"Tetra-eta"-series *(well chosen name?)*

The goal here is to discuss series of this type:

$$v(x) = \{1,x\}^{\wedge\wedge 2} - \{2,x\}^{\wedge\wedge 2} + \{3,x\}^{\wedge\wedge 2} - \{4,x\}^{\wedge\wedge 2} + \ldots - \ldots$$
$$= 1^{1^x} - 2^{2^x} + 3^{3^x} - 4^{4^x} + 5^{5^x} - \ldots + \ldots$$

I'll denote this as special case of

$$TE(h,x) = \sum_{b=1}^{inf} (-1)^{b-1} \{b,x\}^{\wedge\wedge} h$$

where $\{b,x\}^{\wedge\wedge}h$ is the notation for the powertower $b^\wedge b^\wedge b^\wedge \ldots ^\wedge b^\wedge x$ of height h (h-fold occurence of base b) and general, continuous h, and the series under consideration here has $h=2$.

The described approach allows generalization to other heights, for instance the trivial $h=0$, but also $h=1$ (where it describes the alternating zeta-series and yields the correct results) and principally all greater heights, though always only at integer values for h. A formula for $h=3$ is appended, but in no way evaluated.

Let's recall the matrix-approach in short.

I have the matrix $\mathbf{B} = [b_{r,c}] = matrix(_{r=0..inf,c=0..inf} c^r/r!)$ such that the entries of its second column $b_{*,1}$ together with the powers of a variable x and of $log(s)$, where s is a fixed parameter for the base of the powertower, form the exponential-series to obtain s^x :

$$\begin{aligned} s^x &= \sum_{r=0}^{inf} log(s)^r x^r b_{r,1} \\ &= \sum_{r=0}^{inf} \frac{log(s)^r 1^r}{r!} x^r \\ &= e^{log(s)x} \\ &= s^x \end{aligned}$$

in matrix-notation for the scalar result, using the second column of $\mathbf{B}$ only.

I'm using σ for $log(s)$ for notational convenience here and $[,1]$ to denote the second column of $\mathbf{B}$:

$$s^x = V(x) \sim *_d V(\sigma) * B[,1]$$

where $V(x)\sim = [1,x,x^2,x^3,\ldots]$ and ${}_dV(x)$ is its diagonal-arrangement

and for the complete vectorial result, where the other columns of $\mathbf{B}$ contain the required coefficients to obtain also the concecutive powers of s^x:

$$V(s^x) \sim = V(x) \sim *_d V(\sigma) * B$$

Now to have the powertower of height 2 we can iterate

$$\begin{aligned} s^{s^x} &= V(s^x) \sim * \left({}_dV(\sigma) * B\right)[,1] \\ &= \left(V(x) \sim *_d V(\sigma) * B\right) * \left({}_dV(\sigma) * B\right)[,1] \\ &= V(x) \sim * \left({}_dV(\sigma) * B * {}_dV(\sigma) * B\right)[,1] \end{aligned}$$

or denote it as a result of a formal power of $\mathbf{B}$:

$$s^{s^x} \quad = V(x) \sim * \left({}_dV(\sigma) * B\right)^2 [,1]$$

and again for convenience I use in the following the abbreviation:

$\mathbf{B_s} = {}_dV(\sigma) * \mathbf{B}$

To obtain the actual terms for the series-expansion in powers of x, we may explicitely do the multiplication of the involved terms according to the iteration or to the formal matrix-power definition.

First the formula for a single term in row r of the second column of Bs^2 :

From

$$(B*_dV(\sigma)*B[,1])_r = \sum_{k=0}^{oo}\left(b_{r,k} * \sigma^k b_{k,1}\right) = \sum_{k=0}^{oo}\left(b_{r,k} * \frac{(\sigma * 1)^k}{k!}\right)$$

it follows

$$(Bs * Bs[,1])_r = \sum_{k=0}^{oo}\left(b_{r,k} * \sigma^k b_{k,1}\right) = \sigma^r \sum_{k=0}^{oo}\left(\frac{k^r}{r!} * \frac{(\sigma * 1)^k}{k!}\right)$$

and

$$Bs^2[r,1] \quad = \sum_{k=0}^{oo}\left(\frac{k^r}{r!} * \frac{\sigma^{k+r}}{k!}\right)$$

$$= \sum_{k=0}^{oo}\left(k^r \binom{r+k}{k} \frac{\sigma^{k+r}}{(k+r)!}\right)$$

and then the formula for the whole expression, which means: the sum over all rows $r=0..inf$:

$$s^{s^x} \quad = V(x) \sim * Bs^2[,1]$$

$$= \sum_{r=0}^{oo}\sum_{k=0}^{oo}\left(x^r k^r \binom{r+k}{k} * \frac{\sigma^{k+r}}{(k+r)!}\right)$$

In the above we see, that the *log(s)*-coefficient is nicely isolated, so that we may form sums (or in more general: linear combinations) of B_s using different *s*:

$$s_0^{s_0^x} + s_1^{s_1^x} \quad = V(x) \sim *(Bs_0^2[,1] + Bs_1^2[,1])$$
$$= \sum_{r=0}^{oo}\sum_{k=0}^{oo}\left(x^r k^r \binom{r+k}{k} * \frac{(\sigma_0^{k+r} + \sigma_1^{k+r})}{(k+r)!}\right)$$

with the further assumption, that we can build the alternating series:

$$TE(2,x) \quad = 1^{1^x} - 2^{2^x} + 3^{3^x} - + \ldots$$
$$= V(x) \sim *(B_1^2[,1] - B_2^2[,1] + B_3^2[,1] - + \ldots)$$
$$= \sum_{r=0}^{oo}\sum_{k=0}^{oo}\left(x^r k^r \binom{r+k}{k} * \frac{(\log(1)^{k+r} - \log(2)^{k+r} + \log(3)^{k+r} - + \ldots)}{(k+r)!}\right)$$

if the sum of like powers of logarithms can regularly be summed, for instance by Euler-summation.

Alternating series of powers of logarithms: "lambda(p)" or "λ(p)"

Since Euler-summation can regularly sum any geometric series with $q<1$, even where $|q|>1$, the sums of powers of logarithms can also be summed (the quotients of subsequent terms decrease in absolute value) we can evaluate the above alternating sum for any exponent.

Let's call the alternating sum of the *b*'th power of logarithms of consecutive parameters (as in the above formula) as $\lambda(b)$, then we have:

$$TE(2,x) \quad = 1^{1^x} - 2^{2^x} + 3^{3^x} - + \ldots$$
$$= \sum_{r=0}^{oo}\sum_{k=0}^{oo}\left(x^r k^r \binom{r+k}{k} * \frac{\lambda(k+r)}{(k+r)!}\right)$$

which we need to approximate having only finitely many terms.

I don't have a final statement about the bounds for convergence of this formula; however for *x* in the range $-inf<x<1.3$ it seems, that this series converges (conditionally) or is at least Euler-summable with a reasonable order in relation to the accessible finite number of terms for the series (but see a plot in he appendix).

The core question here is the rate of growth of the sequence of the logarithm-sums $\lambda(b)$, for which I don't have definitive bounds or characteristics so far. (It seems to be of the order *exp(b)/b*, or reflecting the changings of signs *sinh(b)/b*)

Numerical computation

The coefficients *λ(r+k)* can be precomputed for instance by Euler-summation of the powers of logarithms; in **Pari/GP** the function "*sumalt()*" is sufficient here up to powers of some hundred, if this is needed to have enough terms to make in turn the series itself summable to a reasonable approximation.

Crucial is here the range for manageable x; the above series representation gives no further hint how I possibly could extend the range for $x>1.3$ for the current problem to be reasonably summable:

$$1^{1^x} - 2^{2^x} + 3^{3^x} - + \ldots = v(x)$$

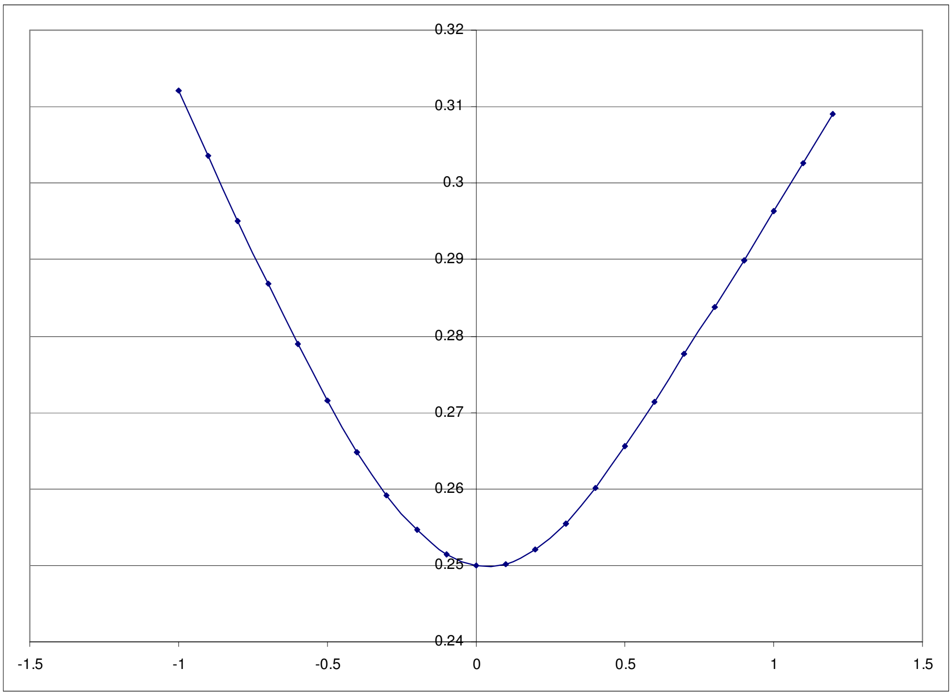
but at least for negative x we can find results for a greater range.

For some x the series of towers of height 2 degenerates to known series, for instance to the eta-series itself (see next page)

Here I give some results of approximation:

Values for v(x) using -9 < x <= 1.2
(difficult approximation of series at upper limit-point x=1.2 . For x->-oo the result to eta(0) = 1/2).

x	v(x)
-inf	-> 0.5
-9	0.49911
-5	0.48164
-3	0.43709
-2	0.38800
-1	0.31214
-0.9	0.30359
-0.8	0.29512
-0.7	0.28685
-0.6	0.27892
-0.5	0.27151
-0.4	0.26484
-0.3	0.25912
-0.2	0.25460
-0.1	0.25150
0	0.25000
0.1	0.25021
0.2	0.25210
0.3	0.25549
0.4	0.26010
0.5	0.26554
0.6	0.27147
0.7	0.27759
0.8	0.28375
0.9	0.28995
1	0.29632
1.1	0.30266
1.2	0.30908
1.3	0.31560



Tetra-eta-series $y = v(x) = 1^{1^x} - 2^{2^x} + 3^{3^x} - + \ldots$ *for some x*

Some identities

x	v(x)	Identities
		$=1^{1^{-\infty}}-2^{2^{-\infty}}+3^{3^{-\infty}}-+\ldots$
-inf	-> 0.5	$=1^{0}-2^{0}+3^{0}-+\ldots$
		$=1-1+1-1+-\ldots=\eta(0)=1/2$
-9	0.49911	
-5	0.48164	
-3	0.43709	
-2	0.38800	
		$=1^{1^{-1}}-2^{2^{-1}}+3^{3^{-1}}-+\ldots$
-1	0.31214	$=1^{1/1}-2^{1/2}+3^{1/3}-+\ldots$
		$=1-\sqrt[2]{2}+\sqrt[3]{3}-\sqrt[4]{4}+-\ldots$
-0.9	0.30359	
-0.8	0.29512	
-0.7	0.28685	
-0.6	0.27892	
-0.5	0.27151	
-0.4	0.26484	
-0.3	0.25912	
-0.2	0.25460	
-0.1	0.25150	
		$=1^{1^{0}}-2^{2^{0}}+3^{3^{0}}-+\ldots$
0	0.25000	$=1^{1}-2^{1}+3^{1}-+\ldots$
		$=1-2+3-4+-\ldots=\eta(-1)=1/4$
0.1	0.25021	
0.2	0.25210	
0.3	0.25549	
0.4	0.26010	
0.5	0.26554	
0.6	0.27147	
0.7	0.27759	
0.8	0.28375	
0.9	0.28995	
		$=1^{1^{1}}-2^{2^{1}}+3^{3^{1}}-+\ldots$
1	0.29632	$=1^{1}-2^{2}+3^{3}-+\ldots$
1.1	0.30266	
1.2	0.30908	
1.3	0.31560	

The question, which motivated this analysis:

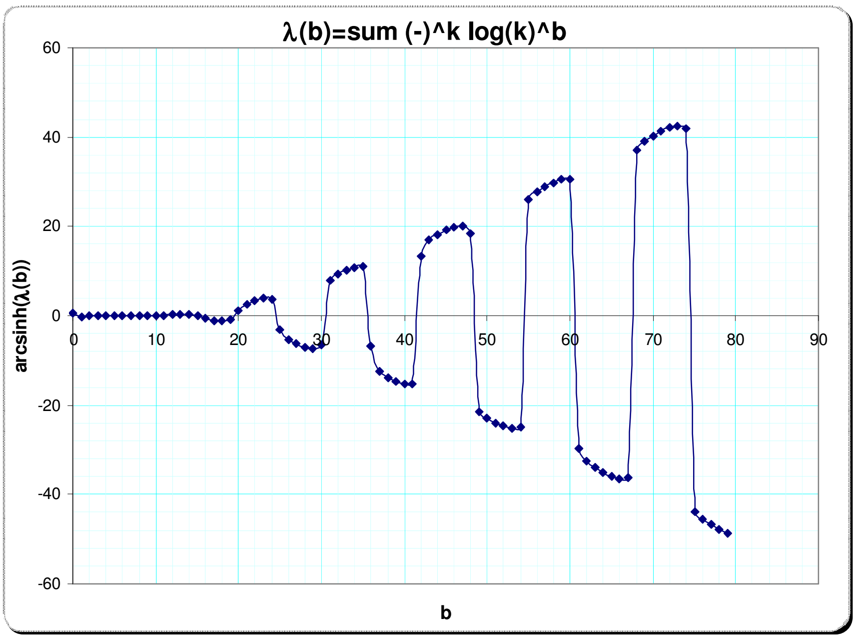
- *Are the values v(x) and v(-x) related in any "closed-form"-way (for instance comparable to the relation between zeta(s) and zeta(1-s)) ?*

Gottfried Helms, 24.10.2007

Appendix 1:

Alternating sums of like powers of logarithms $\lambda(b)=(log(1)^b - log(2)^b + log(3)^b - + ...)$

b	λ(b)	b	asinh(λ(b))	b	asinh(λ(b))
0	0.500000000000	0	0.481211825060	40	-15.1818412465
1	-0.225791352645	1	-0.223915537188	41	-15.3413813773
2	-0.0610291437681	2	-0.0609913227093	42	13.3863855599
3	0.0234746814918	3	0.0234725260307	43	16.9673666064
4	0.0500359565663	4	0.0500151017418	44	18.2545940620
5	0.0374729382845	5	0.0374641737732	45	19.1664982658
6	0.00344063726087	6	0.00344063047254	46	19.8109486731
7	-0.0343472429980	7	-0.0343404931509	47	20.0861784444
8	-0.0582959923401	8	-0.0582630236631	48	18.3188936395
9	-0.0529364492457	9	-0.0529117566876	49	-21.5378070489
10	-0.00915199731305	10	-0.00915186955743	50	-22.9561615486
11	0.0693844777406	11	0.0693289261446	51	-23.9619480421
12	0.158602872841	12	0.157945349799	52	-24.7108335891
13	0.208236018082	13	0.206759717510	53	-25.1593036293
14	0.143787258170	14	0.143296350279	54	-24.8000400231
15	-0.114162594723	15	-0.113916055880	55	26.0555156099
16	-0.601147289355	16	-0.569808442228	56	27.7807084444
17	-1.22072801766	17	-1.02917555126	57	28.9293046667
18	-1.62212630968	18	-1.26065224980	58	29.8092156962
19	-1.08705243426	19	-0.941609319929	59	30.4400745461
20	1.45595926753	20	1.17008289864	60	30.6455513303
21	7.05601010979	21	2.65201089049	61	-29.6278475744
22	15.6419699091	22	3.44412507785	62	-32.6197587377
23	23.8103264829	23	3.86370723184	63	-33.9988805911
24	21.4360588336	24	3.75876529751	64	-35.0382050560
25	-11.7646134485	25	-3.16004475011	65	-35.8460676854
26	-105.784156347	26	-5.35457027785	66	-36.3801258348
27	-286.902828550	27	-6.35229379967	67	-36.2442937016
28	-535.048491795	28	-6.97550543550	68	37.1513142473
29	-690.440473281	29	-7.23047746598	69	39.0714428426
30	-297.313273438	30	-6.38793638438	70	40.3298315636
31	1574.37154035	31	8.05475873105	71	41.3198123893
32	6337.43576832	32	9.44737669946	72	42.0890220202
33	15211.2342489	33	10.3229367108	73	42.5525110769
34	26804.4037721	34	10.8894686538	74	41.9535623588
35	31290.8881088	35	11.0442294002	75	-43.8963528963
36	-408.822229125	36	-6.70642909029	76	-45.5901725807
37	-131095.697428	37	-12.4768300308	77	-46.8053938775
38	-463467.632371	38	-13.7396390090	78	-47.7813848586
39	-1099365.70093	39	-14.6033911165	79	-48.5413865423



It may be worth to note, that inserting these values in the matrix equation

$Y\sim = V(1)\sim * diagonal(\lambda(0), \lambda(1), \lambda(2),...) * B$
(***B*** without exponent gives series of tower-heights *1*)

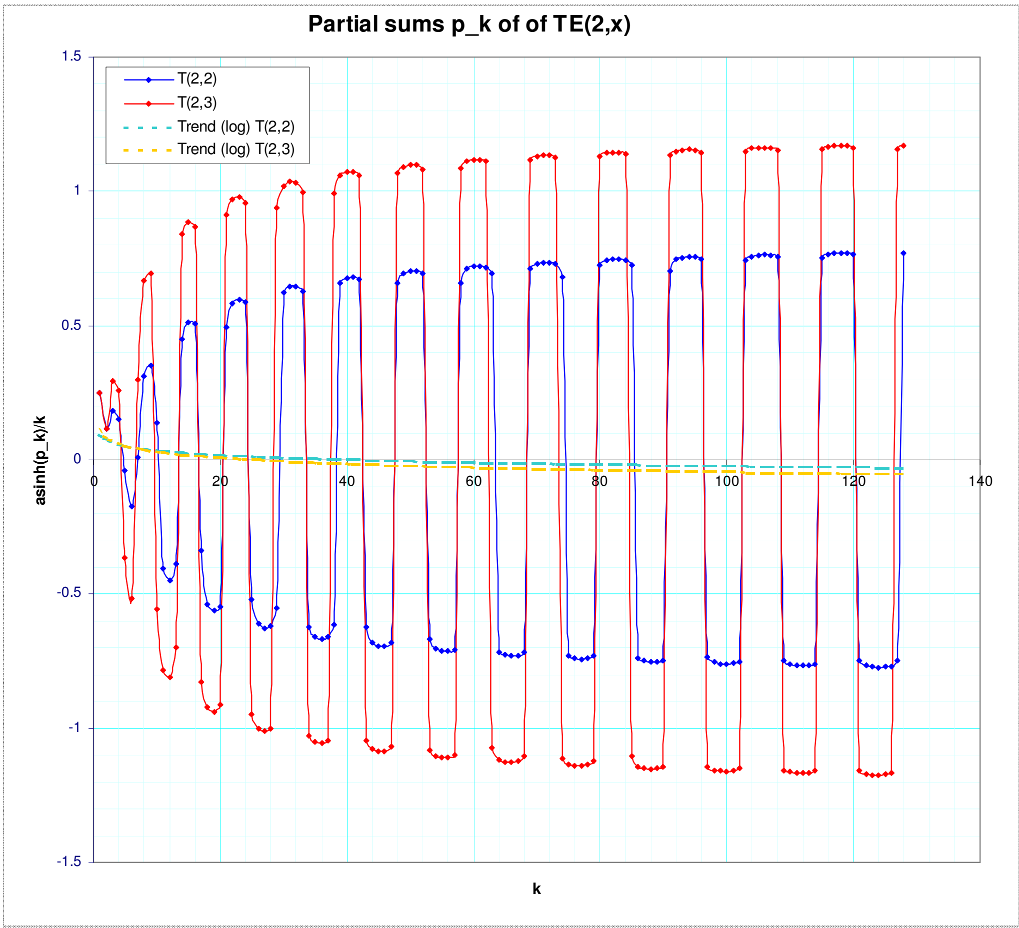
we get the result for the simple evaluation of eta-series

$y\sim = [\eta(0), \eta(-1), \eta(-2),...] = [1/2, 1/4, 0, ...]$

as expected.

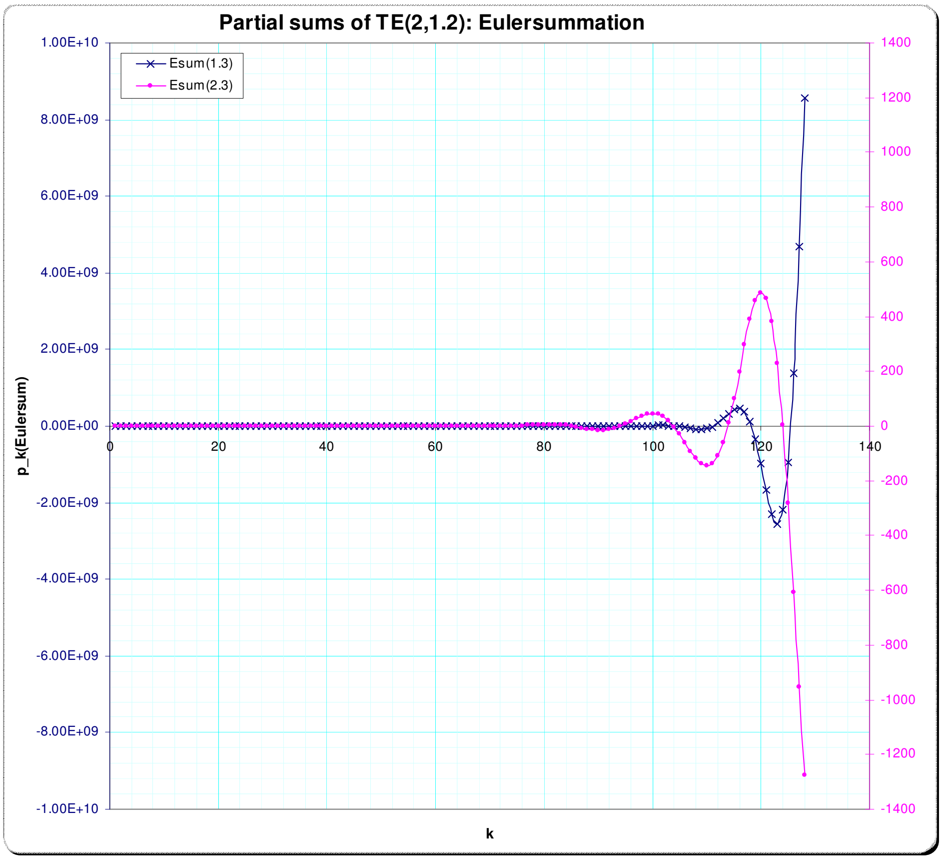
Appendix 2: Trend of *TE(2,x)* for *x>1*

Here the partial sums p_k for the evaluation of the series for *TE(2,x)* for $x=2$ and $x=3$ are shown. The values of p_k are scaled such that $y_k = asinh(p_k)/k$. Logarithmic trends (construction by Excel) are also inserted shown by the dotted lines.

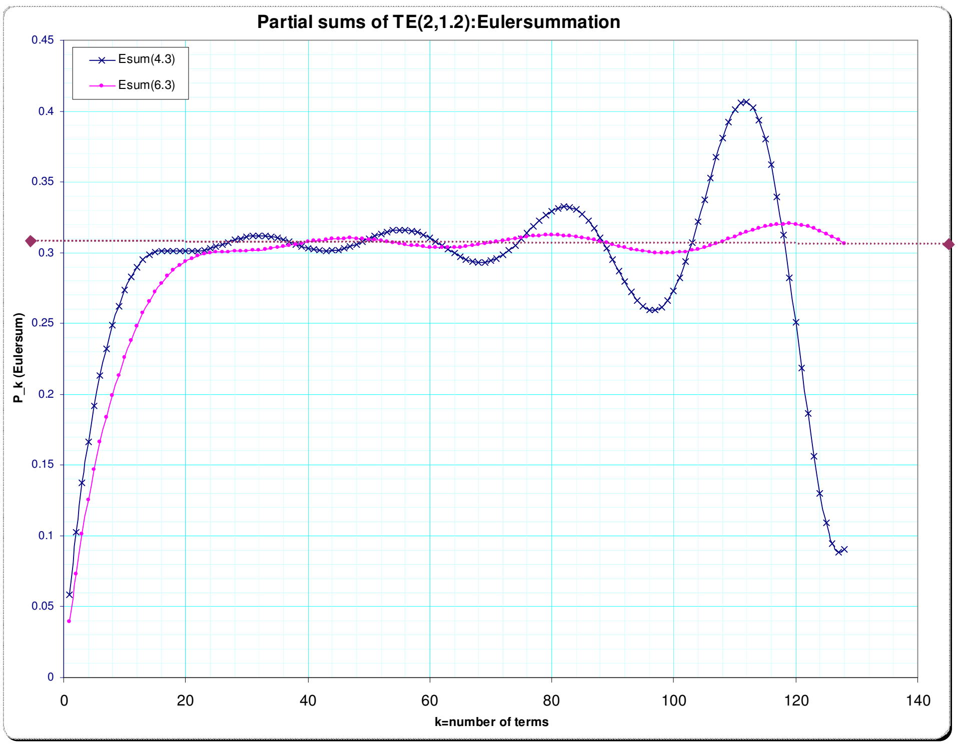


Appendix 2.1: Eulersummation

The simple Eulersummation does not help much here. For low orders of about *Eulersum(1)* (no implicite transformation) and *Eulersum(2)* (binomial transformation) we get the following plot for the approximation using a more moderate value for *x*, namely *x=1.2* :



Higher orders give at least a clue of a final value, but still no apparently converging approximation



Note, that due to the strong divergence of the series the oscillation will diverge again with higher *k*.

Appendix 2.2: additional Stirling-transformation

A problem with this Euler-summation is, that even the center of the oscillation (the local mean) decreases or increases, so an early cut of the partial evaluation of the series at a "good" local mean does not allow to infer the "final" value.

To improve this I try an additional Stirling-transform of the terms before computing the partial sums and the Euler-summation. Still this does not give decisive results, but it may be interesting, that at least the characteristic of the local means seem to be better: they seem to meet a "final" value much better.

The Stirling-transform of the original terms in a column-vector T means, in matrix-notation

$$T_1 = {}_dV(log(1+1)) * (S2 * T)$$

where $S2$ is a factorial scaled matrix of Stirling numbers of second kind. I also tried some iterations of this, such that

$$T_2 = {}_dV(log(1+log(1+1))) * (S2^2 * T)$$
$$T_3 = {}_dV(log(1+log(1+log(1+1)))) * (S2^3 * T)$$

and so on. These transformations are asymptotically regular, since the other way of associativity of the partial expressions

$$V(1)\sim = V(log(1+1))\sim * S2$$
$$V(1)\sim = V(log(1+log(1+1)))\sim * S2^2$$
$$V(1)\sim = V(log(1+log(1+log(1+1))))\sim * S2^3$$

gives asymptotically always the simple $V(1)\sim$ - summation vector for T, such that, writing t_r for the r'th original term we get asymptotically the original sum

$$s_{oo} = V(1)\sim * T = V(1)\sim * T_1 = V(1)\sim * T_2 = ... = \Sigma_{r=0..inf}\, t_r$$

The matrix $S2$ is

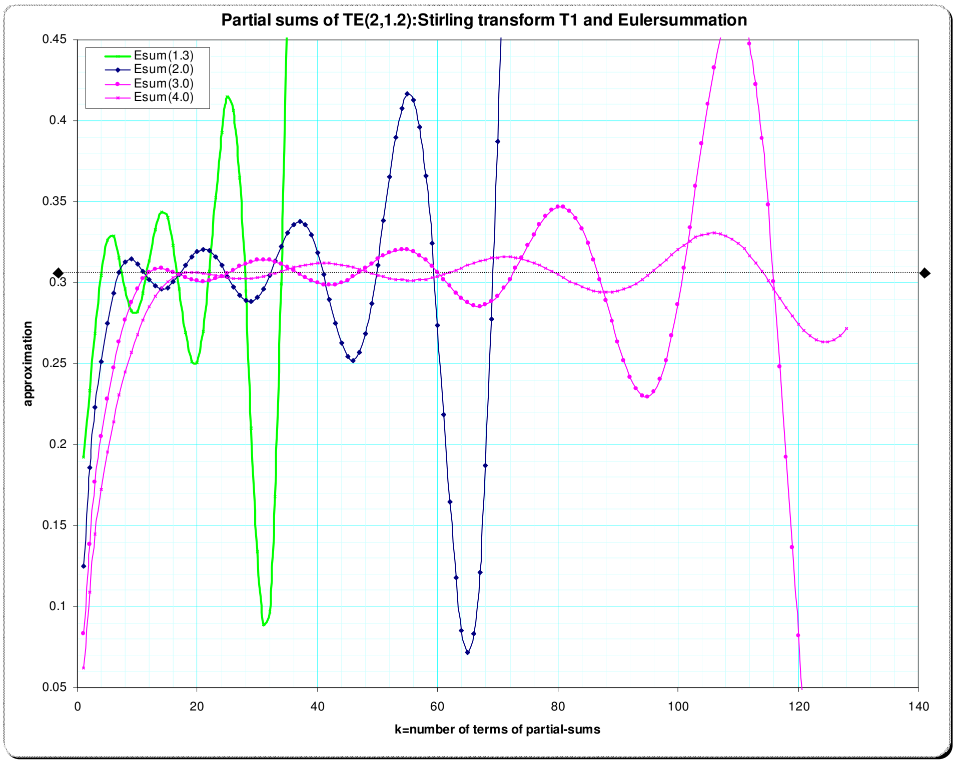
$$S2 := [v_{r,c}] = matrix(\{r,c\} * c! / r!) \qquad (r,c \text{ row and col-index ,beginning at } 0)$$

where $\{r,c\}$ is the stirling-number of second kind and $S2$ looks like

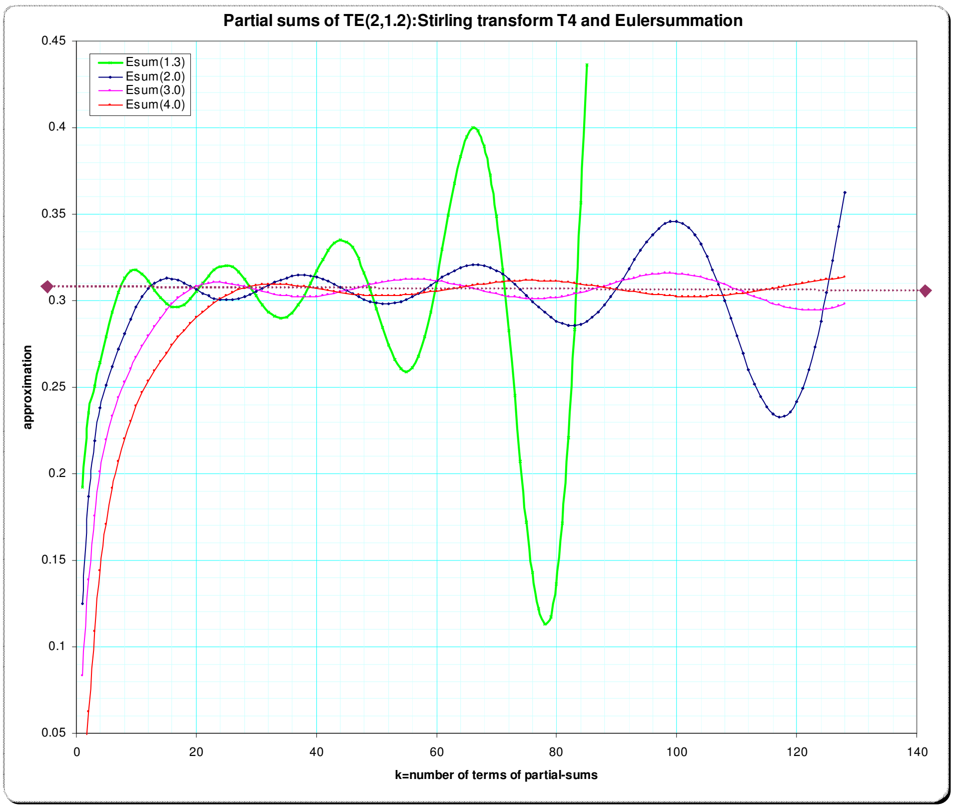
$$S2 = \begin{bmatrix} 1 & . & . & . & . & . \\ 0 & 1 & . & . & . & . \\ 0 & 1/2 & 1 & . & . & . \\ 0 & 1/6 & 1 & 1 & . & . \\ 0 & 1/24 & 7/12 & 3/2 & 1 & . \\ 0 & 1/120 & 1/4 & 5/4 & 2 & 1 \end{bmatrix}$$

Then using the partial sums of T_2, T_3 or T_4 means to implement orders of the Stirling-transform of T and their Euler-sums show a behaviour with a bit better image.

Using T_1:



Using T_4:



where we see, that low orders of Euler-sum still don't limit the oscillating divergence, but the overall picture looks already much better, especially at order 3 or 4, where we may have arrived at an order which will finally give a useful approximate, if more terms were involved.

But all this does not help much. A far better summation-procedure is required...

Appendix 3: change of order of summation

The series *TE(2,x)* can possibly be evaluated with better performance, if order of summation is changed. Here I propose the summation along diagonals of the two-way array of terms of the double-sum in

$$TE(2,x) \qquad = 1^{1^x} - 2^{2^x} + 3^{3^x} - + \dots$$
$$= \sum_{r=0}^{oo} \sum_{k=0}^{oo} \left(x^r k^r \binom{r+k}{k} * \frac{\lambda(k+r)}{(k+r)!} \right)$$

First introduce the sum $d=r+k$ and replace for clarity of indexes $e=r$, such that also $k=d-e$

$$TE(2,x) \qquad = 1^{1^x} - 2^{2^x} + 3^{3^x} - + \dots$$
$$= \sum_{d=0}^{oo} \sum_{e=0}^{d} \left(x^e k^e \binom{d}{d-e} * \frac{\lambda(d)}{d!} \right)$$

then the $\lambda()$-term can be extracted from the inner sum:

$$TE(2,x) \qquad = \sum_{d=0}^{oo} \left(\frac{\lambda(d)}{d!} * \sum_{e=0}^{d} \left(x^e (d-e)^e \binom{d}{e} \right) \right)$$

and, for better computing-performance, the factorial can be cancelled against the binomial:

$$TE(2,x) \qquad = \sum_{d=0}^{oo} \left(\lambda(d) * \sum_{e=0}^{d} \left(\frac{x^e}{e!} \frac{(d-e)^e}{(d-e)!} \right) \right)$$

The inner sum is now finite for any d; I assume, it vanishes always for $d \to inf$ for any x, though I didn't prove this. Critical is then the rate of decay in relation to the rate of increase of the lambda-term.

For the numerical computation I could not exploit this reformulation to get significantly better results for $x>=1$, since we still need high order of Euler-summation (if Eulersummation should suffice at all)

Appendix 4: Tetra-eta-series for height 3

(Text copied from file tetration-intro-short.doc, without editing)

Insert s^x at the position of x in the previous formula for the single powertower:

$$
\begin{aligned}
s^{s^{s^x}} &= \sum_{r_0=0}^{oo}\sum_{r_1=0}^{oo}\left(\frac{log(s)^{r_0+r_1} r_0^{r_1}}{r_0!\, r_1!}(s^x)^{r_1}\right)\\
&= \sum_{r_0=0}^{oo}\sum_{r_1=0}^{oo}\left(\frac{log(s)^{r_0+r_1} r_0^{r_1}}{r_0!\, r_1!}\left(\sum_{r_2=0}^{oo}\frac{log(s)^{r_2} r_1^{r_2}}{r_2!} * x^{r_2}\right)\right)\\
&= \sum_{r_0=0}^{oo}\sum_{r_1=0}^{oo}\sum_{r_2=0}^{oo}\left(\frac{log(s)^{r_0+r_1} r_0^{r_1}}{r_0!\, r_1!}\frac{log(s)^{r_2} r_1^{r_2}}{r_2!} x^{r_2}\right)
\end{aligned}
$$

to arrive at the most concise form:

$$
s^{s^{s^x}} = \sum_{r_0=0}^{oo}\sum_{r_1=0}^{oo}\sum_{r_2=0}^{oo}\left(log(s)^{r_0+r_1+r_2}\frac{r_0^{r_1} r_1^{r_2}}{r_0!\, r_1!\, r_2!} x^{r_2}\right)
$$

If we reorder this again to summation along the antidiagonals, initially introduce the diagonal-counter d, letting $d=r_0 + r_1 + r_2$ and $q_0= d - r_0 = r_1+r_2$ and $q_1=r_2$ then we have

$$
s^{s^{s^x}} = \sum_{d=0}^{oo}\sum_{q_0=0}^{d}\sum_{q_1=0}^{q_0}\left(log(s)^d\frac{(d-q_0)^{q_0-q_1}(q_0-q_1)^{q_1}}{(d-q_0)!(q_0-q_1)!\, q_1!} x^{q_1}\right)
$$

or, reordered for less extensive computation, involving only finite series for the inner double sum:

$$
s^{s^{s^x}} = \sum_{d=0}^{oo}\left(log(s)^d\sum_{q_0=0}^{d}\left(\frac{1}{(d-q_0)!}\sum_{q_1=0}^{q_0}\left(\frac{(d-q_0)^{q_0-q_1}}{(q_0-q_1)!}\frac{(q_0-q_1)^{q_1}}{q_1!} x^{q_1}\right)\right)\right)
$$

The expression for the inner double sum seems to converge to zero after it approaches a certain local maximum, however, as in the case of height $h=2$ I did not explicitely determine the characteristics of this convergence in relation to powers of $log(s)$, dependent on s and x, yet.

The formula for the alternating series $TE(3,x)$ occurs then simply, if the single $log(s)^d$-term is replaced by the $\lambda(d)$-term for the linear combination of powers of logs as in case for height 2:

$$
TE(3,x) = \sum_{d=0}^{oo}\left(\lambda(d)\sum_{q_0=0}^{d}\left(\frac{1}{(d-q_0)!}\sum_{q_1=0}^{q_0}\left(\frac{(d-q_0)^{q_0-q_1}}{(q_0-q_1)!}\frac{(q_0-q_1)^{q_1}}{q_1!} x^{q_1}\right)\right)\right)
$$

- no plot or computations yet -

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