to U_e-tetration

The matrix-logarithm U_L of U_e can exactly be determined, since due to its unit-diagonal, $(U_e - I)$ is nilpotent to the order of its size and the number of (matrix-) terms of the powerseries for logarithm is therefore finite if the final function $U^{\circ h}(x)$ is approximated by finite matrix-size.

(4.1.4.1.) $$U_L = \begin{bmatrix} 0 & . & . & . & . & . \\ 0 & 0 & . & . & . & . \\ 0 & 1/2 & 0 & . & . & . \\ 0 & -1/12 & 1 & 0 & . & . \\ 0 & 1/48 & -1/6 & 3/2 & 0 & . \\ 0 & -1/180 & 1/24 & -1/4 & 2 & 0 \end{bmatrix}$$

If we multiply this with the height parameter h (which has no restriction to be integer now) and compute the exponential again, we get the formal composition of the general h'th power of U_e:

(4.1.4.2.) $$U_e^h = \begin{bmatrix} 1 & . & . & . & . \\ 0 & 1 & . & . & . \\ 0 & 1/2*h & 1 & . & . \\ 0 & 1/4*h^2-1/12*h & h & 1 & . \\ 0 & 1/8*h^3-5/48*h^2+1/48*h & 3/4*h^2-1/6*h & 3/2*h & 1 \end{bmatrix}$$

and the second column provides the coefficients for the powerseries in x for the h'th-iterate of $U_e^{\circ h}(x)$:

(4.1.4.3.) $V(x) \sim * U_e^h[,1] = U_e^{\circ h}(x)$

or

(4.1.4.4.) $U_e^{\circ h}(x) = 1 *x + (0 + 1/2\, h) *x^2 + (0 - 1/12\, h + 1/4\, h^2) *x^3 + ...$

which is exactly the same result as we got using the polynomial interpolation.

It is important to note, that the power series which we get by fractional h have all a zero-radius of convergence (this was proved by I.N. Baker) and as already mentioned in the previous section we cannot approximate exact values for them. We can nevertheless do evaluations using the properties of asymptotic series or using methods of divergent summation to arrive at meaningful approximations.

4.2. General U_t-tetration: Eigensystem-approach / diagonalization

For other bases $t \neq e = exp(1)$, $u=log(t) \neq 1$ the polynomial interpolation as well as the matrix-logarithm cannot be done symbolically, since the first parameter a of the formal powerseries for $U_t(x)$ $[0,a,b,c,d,..]$ as discussed in the beginning occurs with its consecutive powers and the expansion of the matrix-logarithm hasn't nilpotent matrices and must be described as an infinite series. Thus one has to employ the symbolic eigensystem-decomposition of $\mathbf{U_t}$. Fortunately, the matrix $\mathbf{U_t}$ is triangular and we obtain exact solutions for truncations of each size, which are also constant in their top-left truncations across that increasing sizes – so we may usem them as template for the case of infinite size as well.

Let $u=log(t)$ then the matrix-operator $\mathbf{U_t}$, which performs the iteration $x \to t^x - 1$ or

(4.2.1.1.) $U_t = {}^dV(log(t)) * U = {}^dV(u) * U$

is (the infinite extension of)

(4.2.1.2.) $$\mathbf{U_t} = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & u & . & . & . & . \\ 0 & 1/2*u^2 & u^2 & . & . & . \\ 0 & 1/6*u^3 & u^3 & u^3 & . & . \\ 0 & 1/24*u^4 & 7/12*u^4 & 3/2*u^4 & u^4 & . \\ 0 & 1/120*u^5 & 1/4*u^5 & 5/4*u^5 & 2*u^5 & u^5 \end{bmatrix}$$

The index indicates the base $t \neq e$ here. The second column of $\mathbf{U_t}$ provides the coefficients for the powerseries of $U_t(x) = t^x - 1$

The matrix $\mathbf{U_t}$ is triangular and exactly (up to any truncated size) decomposable into an eigensystem even in symbolic notation (where u is kept as variable) [11]:

(4.2.1.3.) $U_t = W_u * D_u * W_u^{-1}$

where $\mathbf{W_u}$ and $\mathbf{W_u^{-1}}$ are also triangular and $\mathbf{D_u}$ is diagonal.

Here $\mathbf{D_u} = {}^dV(u)$ since the eigenvalues of a triangular matrix are just the entries of their diagonal.

I omit the indexes for the matrices $\mathbf{W_u}$ and $\mathbf{D_u}$ in the following for shortness, since they are constants for a given t, u:

(4.2.1.4.) $$\begin{bmatrix} 1 & . & . & . \\ 0 & 1 & . & . \\ 0 & -u/(2*u-2) & 1 & . \\ 0 & (2*u^3+u^2)/(6*u^3-6*u^2-6*u+6) & -u/(u-1) & 1 \end{bmatrix} \begin{bmatrix} 1 & . & . & . \\ . & u & . & . \\ . & . & u^2 & . \\ . & . & . & u^3 \end{bmatrix} \begin{bmatrix} 1 & . & . & . \\ 0 & 1 & . & . \\ 0 & u/(2*u-2) & 1 & . \\ 0 & (u^3+2*u^2)/(6*u^3-6*u^2-6*u+6) & u/(u-1) & 1 \end{bmatrix}$$

It is a nice feature, that this form of $\mathbf{W}$ and $\mathbf{W^{-1}}$ represent matrix-operators themselves, and they contain the coefficients for the Schröder-function (by $\mathbf{W}[,1]$) and its inverse (by $\mathbf{W^{-1}}[,1]$).

Iterates of the function $U_t^{\circ h}(x)$ are interpreted as powers of the matrix-operator $\mathbf{U_t}$ and those are – according to the principles of eigensystem-analysis - computable by powers of $\mathbf{D}$ and their composition using the unchanged matrix-constants $\mathbf{W}$ and $\mathbf{W^{-1}}$.

(4.2.1.5.) $U_t^h = W * D^h * W^{-1} = W * {}^dV(u^h) * W^{-1}$

The coefficients for the terms of the according powerseries in x are then in the second column of the result $\mathbf{U_t^h}$.

(4.2.1.6.) $U_t^{\circ h}(x) = \mathbf{V}(x)\sim * \mathbf{U_t^h}[,1]$

[11] Aldrovandi/Freitas state in [AF97] 1997, p.16 "*Bell matrices are not normal, that is, they do not commute with their transposes. Normality is the condition for diagonalizability. This means that Bell matrices cannot be put into diagonal form by a similarity transformation. (...)*" This remark is a bit misleading; the normality-criterion applies only, if a ***orthonormal*** similarity transform is requested, which is usually also called a rotation. But here we are able to do a similarity transform using triangular matrices.

4.3. Coefficients for the general U_t-tetration (Eigensystem-based)

Let $U_t^{\circ h}(x)$ denote the h'th iterate of $U_t(x)$, then its powerseries is:

(4.3.1.1.) $$U_t^{\circ h}(x) = a_1 \frac{x}{1!} + a_2 \frac{u}{u-1} \frac{x^2}{2!} + a_3 \frac{u^2}{(u-1)(u^2-1)} \frac{x^3}{3!} + \dots + a_k \frac{u^{k-1}}{\prod_{j=1}^{k-1}(u^j-1)} \frac{x^k}{k!} + \dots$$

where

(4.3.1.2.) $a_1 = \quad 1\, u^h$

$$\begin{aligned}
a_2 = &- (1)\, u^h \\
&+ (1)\, u^{2h}
\end{aligned}$$

$$\begin{aligned}
a_3 = &\quad (1 + 2u)\, u^h \\
&- (3 + 3u)\, u^{2h} \\
&+ (2 + 1u)\, u^{3h}
\end{aligned}$$

$$\begin{aligned}
a_4 = &- (1 + 6u + 5u^2 + 6u^3)\, u^h \\
&+ (7 + 18u + 18u^2 + 11u^3)\, u^{2h} \\
&- (12 + 18u + 18u^2 + 6u^3)\, u^{3h} \\
&+ (6 + 6u + 5u^2 + 1u^3)\, u^{4h}
\end{aligned}$$

$$\begin{aligned}
a_5 = &\quad (1 + 14u + 24u^2 + 45u^3 + 46u^4 + 26u^5 + 24u^6)\, u^h \\
&- (15 + 75u + 130u^2 + 180u^3 + 165u^4 + 105u^5 + 50u^6)\, u^{2h} \\
&+ (50 + 145u + 230u^2 + 275u^3 + 215u^4 + 130u^5 + 35u^6)\, u^{3h} \\
&- (60 + 120u + 170u^2 + 180u^3 + 120u^4 + 60u^5 + 10u^6)\, u^{4h} \\
&+ (24 + 36u + 46u^2 + 40u^3 + 24u^4 + 9u^5 + 1u^6)\, u^{5h}
\end{aligned}$$

4.3.2. Basic observations:

For $h=0$ all terms except a_1 collapse to zero, so $U_t^{\circ 0}(x) = x$, for $h=1$ the a_k-coefficients cancel against the product in the denominator except one factor $u^h = u$, which combines with u^{k-1} to u^k and produces the exponential-series for $U_t(x) = t^x-1$.

For all integer h the a_k-coefficients contain the product of the associated denominator as factor and build integer functions of u when cancelled with the denominators.

For fractional h the denominators do not cancel. So for fractional h it must be that $|u|<>1$

If $u=1, t=e$ we have $0/0$ – expressions, and the function must be evaluated with other methods (as shown in the description of polynomial interpolation and matrix-logarithm above).

The numerical coefficents in each a_k-expression form matrices A_k, which seem to be computable even without the symbolic eigen-decomposition.

4.3.3. Hypotheses:

1) The first column of the matrices A_k are Stirling-numbers 2'nd kind, scaled by factorials, signed (rows taken, see next page) .

2) The last column of the matrices A_k are Stirling-numbers 1'nd kind (rows taken, see next page).

3) The shifting of the rows by integer values of the height-parameter h provides polynomials in u, whose sums according to the above scheme are multiples of the denominator of the current term of the powerseries in x.

4) The combination of ***1)*** and ***2)*** gives initial conditions, which in connection with ***3)*** allow to determine the remaining columns in A_k uniquely.

Example-computation for coefficient a_4 (denoting it here as A), using the hypotheses

We assume the left and right columns as known (hypothesis ***1)*** and ***2)***), and the property, that integer h provide integer multiples of the denominator (hypothese ***3)***). Let us call the a_4-coefficient of the powerseries as A to prevent confusion of notation here.

denominator at A (=a_4) omitting the factorial

(4.3.3.1.) $D = (u^3-1)(u^2-1)(u-1) = 1 u^6 - 1u^5 - 1 u^4 + 0u^3 + 1u^2 + 1u - 1$

For convenience of display I also rewrite A in reverse order of powers of u:

```
A =    - (  6u^3  +  5u^2 +  6u+  1)  u^h    // this is what we not yet know !
       + ( 11u^3  + 18u^2 + 18u+  7)  u^2h
       - (  6u^3  + 18u^2 + 18u+ 12)  u^3h
       + (  1u^3  +  5u^2 +  6u+  6)  u^4h
```

Rewritten showing the hypothese and the remaining unknowns

```
A =    - (  6u^3 + a1u^2 + b1u +  1)  u^h
       + ( 11u^3 + a2u^2 + b2u +  7)  u^2h
       - (  6u^3 + a3u^2 + b3u + 12)  u^3h
       + (  1u^3 + a4u^2 + b4u +  6)  u^4h
```

---

setting $h = 0$, rewritten wrt to required column-sums which must give $k*D = 0*D=0$

```
A =    - (  6u^3 + a1u^2 + b1u +  1)
       + ( 11u^3 + a2u^2 + b2u +  7)
       - (  6u^3 + a3u^2 + b3u + 12)
       + (  1u^3 + a4u^2 + b4u +  6)
       =k*D
```

Obviously k=0 and

(4.3.3.2.) $a_1-a_2+a_3 = a_4$ $\qquad$ $b_1-b_2+b_3 = b_4$

---

setting $h = 1$, (*irrelevant powers of* u *removed)*, yellow marked entries can directly be determined by known column-sums:

```
A =    -  (                         6u^3 + a1u^2 + b1u +  1)
       +  (                11u^4 + a2u^3 + b2u^2 + 7u     )
       -  (        6u^5 + a3u^4 + b3u^3 + 12u^2           )
       +  (  1u^6 + a4u^5 + b4u^4 +  6u^3                 )
    =k*D
       = 1*(1 u^6  - 1u^5 - 1 u^4 + 0u^3 +  1u^2 +  1u - 1 )
```

1) Because of coefficient at highest and lowest power of u follows k = 1

2) a_4 = 6-1 = 5 $\qquad$ *b_1=7-1 = 6*

Setting $h = 2$, *(irrelevant powers of u removed)*, yellow marked entries can directly be determined:

```
A =    -  (                                   6u^3 + a1u^2 + b1u +  1 )
       +  (                          11u^5 + a2u^4 + b2u^3 +  7u^2      )
       -  (              6u^7 + a3u^6 + b3u^5 + 12u^4                   )
       +  (  1u^9 + a4u^8 + b4u^7 +  6u^6                               )
       =  (k1*u^3 +k2*u^2 + k3*u + k4)  * D

       = 1*( 1u^9  - 1u^8 -  1u^7 +  0u^6 +  1u^5 +  1u^4 - 1u^3             )
       +k2*(         1u^8 -  1u^7 -  1u^6 +  0u^5 +  1u^4 + 1u^3 - 1u^2      )
       +k3*(                 1u^7 -  1u^6 -  1u^5 +  0u^4 + 1u^3 + 1u^2 - 1u  )
       + 1*(                         1u^6 -  1u^5 -  1u^4 + 0u^3 + 1u^2 + 1u - 1 )
```

1) Because of coefficient at highest and lowest powers of u follows $k_1 = 1$, $k_4=1$

2) Because a_4 is known, k_2 can be determined by second column-sum; analoguously b_1 and k_3 : $k_2=6$, $k_3=7$

3) Since all k are known, all column-sums are known and all remaining entries can be determined:

b_4 = 6, a_3= 18 b_3= 18 a_2=18 b_2=18 a_1=5

Hypothese for the border-coefficients: *they are always taken from rows of the matrices of Stirling numbers 2nd and 1st kind.*

*St2 *diag(0!,1!,2!,...) signed* (S2F)

1	.	.	.	.	.
-1	1	.	.	.	.
1	-3	2	.	.	.
-1	7	-12	6	.	.
1	-15	50	-60	24	.
-1	31	-180	390	-360	120

Stirling kind 1 (no shift) (St1)

1	.	.	.	.	.
-1	1	.	.	.	.
2	-3	1	.	.	.
-6	11	-6	1	.	.
24	-50	35	-10	1	.
-120	274	-225	85	-15	1

4.4. Coefficients as dependent on the height-parameter h in u^h

a) Let $v = u^h$, then the coefficients c_k at each k'th power of v are ***series*** of the following structure:

$$\begin{array}{lll} v & (x/1! - u/(u-1) *x^2/2! + (1+2u) u^2 /(u-1)(u^2-1)*x^3 / 3! +....) & =u^h * c_1 (x,u) \\ v^2 & (\quad u/(u-1) *x^2/2! + (3+3u) u^2 /(u-1)(u^2-1)*x^3 / 3! + ...) & =u^{2h} * c_2 (x,u) \\ v^3 & (\quad (2+ u) u^2 /(u-1)(u^2-1)*x^3 / 3! + ...) & =u^{3h} * c_3 (x,u) \\ ... & & \end{array}$$

and the value $U_t^{\circ h} (x)$ is then a trivariate function in u,h,x, where we may assume a given u and x:

$$U_t^{\circ h} (x) = f_{u,x}(h) = c_1(x,u) * u^h + c_2 (x,u) * u^{2h} + c_3 (x,u)* u^{3h}$$

For all $c_k(x,u)$ it is for $x=0$ $c_k(0,u) = 0$ and thus, as expected

$$U_t^{\circ h} (0) = (t^0 - 1)^{\circ h} = 0$$

b) Let $x=1$ then this is a shorter form for the usual t^^h – notation ($u=log(t)$)

$$\begin{array}{lll} v & (1/1! - u/(u-1)/2! + (1+2u) u^2 /(u-1)(u^2-1)/ 3! +....) & =u^h * c_1 (u) \\ v^2 & (\quad u/(u-1)/2! + (3+3u) u^2 /(u-1)(u^2-1)/ 3! + ...) & =u^{2h} * c_2 (u) \\ v^3 & (\quad + (2+ u) u^2 /(u-1)(u^2-1)/ 3! + ...) & =u^{3h} * c_3 (u) \\ ... & & \end{array}$$

c) For $|u|>1$ this may be rewritten as

$$\begin{array}{lll} v & (1/1! - 1/(1-1/u) /2! + (1/u+2)/(1-1/u)(1-1/u^2)/ 3! +...) & =u^h * c_1 (u) \\ v^2 & (\quad 1/(1-1/u) /2! + (3/u+3)/(1-1/u)(1-1/u^2)/ 3! + ...) & =u^{2h} * c_2 (u) \\ v^3 & (\quad (2/u+1)/(1-1/u)(1-1/u^2)/ 3! + ...) & =u^{3h} * c_3 (u) \\ ... & & \end{array}$$

If u is a rational unit-root on the complex unit-circle, then we get periodically infinities in this series (because some denominators evalute to zero), but it seems, that if u is an irrational complex unit-root, then the series doen't show this effect and can possibly evaluated.

4.5. Conclusion/ perspective for U- and T- tetration if hypotheses hold:

If the hypothese ***4)*** in the previous holds, then each term-matrix A_k can be uniquely determined individually - without need of the eigensystem-decomposition of the matrix-operator U_t and even without other terms being involved. It provides a computation scheme for arbitrary many terms for the powerseries for fractional iterates of the function $U_t^{\circ h}(x)$ in sequential or random order.

Since the **U**-tetration $U_t(x) : x \to t^x - 1$ and **T**-tetration $T_b(x) : x \to b^x$ can be converted into each other by shift and rescale of the x-parameter and by relating the bases-parameters b and t

(4.5.1.1.) $T_b^{\circ h}(x) = (U_t^{\circ h}(x/t - 1))+1)*t$
where
$b = t^{1/t}$

this provides also a systematic access to the powerseries for and the characteristics of the **T**-tetration, which is the commonly understood "tetration" if its parameter x is $x=1$.

Since we had no special restrictions (except the final convergence consideration) on the parameters b, t u and h, and since for all b (with some, at most enumerable infinitely many, exceptions) we can determine fixpoints t (possibly using complex values and then using their principal branches of their logarithms for u), the above describes a very general framework – perhaps the most complete one – for the problem of continuous extension of integer tetration.

The surprising possibility to be able to compute the terms for the powerseries in x, as indicated in the previous paragraphs asks for an iterated gaussian-elimination-procedure, which may come out to be a new, but basic process, which needs description of the recursive algorithm. Also the computed numbers $a_1,a_2,\ldots$ $b_1,b_2,\ldots$ seem to be basic constants with some flavour of being somehow "eigen-numbers" of the sequences of Stirling-numbers in the related rows of the matrices of Stirling-numbers of 1'st and 2'nd kind. Since they are ultimately derived from the Taylor-series-expansion for the function $f(x) = e^x-1$ iterations of other function of the same type ($f(x) = \sin(x)$) may be similarly dependend on such typical "eigen-numbers" accordingly to the coefficients in their series-expansion. But also it may be possible to find another process, which describes these numbers with less effort... This remains open here for further investigation.

Both methods:
- matrix-logarithm for bases t where $|u|=1$, (tetration for base $b=t^{1/t}$)
- eigensystem-decomposition for other bases

together answer some strange properties of tetration, if the above hypotheses hold:

Q: why does tetration converge for $1/e^e < b < e^{1/e}$ but diverge for other b?

A: (Eigensystem): because then $1/e<t<e$ and $|u|<1$. Then the sequence of absolute values of eigenvalues $[1,u,u^2,\ldots]$ is a convergent sequence and thus powers of the diagonalmatrix $D^h = diag(1,u,u^2,\ldots)^h = diag(1,u^h,u^{2h},\ldots)$ provide convergent sequences. For $|u|>1$ the diagonalmatrix of eigenvalues (as well as its positive powers) contain divergent sequences.

Q: why does tetration oscillate if $b<1/e^e$?

A: (Eigensystem): because $u<-1$. Say, $u=-k$, (where k is assumed as positive number>1), then the set of eigenvalues is $[1,-k,k^2,-k^3,k^4,-k^5,\ldots]$ and is used as diagonalmatrix D. Even powers of D, say D^w, where $w=2*n$ give $[1,k^w,k^{2w},k^{3w},\ldots]$ and odd powers D^v, where $v= w+1$ gives $[1,-k^v,k^{2v},-k^{3v},\ldots]$. The signs of each second entry in the resulting diagonal-matrices is alternating between v and w and since k>1 the sequences of powers of eigenvalues are also divergent, this leads to the oscillation of values/bifurcation for even/odd integer heights for $U_t^{\circ h}(x)$ or $T_b^{\circ h}(x)$ (U- and (T-) tetration)

Q: ... (to be continued)

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19.10.2017 *(first version: 18.01.2008)*

5. Appendices

5.1. Indeterminacy of reciprocal in case of infinite matrices

It was mentioned, that with infinite matrices the reciprocal may not be uniquely determined. This is true even for the case of rowfinite (triangular) matrices, and should be considered in more detail. Here I give a simple heuristic.

We know, that the logarithm-function is multivalued in **C**, and with tetration we need an apriori consideration of this property.

For the inverse of the ***U***-matrix (= ***fS2F***-matrix in my other articles), which I may denote here as ***S1***, this has the following consequence.

Recall the powerseries for logarithms, (as it occurs as well with ***S1***) as

(5.1.1.1.) $log(1+x) = 1/x - 1/2x^2 + 1/3x^3 - ...$

and $exp(log(1+x)) -1 = x$. The multivaluedness can then be described from

(5.1.1.2.) $exp(log(1+x) + k*2 \pi i) -1 = x$ //k integer

Since the $k*2 \pi i$ – term is constant w.r.t. x, we may add this term to the powerseries

(5.1.1.3.) $log(1+x)+2k \pi i = 2 k \pi i + 1x - 1/2 x^2 + 1/3 x^3 - ...$

and formally we have to do the construction of the matrix-operator $S1_k$ for the current branch of logarithm according to *(1.1)* by the series-expansion-process, based on the coefficients of the formal powerseries

(5.1.1.4.) $f(x) = K + a x + b x^2 + c x^3 + d x^4 + ...$

with $K= k*2 \pi i =/=0$, $a=1$, $b=-1/2$, $c = 1/3$ and so on.

Let's denote $f_k(x)$ for the function, where a certain k is selected, in this example $k=-1$

The first four powers of $f_k(x)$ are then

(5.1.1.5.)
$$\begin{aligned}
f_k(x)^0 &= 1 \\
f_k(x)^1 &= K + 1\ x - 1/2\ x^2 + 1/3\ x^3 - 1/4\ x^4 + ... \\
f_k(x)^2 &= K^2 + (2K)\ x + (1 - 2K/2)\ x^2 + (2K/3 - 1)\ x^3 + .. \\
f_k(x)^3 &= K^3 + (3K^2)x + (3K - 3/2K^2)x^2 + (1 - 3K + K^2)\ x^3 \quad ... \\
... &= ...
\end{aligned}$$

and the entries of the columns of $S1_k$ must now be determined by evaluation of the parentheses.

Numerically this gives about

$S1_1=$

1.000	-6.283*I	-39.48	248.1*I	1559.
0	1.000	-12.57*I	-118.4	992.2*I
0	-0.5000	1.000+6.283*I	59.22-18.85*I	-236.9-496.1*I
0	0.3333	-1.000-4.189*I	-38.48+18.85*I	236.9+305.6*I
0	-0.2500	0.9167+3.142*I	28.11-17.28*I	-216.1-210.4*I

To check, whether this is a valid reciprocal for ***U*** (=***S2***) in the sense that first we compute the first branch-logarithm and then exponentiate to get the original value, we need to matrix-multiply the row-vectors of $S1_1$ with the col-vectors of ***S2***.

The first nontrivial column in ***S2*** is the second, which contains the coefficients of the powerseries for $exp(x)-1$. Using the first row in $S1_1$ this gives

(5.1.1.6.) $1*0 + K/1! + K^2 /2! + K^3/3! + ... = exp(K) - 1 = exp(2 \pi i) - 1 = 0$

Next row gives

(5.1.1.7.)
$$\begin{aligned}
&0*0 + 1/1! + 2K/2! + 3K^2/3! + \\
&= 1/0! + K/1! + K^2 /2! + \quad = exp(K) = exp(2 \pi i) = 1
\end{aligned}$$

and it needs then be proved by induction, that indeed the equality

(5.1.1.8.) $S1_k * S2 = I$ // k=/=0

holds for the case of infinite sized matrices to verify the correctness for the *2*'nd column of ***S2***.

Numerically, size of *64x64* suffices, to get good approximation in the top left using *k=-1*:

$$\text{(5.1.1.9.)}\quad S1_{-1} * S2_{64x64} = \begin{bmatrix} 1.00000 & 0 & 0 & -0.0000000302794-0.0000000880113*I \\ . & 1.00000 & 0 & -0.0000000908382+0.000000307680*I \\ . & . & 1.00000 & 0.00000158405+0.000000307680*I \\ . & . & . & 1.00000-0.00000540649*I \\ . & . & . & -0.0000130919+0.0000413178*I \end{bmatrix}$$

Because the generating-formula shows, that the entries along the rows in $S1_k$ grow only geometrically with *k*, but the entries of the columns in ***S2*** eventually decrease hypergeometrically, all occuring sums are convergent and the precision of the entries can be increased, if convergence-acceleration is used, for instance Euler-summation.

I didn't consider the problem of iterations here; but it should be mentioned, that we have then to deal with divergent summation with complex series, and heuristics indicate, that simple Euler-summation may not suffice for that.

For the second iterate we get by such expansions of powers of $f(x)$:

$$\text{(5.1.1.10.)}\quad \begin{aligned} f^{\circ 2}(x) &= K \\ &+ a\,(K + a\,x + b\,x^2 + c\,x^3 + d\,x^4 + \dots) \\ &+ b\,(K^2+2Ka\,x+(a^2+2Kb)\,x^2+(2Kc+2ba)\,x^3+(2ac+b^2+2Kd)\,x^4 +(2bc+2da+2Ke)\,x^5 \quad) \\ &+ c\,(K^3+3K^2a\,x+(3Ka^2+3K^2b)\,x^2+(3K^2c+(a^3+6Kba))\,x^3+\dots) \\ &+ \dots \end{aligned}$$

Collect like powers of *x*:

$$\text{(5.1.1.11.)}\quad \begin{aligned} f^{\circ 2}(x) &= K + \quad (aK + \; bK^2 + cK^3 + \dots) \\ &+(\quad a \quad (1a + 2bK + 3cK^2 + \dots)) && * \mathbf{x} \\ &+(\quad (a^2 \quad (1b + 3cK + 6dK^2 + 10eK^3 + \dots) \\ &\qquad + b \quad (1a + 2bK + 3cK^2 + \; 4dK^3 + \dots) && * \mathbf{x^2} \\ &+(\quad (a^3 \quad (1c + 4dK + 10eK^2 + \dots) \\ &\qquad + 2ab \;\; (1b + 3cK + 6dK^2 + 10eK^3 + \dots) \\ &\qquad + c \quad (1a + 2bK + 3cK^2 + \; 4dK^3 + \dots)) && * \mathbf{x^3} \\ &+(\quad (a^4 \quad (1d + 5eK + 10fK^2 \; + \dots) \\ &\qquad + 3a^2b \quad (1c + 4dK + 10eK^2 \; + 20fK^3 + \dots) \\ &\qquad + (2ac + b^2)(1b + 3cK + \; 6dK^2 + 10eK^3 + \dots) \\ &\qquad + \; d \quad (1a + 2bK + \; 3cK^2 \; + \; 4dK^3 + \dots)) && * \mathbf{x^4} \\ &+ \dots \end{aligned}$$

write $g(k) =(K + aK + bK^2 + cK^3 + \dots)$ and the derivative of *g* at *K* $g'(K) = dg(K)/dK$ then

$$\text{(5.1.1.12.)}\quad \begin{aligned} f^{\circ 2}(x) &= g(K) \\ &+ \; a\,g'(K) && * \mathbf{x} \\ &+(a^2 /2!\, g''(K) \; + \; b \quad g'(K)) && * \mathbf{x^2} \\ &+(a^3 /3!\, g'''(K) \; + 2ab /2!\; g''(K) \quad + c \quad g'(K)) && * \mathbf{x^3} \\ &+(a^4 /4!\, g''''(K) \; + 3a^2b /3!\, g'''(K)) \; +(2ac+b^2)/2!\, g''(K) \quad + d\, g'(K)) && * \mathbf{x^4} \\ &+ \dots \end{aligned}$$

This all may be written more clearly in a sketched matrix-notation (I omit *(K)* at *g* here):

$$ {}^dV(x)* \quad \begin{matrix} [1 & . & . & . & . & . & . &] \\ [0 & a & . & . & . & . & . &] \\ [0 & b & & 1a^2 & . & . & . &] \\ [0 & c & 2ab & & 1\,a^3 & & . &] \\ [0 & d & (2ac+1b^2) & & 3a^2b & & 1\,a^4 &] \end{matrix}$$

$$* \operatorname{diag}(g(K), g'(K)/1!, g''(K)/2!, g'''(K)/3!, g''''(K)/4! ,\dots)$$

Inserting values $a=1$, $b=-1/2$, $c=1/3$, $d=-1/4$ gives then

(5.1.1.13.)
$$
\begin{aligned}
f^{\circ 2}(x) &= K + g(K) \\
&+ \quad g'(K) && *x \\
&+(1/2!\, g''(K)) \quad -1/2\, g'(K)) && *x^2 \\
&+(1/3!\, g'''(K) \quad -1/2!\, g''(K) \quad +1/3\, g'(K)) && *x^3 \\
&+(1/4!\, g''''(K) \quad -1/4\, g'''(K) \quad +11/24\, g''(K)) \quad -1/4\, g'(K)) && *x^4 \\
&+ \ldots.
\end{aligned}
$$

Now set $K_1 = K+1$ and set $g(K) = \log(K_1)$, $g'(K)=1/K_1$, $g''(K)=-1/K_1^2$, $g'''(K)=2!/K_1^3$, ...allowing divergent summation, and reorder in the rows, then

(5.1.1.14.)
$$
\begin{aligned}
f^{\circ 2}(x) &= K + \log(K_1) \\
&+ \ 1/K_1 && *x \\
&- \ (1/K_1 \ + \ 1 \ /K_1^2) && *x^2/2 \\
&+ (1/K_1 + 3/2/K_1^2 + 1/K_1^3) && *x^3/3 \\
&- (\ 1/K_1 + 11/6/K_1^2 + 2/K_1^3 + 1/K_1^4) && *x^4/4 \ldots
\end{aligned}
$$

The view on the columns suggest to use column-wise summation to find the final value.

If indeed the columns follow the Stirling-numbers 1'st kind, we would get

(5.1.1.15.)
$$
\begin{aligned}
f^{\circ 2}(x) &= K + \log(K_1) + \log(1+x)/K_1 + (\log(1+x)/K_1)^2 + (\log(1+x)/K_1)^3 + \ldots \\
&= K + \log(K_1) + \ \log(1+x)/K_1/(1 - \ \log(1+x)/K_1) \\
&= K + \log(K_1) + 1/(K_1/\log(1+x) - 1)
\end{aligned}
$$

5.2. Experiments with $f(x) = x*e^x$ and $f^{-1}(x)$="Lambert-W"

Considering the function $f(x)=x\,e^x$ exhibits some easyness implied by this matrix-method of inversion and another aspect of different behave of a function at integer and fractional arguments.

First note the matrix-operator for $f(x)$. It is generated from

(5.2.1.1.) $f(x) = x*(1+x/1! + x^2/2! + x^3/3! + \dots)$

by the previously decribed method. The occuring matrix-operator has the form:

(5.2.1.2.) $F =$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 1 & 1 & . & . & . \\ 0 & 1/2 & 2 & 1 & . & . \\ 0 & 1/6 & 2 & 3 & 1 & . \\ 0 & 1/24 & 4/3 & 9/2 & 4 & 1 \end{bmatrix}$$

and the coefficients of the powerseries are in the second column, such that

(5.2.1.3.) $V(x)\sim * F = V(x\,e^x)\sim = [1, xe^x, (xe^x)^2, (xe^x)^3,\dots]$

The triangular inverse can easily be determined, we get

(5.2.1.4.) $F^{-1} = W =$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & -1 & 1 & . & . & . \\ 0 & 3/2 & -2 & 1 & . & . \\ 0 & -8/3 & 4 & -3 & 1 & . \\ 0 & 125/24 & -25/3 & 15/2 & -4 & 1 \end{bmatrix}$$

which contains - the coefficients for the ***Lambert-W***-function!

The required matrix-operation is then

(5.2.1.5.) $V(x)\sim * W = V(w(x))\sim$

to determine the Lambert-W-value for x (for x in range of convergence or using divergent summation).

The famous coefficients can better be seen, if the matrix is factorially scaled

(5.2.1.6.) ${}^dFac(1)*F^{-1}\,{}^d*Fac(-1)= FWf =$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & -2 & 1 & . & . & . \\ 0 & 9 & -6 & 1 & . & . \\ 0 & -64 & 48 & -12 & 1 & . \\ 0 & 625 & -500 & 150 & -20 & 1 \end{bmatrix}$$

and we see the coefficients $1^0,2^1,3^2,4^3,5^4,\dots$ in the second column, and this agrees with the series-description of the Lambert-W:

(5.2.1.7.) $W(x) = 1^0/1!\,x - 2^1/2!\,x^2 + 3^2/3!\,x^3 - 4^3/4!\,x^4 + \dots - \dots$

This is a very easy access to this famous function; and by powers of the matrix we can even iterate the function F and W (while we get even more divergence with the iterates of W, but this doesn't matter here).

The half-iterate of the Lambert-W begins with

(5.2.1.8.)
$W^{\circ 0.5}(x)=1\,x -1/2\,x^2 +1/2\,x^3 -31/48\,x^4 +91/96\,x^5 -2873/1920\,x^6 +2845/1152\,x^7 -150327/35840\,x^8 +O(x^9)$

or

(5.2.1.9.)
$W^{\circ 0.5}(x)= 1.0\,x -0.5\,x^2 +0.5\,x^3 -0.645833\,x^4 +0.947917\,x^5 -1.49635\,x^6 +2.46962\,x^7 -4.19439\,x^8 +7.26496\,x^9 - 12.7707\,x^{10} +22.7309\,x^{11} -40.9236\,x^{12} +74.4549\,x^{13} -136.697\,x^{14} +252.797\,x^{15} + O(x^{16})$

Moreover, there is a curiosity with the function F itself. If we define two variants as

(5.2.1.10.) $F_1(x,r) = x^r e^x \qquad F_2(x,r) = x^r e^{-x}$

and the alternating sums

(5.2.1.11.) $AF_1(r) = \sum_{k=0}^{oo} (-1)^k * k^r e^k \qquad AF_2(r) = \sum_{k=0}^{oo} (-1)^k * k^r e^{-k}$

then, first, for the exponent $r=1$ we get the surprising result, that

(5.2.1.12.) $AF_1(1) - AF_2(1) = 0$

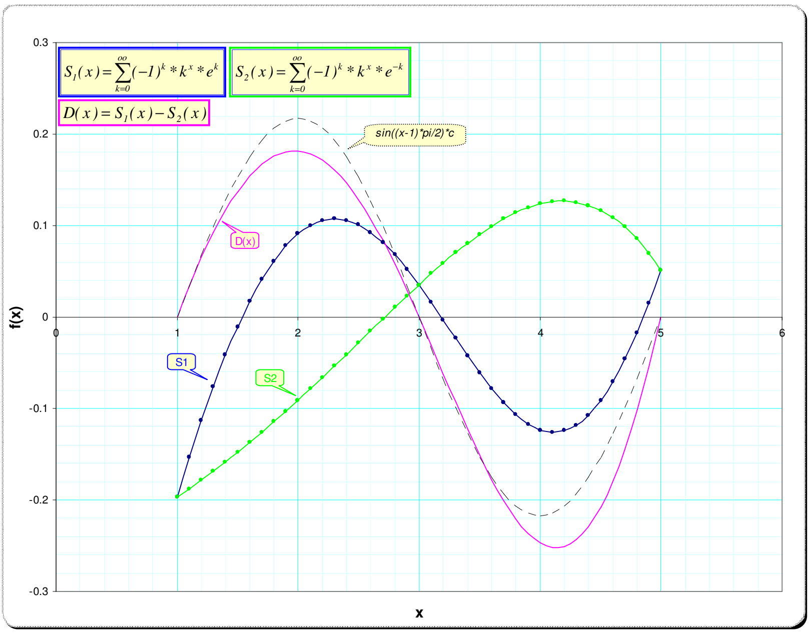
where the divergent series $AF_1()$ is Euler-summed. This is much interesting, and it seems, that generally

(5.2.1.13.) $AF_1(k) + (-1)^k AF_2(k) = 0$

for ***integer*** k and has always a sinusiodal characteristic as function $d(x)$ of ***continuous*** x.

(5.2.1.14.) $AF_1(x) - AF_2(x) = d(x)$ // $d(x)$ sinusoidal periodic with period $\pi/2*x$

Here is a graph, which shows $AF_1(x)$ (blue, "S_1" in the plot), $AF_2(x)$ (green, "S_2" in the plot), $AF_1(x)-AF_2(x)$ (magenta "D" in the plot) and a scaled sinus-curve (dotted, black):



Although I didn't get into more depth with this yet, it reminds of a similar effect with the alternating Tetra-series, and also of the difference, which occurs, if fractional tetration is computed via different fixpoints. The sinusoidal effect of differences, if fractional arguments are involved, seems to be ubiquitous...

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18.3.2008

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