Finding roots for *f(z) = z^z^z+1*

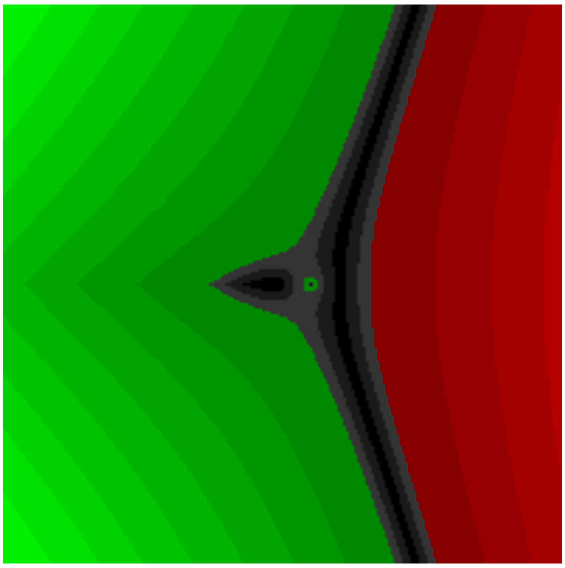
Roots for *g(z)*

To get a first insight in the problem I made computations easier by looking at the double logarithm of *f(z)*:

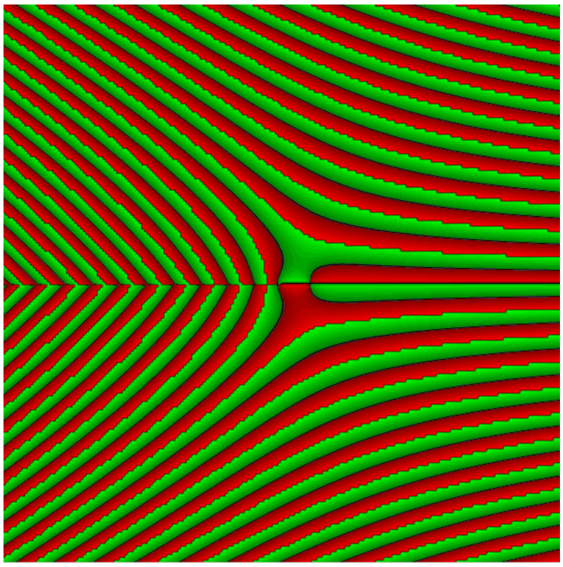
$$
\begin{aligned}
g(z) \quad &= \log(\log(z^{z^z})) - \log(\log(-1)) \\
&= \log(lz \cdot \exp(lz \cdot z)) - ll\_1 \\
&= llz + lz \cdot z - ll\_1
\end{aligned}
$$

The real and imaginary parts for the evaluation of *g(z)* at complex *z* are shown as distances from *zero*, where green color indicates negative values, red color positive value, and gray/black color values near *zero*. Darker grey is nearer *zero*, lighter green/red farther from *zero*.

REAL (g(z))



IMAG (g(z))



In each picture the respective zeros seem to be arranged in certain shapes because they must occur on the borders between green and red (thus in the black regions).

In the left picture, on the real values of *g(z)*, we see that the potential candidates for the roots, (numbers *z* for which the ***real part*** of *g(z)* becomes zero) are found in that black region which looks vertically like a boomerang having a triangular appendix to the left.

In the right picture, the candidate roots (which make the ***imaginary parts*** of *g(z)* zero) follow lines which are in the tendency horizontally (and appear also curved). They occur with some periodicity.

Because zeros of *g()* can only occur, where both pictures show their zeros ***simultanously***, an overlay of that two pictures should make it better visible, where roots of the function *g()* can reside at all: in that boomerang region and vertically with discrete distance and roughly periodically.

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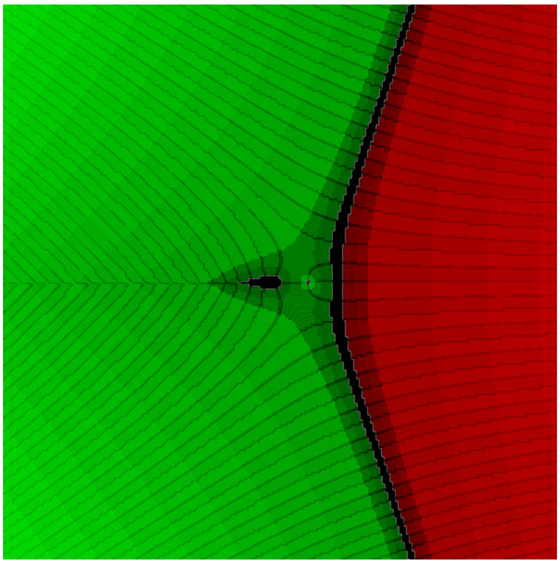
*This exercise was triggered by the question "**Solve x^x^x=-1**" on math.stackexchange.com 30. Aug 2015 ,see* https://math.stackexchange.com/q/1415029/ *and improves and supersedes my there given answers!*

Overlay of the left picture with the black lines of the right picture (the black region of the right picture is also a bit more detailed)

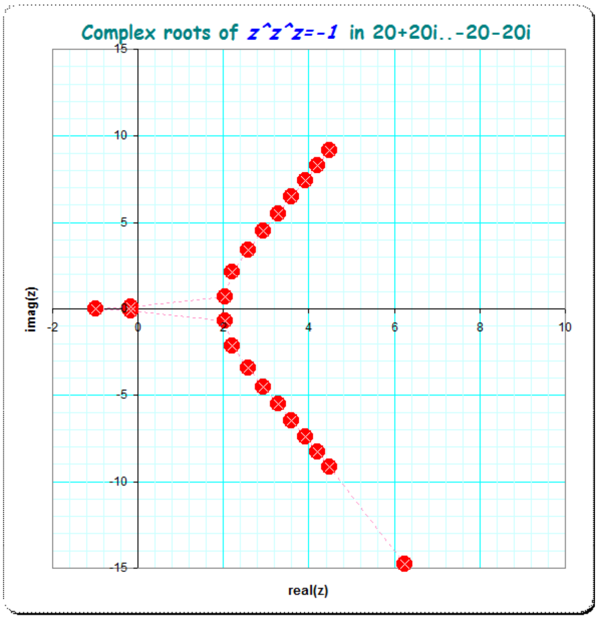


Important disclaimer: the pictures are not "exclusive" w.r.t the existence of roots for *f(z)* : I only compare ***the equality of the double-logarithms*** which occur in the evaluation of *g(z)* and because of multivaluedness of logarithms they form only a subset of all roots for *f(z)* !

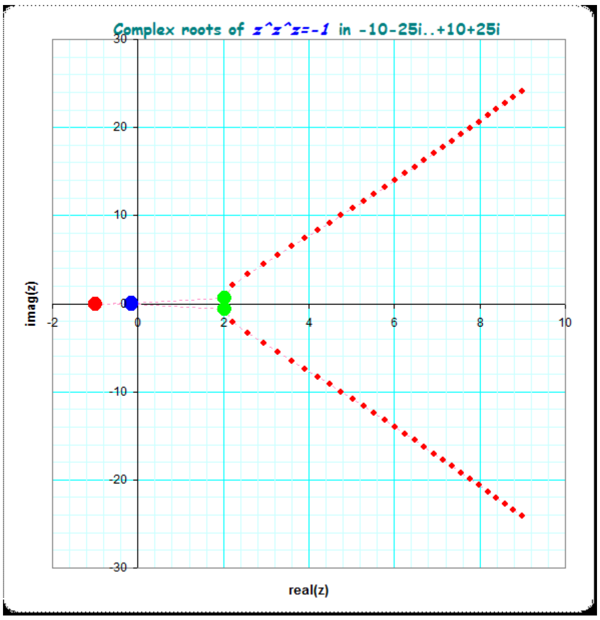
Tracing for more roots for *g(z)*

Based on that visible impression, I built a tracer which searches further roots for *g()*. First it found two more roots ρ_3,ρ_4 towards the right halfplane nearly on a linear line with ρ_2. From that it extrapolates the initial list of three roots ρ_2,ρ_3,ρ_4 based on the direction and distances of their occurences, and traces the way along the boomerang-region to outwards. That reduced the searchspace much and made the numerical search much more efficient. Quickly one could find first -say- *20* new roots, and getting trustful on the mathematical principle even simply new *200* roots, suggesting that this is an obvious principle generalizable for an infinite set of roots, all roughly (but increasingly!) linearly aligned.

```
      ρk
 -----------------------------------------
ρ0  = -1
ρ1  ≈ -0.15890875158 + 0.0968231909176*I
ρ2  ≈  2.03426954187 + 0.678025662373*I
ρ3  ≈  2.21022616044 + 2.14322152216*I
ρ4  ≈  2.57448299040 + 3.39212026316*I
ρ5  ≈  2.93597198855 + 4.49306256310*I
ρ6  ≈  3.27738123699 + 5.51072853255*I
ρ7  ≈  3.60013285730 + 6.47345617876*I
ρ8  ≈  3.90713751281 + 7.39619042452*I
ρ9  ≈  4.20091744993 + 8.28794173821*I
ρ10 ≈  4.48346951212 + 9.15465399776*I
ρ11 ≈  4.75636133031 + 10.0005052039*I
  ...               ...
  zk+1 ≈ ρk + (ρk - ρk-1)*0.96
                as initial value for the Newton algorithm on f(z)= ³z - (-1)
 Newton(zk+1) => ρk+1
```



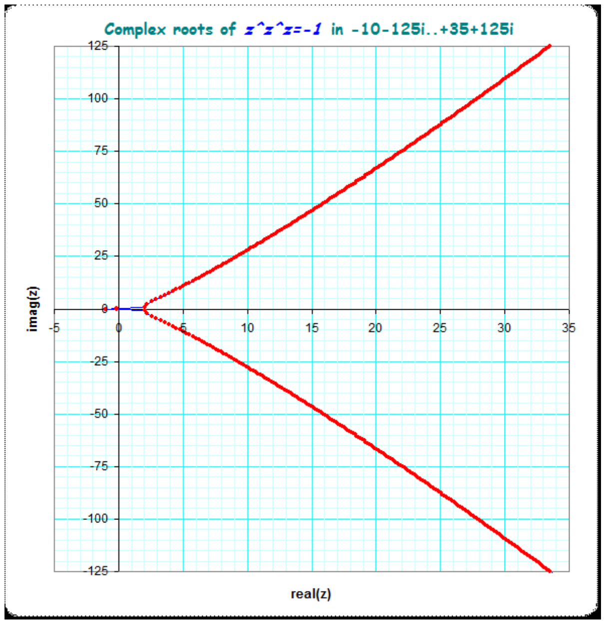
After this, proceeding with that idea one finds easily more roots



The first 21 (plus 20 conjugates) roots. The big points are that first three roots which I found initially using the Newton-iteration on g(z).

Their colours indicate the initial regions from where the Newton-iteration discovers them (see next picture)

and even the suggestion of existence of infinitely many more ...



Here are the first 201 roots (plus 200 conjugates)

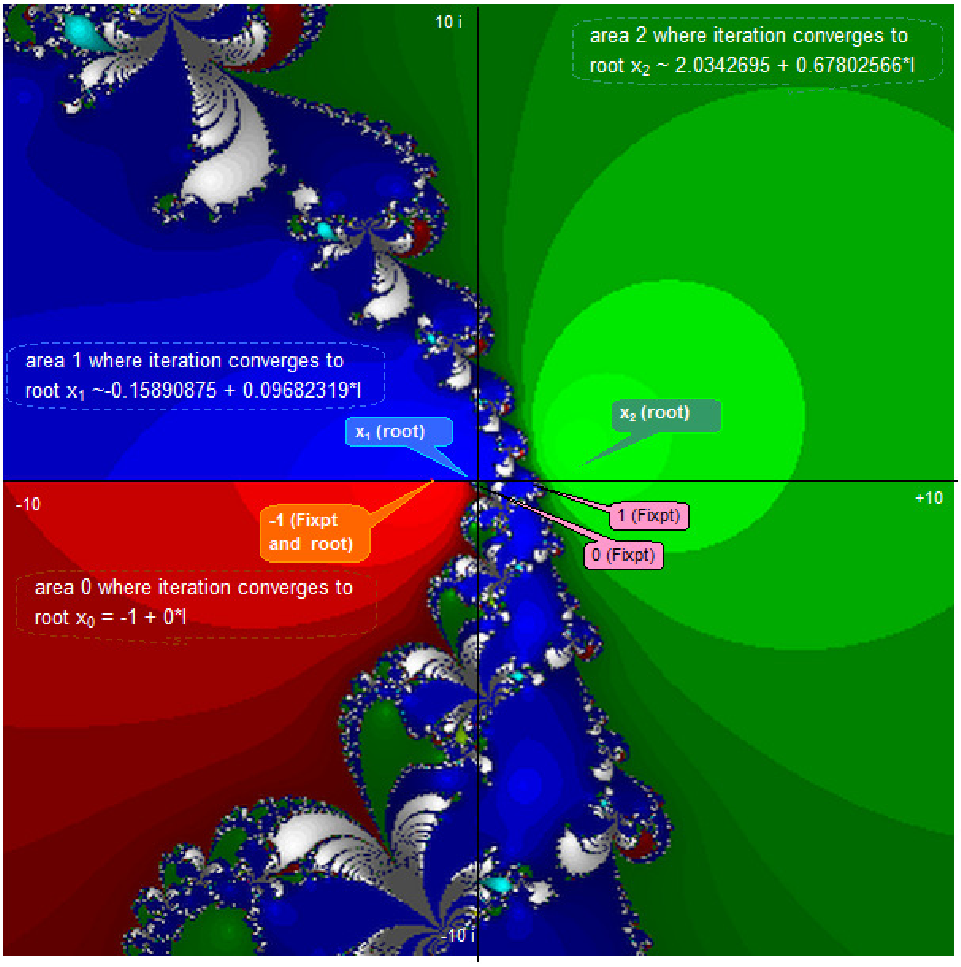
A Newton Fractal

The term "Newton-fractal" (*https://de.wikipedia.org/wiki/Newtonfraktal*) is a fairly common term which means just what I intuitively made here: it documents the orbit of some initial complex value z_0 when the Newton-iteration is applied to some function $f(z)$ to find a root. Then the found root gets a certain color associated and this point of color is placed in a complex scatterplot at the coordinate of the initial value z_0. This gives then a visual impression of at which roots – starting from any z_0 – one arrives. There are pictures of impressive fractals available in the internet, so far as I have seen them only for $f(z)$ being a polynomial, though.

Because I didn't know that this is also a well known and well studied process, and thus had to develop my algorithms myself, the results might not be the most accurate ones, but I think the following picture still gives a valueable impression of even a fractal structure.

Newton-fractal for *g(z)*

The first 3 roots known were ρ_0=-1, ρ_1~-0.1589 + 0.0968î , ρ_2~2.034+0.678î



This picture might be a bit misleading - while evaluating for $g(z)$ it doesn't evaluate for the complex conjugates of the roots for $f(z)$ which must also appear. I've not yet a properly updated version of this picture.

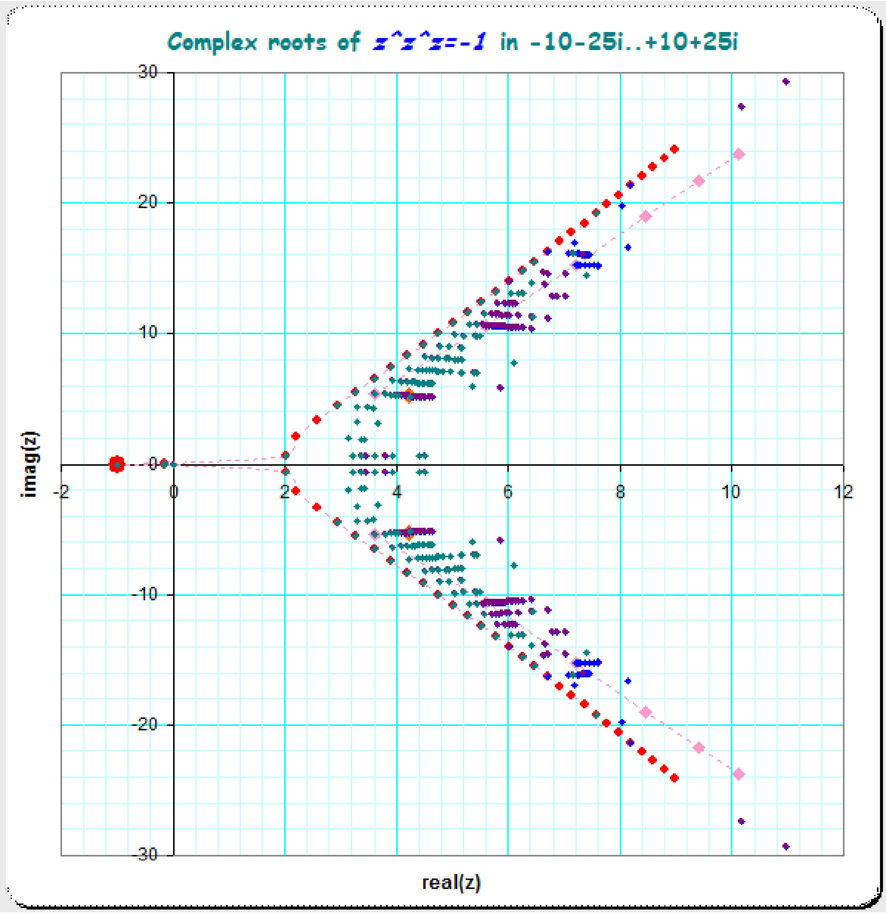
Roots ρ for *f()* (meaning $f(\rho)=0$)

After sneaking into the general situation via considering *g(z)* only, a second approach which applied the Newton-iteration for finding roots now actually for *f()* gave a set of additional roots ρ, such that $f(\rho)=0$. On first sight the localization of this set looked completely chaotic - besides the surprising observation that the new found roots occur only in the right-hand open parabola limited by the shape which has been found for roots of *g()* already.

Then searching with even finer intervals for the initial values z_0 for the Newton-iteration gave always more and more roots ρ_{z0}, and that increasing number of roots begin to show some new structure: see the dotted, but approximately linear lines, on which the roots seem to accumulate. This is not yet sufficiently studied though:

Complex roots ρ of *f()* such that $f(\rho)=0$

Remark: different colors are applied due to different searching episodes with slightly changed specific sets of initial values z_0 *leading to according zeros* ρ_{z0}*. Red dots are the roots already known from* $g(\rho_z)=0$



Detail which shows from what small interval of initial values z_0 for the Newton-iteration new roots ρ_{z0} can be found! Starting at some known root ~3.6+5.8î scanning vertically one imaginary unit upwards in steps of 0.001 finds 105 new roots!